PROGRESSIVE
ALGEBRA
FOR
INDIAN HIGH SCHOOLS

NEW REVISED & ENLARGED EDITION

Bharat

Suraj Balram Sawhney

Progressive Algebra

FOR

Indian High Schools

BY

A Board of Eminent Mathematicians

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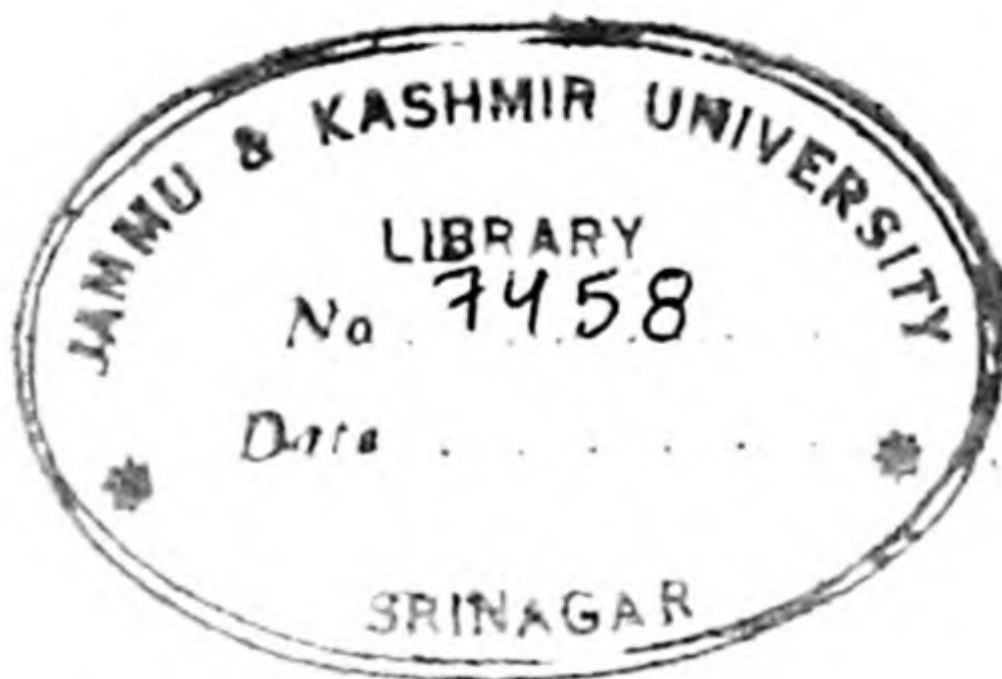
PREFACE TO THE ELEVENTH EDITION.

This is a completely over-hauled edition of the book and presents the matter in an entirely new form. The authors claim to have introduced the following distinguishing features in it :—

(1) The various exercises, groups and examples in the book, have been graded according to difficulty with scrupulous care.

(2) The division of each excercise into different groups is a provision for teachers of two extreme views. Those who are anxious to cover the course hurriedly at first, can easily do so by taking the first two questions of each group, while those who want to do everything exhaustively on the every first reading, can do so by taking all questions in full.

(3) At least one question of each group (generally the first) has been solved and hints have been provided for various others. These solutions and hints have been put *at the end of* each exercise, so that the students may consult them only when they feel the necessity of doing so after an honest but unsuccessful self-attempt. Each exercise is preceded by a clear exposition of the principles and methods needed for it, but the authors are against telling too much beforehand and thus damping the originality of the students.

In short, a sincere attempt has been made by the authors to educate the students in the real sense of the word. It is confidently hoped that the work will be appreciated by the teachers and the taught alike. Any suggestion for further improvement of the book will be thankfully received.

Feb. 21. 1950.

Authors.

CONTENTS

LIST OF FORMULAE

1. $(a+b)^2 = a^2 + 2ab + b^2.$
2. $(a-b)^2 = a^2 - 2ab + b^2.$
3. $(a+b)^2 + (a-b)^2 = 2(a^2 + b^2).$
4. $(a+b)^2 - (a-b)^2 = 4ab.$
5. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$
6. $(a+b+c+d+e+ \ldots\ldots\ldots\ldots)^2$
$$= a^2 + b^2 + c^2 + d^2 + e^2 \ldots\ldots\ldots \ldots$$
$$+ 2a(b+c+d+e+\ldots\ldots\ldots\ldots)$$
$$+ 2b(c+d+e+\ldots\ldots \ldots\ldots)$$
$$+ 2c(d+e+\ldots\ldots\ldots\ldots)$$
$$+ \ldots\ldots \quad \ldots\ldots\ldots$$

7. $(a+b)(a-b) = a^2 - b^2.$
8. $(x+a)(x+b) = x^2 + (a+b)x + ab$
9. $(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc.$
10. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ or $a^3 + b^3 + 3ab(a+b).$
11. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ or $a^3 - b^3 - 3ab(a-b).$
12. $(a+b)(a^2 - ab + b^2) = a^3 + b^3.$
13. $(a-b)(a^2 + ab + b^2) = a^3 - b^3.$
14. $(a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc) = a^3 + b^3 + c^3 - 3abc.$

-----

15. $a^2(b-c) + b^2(c-a) + c^2(a-b) = -(a-b)(b-c)(c-a).$
16. $ab(a-b) + bc(b-c) + ca(c-a) = -(a-b)(b-c)(c-a).$
17. $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2) = (a-b)(b-c)(c-a)$
18. $a^3(b-c) + b^3(c-a) + c^3(a-b) = -(a-b)(b-c)(c-a) \times$
$$(a+b+c).$$
19. $a(b^3 - c^3) + b(c^3 - a^3) + c(a^3 - b^3) = (a-b)(b-c)(c-a) \times$
$$(a+b+c).$$
20. $a^4(b-c) + b^4(c-a) + c^4(a-b) = -(a-b)(b-c)(c-a) \times$
$$(a^2 + b^2 + c^2 + ab + bc + ca).$$
21. $a^3(b^2 - c^2) + b^3(c^2 - a^2) + c^3(a^2 - b^2) = -(a-b)(b-c)(c-a) \times$
$$(ab + bc + ca).$$
22. $a^2(b^3 - c^3) + b^2(c^3 - a^3) + c^2(a^3 - b^3) = (a-b)(b-c)(c-a) \times$
$$(ab + bc + ca).$$
23. $ab(a^3 - b^3) + bc(b^3 - c^3) + ca(c^3 - a^3) = -(a-b)(b-c)(c-a)$
$$\times (a^2 + b^2 + c^2 + ab + bc + ca).$$

-----

24. $a^m \times a^n = a^{m+n}$.
25. $a^m \div a^n = a^{m-n}$.
26. $(a^m)^n = a^{mn}$.
27. $(ab)^n = a^n . b^n$.

28. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$.

29. $a^0 = 1$.

30. $a^{-n} = \dfrac{1}{a^n}$.

———

$\sqrt{a}$ means the square root of a.

$a^{\frac{1}{p}}$ „ the pth root of a.

$a^{\frac{p}{q}}$ „ the qth root of a^p or the pth power of the qth

root of a ; that is, $\sqrt[q]{a^p}$ or $\left(\sqrt[q]{a}\right)^p$.

———

In the following results a is a finite quantity, not equal to zero (a finite quantity is that which can be measured) :

$1+0=1,\ 1-0=1,\ 0-1=-1$.
$0\times1=0,\ 0\times a=0,\ 0\times0=0$.

$\dfrac{0}{1}=0,\quad \dfrac{0}{a}=0,\quad \dfrac{a}{a}=1$.

$\dfrac{1}{0}=$ infinity (*i.e.*, a limitless large quantity which cannot be measured. The symbol ∞ is used to denote it.)

$\dfrac{a}{0}=\infty$

$a^0=1,\ 0^a=0$,

$\dfrac{0}{0}$ is an *indeterminate quantity*, that is, a quantity, which has no fixed value. A complete discussion of such quantities is beyond the scope of this book.

———

PROGRESSIVE ALGEBRA

CHAPTER I

PRIMARY DEFINITIONS & NOTATIONS ; NUMERICAL

SUBSTITUTIONS

1. ALGEBRA is *generalised Arithmetic*.

Besides the Arithmetical figures (1, 2, 3, etc.), it makes use of letters (like *a, b, c,* etc.,) as well, called *symbols of quantity*. These symbols stand for arithmetical numbers and may have any values whatever. However, in a particular piece of work, a particular letter is supposed to keep the same value throughout.

Algebra also involves the use of *negative quantities* which have no place in Arithmetic. The idea of such quantities will be given in the next chapter.

The five signs $+$, $-$, $\times$, $\div$ and $=$ have the same meanings in Algebra as in Arithmetic and are called *symbols of operation*. A dot can replace the symbol $\times$, as in 3.4, which means 3×4. (Distinguish this from 3·4, in which the dot is placed higher and stands for the decimal point.)

2 The student is advised to study the following facts very carefully :—

(i) $2a$ stands for $2 \times a$ or $2.a$

   ab ,,   ,, $a \times b$ or $a.b$

   $4xyz$,,   ,, $4 \times x \times y \times z$ or $4.x.y.z$, etc., etc.

Also note that :—

$4xyz$ is the *product of* the four *factors* 4, *x*, *y* and *z*.

   4 is the *coefficient* of xyz.

   $4x$,,   ,, yz.

$4xy$,,   ,, z.

(*ii*) a^2 stands for $a \times a$.

    a^3 ,, $a \times a \times a$.

    x^4 ,, $x \times x \times x \times x$.

Also note that :—

a^2 is called the *second power of a* and is usually read "*a* squared." '*a*' is the *base* of this power and 2 is the *index* or *exponent*.

a^3 is called the *third power of a* and is usually read "*a* cubed." Here also the base is '*a*', but index$=3$.

x^4 is called the *fourth power of x* and is usually read "*x* to the fourth." Here base is x and index$=4$, etc., etc.

(*iii*) $\sqrt{a}$ stands for a number whose second power (or square) is equal to a. It is called *the square root of a*. For example, $\sqrt{9}$ (*i.e.*, square root of 9)$=3$, because the square of 3 is 9.

$\sqrt[3]{a}$ stands for a number whose third power (or cube) is equal to a. It is called the *cube root of a*. For example, $\sqrt[3]{125}$ (*i.e.*, cube root of 125)$=5$, because the cube of 5 is 125.

Similarly $\sqrt[4]{81}$, (*i.e.*, fourth root of 81)$=3$; etc., etc.

Note 1. It is immaterial in what order the factors of a product are written. Thus ab and ba have the same value, so also have abc, acb, bca, bac, cab and cba the same value. However, the *numerical factor* (*i.e.*, the factor which is an arithmetical number, should always be written first. For example, $3a$ should not be written as $a3$. However, it may be written as $a \times 3$ or $a.3$.

Note 2. When the factor or index is unity (*i.e.*, 1) it is usually omitted. Thus "$1a$", as well as "a^1", is usually written as "a".

Note 3. Fractional co-efficients which are greater than unity are usually kept in the form of improper fractions. For example, we usually write $\frac{11}{4}a$, and not $2\frac{3}{4}a$.

3. Substitution.

Replacing one quantity by another equivalent quantity is called *substitution*. In this chapter we shall consider only

numerical substitutions, that is, the substitutions of algebraical symbols (a, b, c, etc.,) by their given numerical values.

EXERCISE 1

If $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, find the value of :—

1. $8c$. [Solved]
2. $9d$.
3. $12a$.
4. $5be$. [Solved]
5. $6bd$.
6. abc.
7. $2d^3$. [Solved]
8. $5e^2$
9. $6b^4$.
10. $4c^3$
11. $5a^5$.
12. b^c. [Solved]
13. c^b.
14. e^d.
15. a^e.

If $x=2$, $y=3$. $z=0$, $p=1$, $q=4$, find the value of :—

16. $\frac{3}{4}x^3$. [Solved]
17. $\frac{1}{8}q^2$.
18. $\frac{4}{9}y^4$.
19. $\frac{1}{8}y^2x^3$. [Solved]
20. $\frac{7}{13}q^2p^4$.
21. $\frac{1}{8}y^2q^3$.
22. $\frac{1}{4}x^2yz$. [Solved]
23. $\frac{4}{9}y^2z^2q$.
24. $\frac{10}{7}x^2y^3z^4$.
25. $\dfrac{9x^3y^2}{2q}$. [Solved]
26. $\dfrac{9p^2q^3}{8y^3x^3}$.
27. $\dfrac{3y^2zq^2}{5x}$.
28. $\dfrac{3x^y}{2q^p}$.
29. $\dfrac{5z^q}{6p^y}$.
30. $\dfrac{72}{3pq^2x^3}$.

If $a=0$, $b=1$, $x=4$, $y=8$, $l=9$, evaluate :—

31. $\sqrt{2xy}$. [Solved]
32. $\sqrt{bxl}$.
33. $\sqrt{3abx}$.
34. $\sqrt[3]{\dfrac{2xb}{3l}}$ [Solved]
35. $\sqrt[3]{\dfrac{3bl}{2xy}}$
36. $\sqrt[3]{\dfrac{9l}{4bx}}$
37. $\sqrt[3]{\dfrac{bxy}{27l}}$.
38. $\sqrt[3]{\dfrac{abx}{yl}}$.
39. $2x\sqrt{\dfrac{xl}{2y}}$.

[Solved]

40. $5b\sqrt{\dfrac{2xy}{25l}}$.
41. $a\sqrt{\dfrac{ly^2}{x}}$.
42. $2y\sqrt{\dfrac{bx^3}{2y}}$.
43. $\sqrt{\dfrac{b^x}{x^b}}$.
44. $\sqrt[3]{\dfrac{a^y}{b^x}}$.
45. $\sqrt[3]{\dfrac{3ly^2x^3}{8b}}$.

SOLUTIONS & HINTS—EXERCISE 1

1. $8c=8\times c=8\times 3=24$. 4. $5be=5\times b\times e=5\times 2\times 5=50$.
7. $2d^3=2\times d\times d\times d=2\times 4\times 4\times 4=128$.

12. $b^c = 2^3 = 2 \times 2 \times 2 = 8$.

16. $\frac{3}{4}x^3 = \frac{3}{4} \times 2^3 = \frac{3}{4} \times 2 \times 2 \times 2 = 6$.

19. $\frac{1}{6}y^2 x^3 = \frac{1}{6} \times 3^2 \times 2^3 = \frac{1}{6} \times 3 \times 3 \times 2 \times 2 \times 2 = 12$.

22. $\frac{1}{4}x^2 yz = \frac{1}{4} \times 2 \times 2 \times 3 \times 0 = 0$.

[**Remember** that when any factor is zero, the product is zero.]

25. $\dfrac{9x^3 y^2}{2q} = \dfrac{9 \times 2 \times 2 \times 2 \times 3 \times 3}{2 \times 4} = 81$.

31. $\sqrt{2xy} = \sqrt{2 \times 4 \times 8} = \sqrt{64} = 8$

34. $\sqrt[3]{\dfrac{2xb}{3l}} = \sqrt[3]{\dfrac{2 \times 4 \times 1}{3 \times 9}} = \sqrt[3]{\dfrac{8}{27}} = \dfrac{2}{3}$.

39. $2x\sqrt{\dfrac{xl}{2y}} = 2 \times 4 \times \sqrt{\dfrac{4 \times 9}{2 \times 8}} = 8 \times \sqrt{\dfrac{9}{4}} = 8 \times \frac{3}{2} = 12$.

4. Expressions.

An *algebraical expression* is a collection of symbols. It may consist of one or more *terms*, which are separated from each other by the signs $+$ and $-$. Thus $2a - 3b + 4xy + z + 5t^3$ is an expression consisting of five terms.

An expression consisting of one term only is called a *simple expression* or a *monomial*, as $2a$ or $4xy^2z$.

Expressions consisting of more than one term are called *compound expressions*.

The following sub-divisions of compound expressions may be noted :—

A *binomial* consists of two terms, as $3a - 4b$.

A *trinomial* consists of three terms, as $3a - 4b + 9c$.

A *polynomial* or *multinomial* consists of more than three terms, as $a - 2b + 3c + d$, $ab + bc - c - a + d$, etc.

5. In working examples, the student is advised to pay attention to the following instructions :—

(*i*) Be very neat. This will ensure accuracy.

(*ii*) Learn the proper use of the sign $=$. It should be used only to connect the quantities which are equal and not employed in any vague or inexact sense. For example, the following process is defective :—

Q. If $a=2$, $b=3$, $c=4$, find the value of $5abc-6c^2$.

Sol. $5abc-6c^2 = 5\times2\times3\times4$
$$= 120-6\times4\times4$$
$$= 120-96$$
$$= 24.$$

The answer is correct, but the process is wrong, for, in the first line of the solution, the sign $=$ does not connect equal quantities. In the second line, too, this sign is wrongly used.

(*iii*) In the steps of the work, place the signs of equality one below the other, as shown in the above process. However, if the expressions are very short, this restriction may be ignored.

EXERCISE 2

If $a=1$, $b=2$, $c=3$, $d=0$, $x=4$, $y=5$, find the numerical value of :—

1. $a-2b+3c+4d$. [*Solved*]
2. $3a-b-c+2x$.
3. $3x+4y-5a-6b$.
4. $y-2x+4c-3b$.

5. $x^2+y^2-3b^2$ [*Solved*]
6. $13b^2+4c^2-5x^2$.
7. $2x^2+3y^2-4d^2$.
8. $y^2-a^2-b^2-c^2$.
9. $x^3-2a^3-3b^3$.
10. $c^4-2b^4-3a^4$.
11. $a^2b+b^2c+c^2d+d^2a$.
12. $abc+bcd-a^3-d^3$.

13. $\frac{1}{2}b^2c^3-a^3-b^3-\frac{3}{4}ab^3c$. [*Solved*].
14. $\frac{1}{3}c^3+\frac{1}{8}a^2d^4-3a^3+b^2d^2$.

15. $3ab-\frac{2}{3}ac^2-a+\dfrac{b^4}{16}-\frac{3}{2}d+\frac{2}{3}c^2$.

16. $ab-\frac{3}{8}b^3+\frac{8}{3}ac-c-\dfrac{d}{2}+\frac{3}{18}ad$.

17. $x^2(y-a)-b^2(c-a)$. [*Solved*].
18. $ab(c-d)+bd(y-x)$.
19. $(y^2-x^2)(c^2-b^2)-a^2$.
20. $yx(y-x)+ba(b-a)+cd(c-d)$.

21. $\dfrac{c^2}{b^2}+\dfrac{6(y^2-x)}{7(c^2-a)}$. [*Solved*].
22. $\dfrac{(b-a)^2}{(c-b)^2}-\dfrac{(y-x)^3}{abc}$.

23. $\dfrac{(a+b+c)^2}{x(x+y)}-\dfrac{d}{y(b+c)}$.
24. $\dfrac{(b-c+x)^3}{3ac}-\dfrac{x^2+y^2-a^2}{xy}$.

25. If $a=3$, $b=1$, $c=0$, $x=5$, prove that the expressions $a^2-2bc+5ax$ and $3x^2+a^2+c^2$ are equal. [*Solved*].

26. If $x=1$, $y=2$, $z=3$, prove that the expressions $x^3+y^3+z^3$ and $6xyz$ are equal.

27. Shew that x^3+8x is equal to $7x^2-2x$ for each of the values $x=0,2,5$.

28. Shew that $x(x^2+11)$ is equal to $6(x^2+1)$ for each of the values $x=1, 2, 3$.

29. Shew that the expression $x^3-13x^2+44x-32$ vanishes when $x=1, 4$ or 8. [*Hint*].

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30. Prove that the relation $2^{x+6}=4^{2x}$ is true for $x=2$. [*Solved*].

31. Shew that the relation $3^{2x+2}=9^{3x-1}$ is satisfied for $x=1$.

32. Prove that $25^{a-2}-\tfrac{1}{4}\,10^{a-1}$ vanishes for $a=3$.

33. If $a=2$ and $b=3$, show that both of the following relations are true :—
$$\left.\begin{array}{l}5a-3b-1=0\\5b-7a-1=0\end{array}\right\}$$

34. If $x=1$, $y=2$, $z=3$, prove that all the following relations hold good :—
$$\left.\begin{array}{l}2x+\ y+\ z-7=0\\2x-3y+4z-8=0\\3x-\ y-\ z+2=0\end{array}\right\}$$

----

35. If $b=13$, $a=14$ and $c=15$, find the value of
$$\sqrt{\frac{s(s-a)}{(s-b)(s-c)}},$$
given that $2s=a+b+c$. [*Solved.*]

36. If $a=3$, $b=4$, $c=5$, find the value of
$$\sqrt{s(s-a)(s-b)(s-c)},$$
given that $2s=a+b+c$.

37. If $2s$ stands for $a+b+c$ and the values of a, b and c are respectively 5, 12 and 13, find the value of
$$\frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}.$$

38. If $a=5$, $b=3$, $c=1$, evaluate
$$\sqrt{a^2+b^2+c^2+2(ab-bc-ca)}.$$

39. If $x=6$ and $y=2$, evaluate $\sqrt[3]{x^3-3x^2y+3xy^2-y^3}$.

40. If $a=5$ and $b=1$, evaluate $\sqrt[4]{a^4-4a^3b+6a^2b^2-4ab^3+b^4}$.

41. If $a=4$ and $b=1$, find the value of $\sqrt{\dfrac{a^3-b^3}{27(a^2+ab+b^2)}}$.

42. If $x=3, y=5, z=0$, evaluate

$$\sqrt[3]{\frac{x^3+y^3+z^3-3xyz}{x^2+y^2+z^2-(xy+yz+zx)}}.$$

SOLUTIONS & HINTS—EXERCISE 2

1. Given Exp. $= a-2b \quad +3c \quad +4d$

$$= 1-2\times2+3\times3+4\times0$$
$$= 1- \quad 4 \ + 9 \ + \quad 0$$
$$= 6.$$

[*Note carefully the neatness of style and arrangement in the above work. Such things pay in Mathematics.*]

5. Given Exp. $= x^2 \quad +y^2 \quad -3b^3$

$$= 4\times4+5\times5-3\times2\times2$$
$$= 16 \quad +25 \ -12$$
$$= 29.$$

13. Given Exp $= \tfrac{1}{2}b^2c^3 \quad -a^2-b^2-\tfrac{3}{4}ab^3c$

$$= \tfrac{1}{2}\times4\times27-1 \ -8 \ -\tfrac{3}{4}\times1\times8\times3$$
$$= 54 \qquad -1 \ -8 \ -18$$
$$= 27.$$

17. Given Exp. $= x^2(y-a)-b^2(c-a)$

$$= 16(5-1)-4(3-1)$$
$$= 16\times4 \quad -4\times2$$
$$= 64 \qquad 8$$
$$= 56.$$

21. Given $\text{Exp.} = \dfrac{c^2}{b^2} - \dfrac{6(y^2-x)}{7(c^2-a)}$

$$= \frac{9}{4} - \frac{6(25-4)}{7(9-1)}$$

$$= \frac{9}{4} - \frac{6 \times 21}{7 \times 8}$$

$$= \frac{9}{4} - \frac{9}{-4}$$

$$= 0.$$

25. 1st Exp. $= a^3 - 2bc \qquad + 5ax$

$$= 9 - 2 \times 1 \times 0 + 5 \times 3 \times 5$$

$$= 9 - \quad 0 \quad + 75$$

$$= 84. \qquad\qquad\qquad\qquad \dots\dots(i)$$

2nd Exp. $= 3x^2 \qquad + a^2 + c^2$

$$= 3 \times 25 + 9 + 0$$

$$= 75 \qquad + 9$$

$$= 84. \qquad\qquad\qquad\qquad \dots\dots(ii)$$

From (i) and (ii) we have 1st Exp. = 2nd Exp.

29. **Hint.**—"Vanishes" means "is equal to zero".

30. L.H.S. $= 2^{x+6} = 2^{2+6} = 2^8 = 256.$

R.H.S. $= 4^{2x} = 4^{2 \cdot 2} = 4^4 = 256.$

$\because$ Each $= 256$

$\therefore$ L.H.S. = R.H.S.

35. Let us first find the value of s.

Now $2s = a + b + c$

$$= 13 + 14 + 15$$

$$= 42$$

$\therefore s = 21.$

$$\therefore \text{ Given Exp.} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$
$$= \sqrt{\frac{21(21-13)}{(21-14)(21-15)}}$$
$$= \sqrt{\frac{21\times 8}{7\times 6}}$$
$$= \sqrt{4} = 2.$$

6. Recapitulation.

Before proceeding to the next chapter, the student should see that the meanings of the algebraical terms, signs, symbols and notations of this chapter are quite clear to him. Let him test his grasp of these fundamental notions by trying to answer the following oral questions independently and intelligently :—

EXERCISE 3. (Oral)

1. (a) Are ab and $a.b$ the same quantities ?
   (b) Are 35 and 3.5 the same quantities ?
   (c) Distinguish between 3.5 and 3·5.
2. Are $25ab$ and $2.5ab$ the same quantities ?
   If not, find the difference between them for $a=1$, $b=2$.
3. Express the product of x and y in six different ways.
4. Find the difference between "twice three" and "three squared".
5. Find the difference between 2^3 and 3^2.
6. Distinguish between "four times x" and "x to the fourth". Give their respective values when $x=2$.
7. (a) Express the product of three factors, each equal to x.
   (b) Express the product of n factors, each equal to x.
8. (a) What is the co-efficient of ab in $5ab$?
   (b) What is the co-efficient of b in $5ab$.
   (c) What is the numerical co-efficient of a ?
9. (a) What are the factors of which $5ab$ is composed ?
   (b) What different expressions can separately be the factors of $5ab$?

10.   (a)   Write in symbols "the fourth power of x". What is the index of this power ?

(b)   If x is regarded as a power, what is its index ?

11.   Write in symbols "the fifth root of $2a$".

12.   (a)   Is there any difference between ax and xa ?

(b)   Is there any difference between x^2yz and zx^2y ?

(c)   Is there any difference between c^2d and cd^2 ?

13.   How many terms has the expression $5a^2b^3cx^4y$? What kind of expression is it ?

14.   How many terms has the expression $3a^2-4abc+c$? What kind of expression is it ?

15.   How many terms has the expression $4 \times a \times b^2-c \div d$? And, how many has $6 \times a \times d \div x \times y \div z$? Give their kinds.

16.   What defect do you find in the following process ?

Q.   If $a=1, b=2, c=3$, find the value of $\dfrac{a^3+b^3+c^3}{a+b+c}$

Sol. $\dfrac{a^3+b^3+c^3}{a+b+c}=1 \times 1 \times 1+2 \times 2 \times 2+3 \times 3 \times 3=1+8+27$

$$=\frac{36}{1+2+3}=\frac{36}{6}=6.$$

————

CHAPTER II

POSITIVE & NEGATIVE QUANTITIES ; ADDITION

7   Positive and Negative Quantities.

Algebraical quantities which are preceded by the sign + are said to be *positive*, while those to which the sign − is prefixed are said to be *negative*. In this sense the signs + and − are not symbols of operation ; they are used to denote that the quantities are *opposite in character*.

For example, if Rs. (+5) represent a *gain* of Rs. 5, then Rs (−5) will represent a *loss* of Rs. 5 ; if +20 yds. denote a distance of 20 yds. from a fixed place *due east*, then −20 yds. will denote a distance of 20 yds. from the same place *due west*.

It should be noted that there is no harm in regarding Rs. (+5) to stand for a loss of Rs 5, but then Rs. (−5) will stand for a gain of Rs. 5. However, it is customary to mark such quantities positive as *raise or strengthen* our position and to mark such quantities negative as *lower or weaken our position.* Thus *gain, income, rise*, etc. are taken as positive and *loss. expenditure, fall*, etc., are taken as negative. Also motion and distance to the *right hand*, motion and distance to the *east*, motion and distance to the *north* are taken as positive ; whereas motion and distance to the *left hand*, motion and distance to the *west*, motion and distance to the *south* are taken as negative.

Note 1. The + sign of a positive quantity is often omitted

Note 2 The *absolute value* of a quantity is its value when its sign is disregarded ; *e. g.*, the absolute value of +7 or −7 is 7.

8. A statement containing a negative quantity can be easily interpreted by making the quantity positive and reversing the character of the quantity. For example :—

A gain of Rs. (−8) means a loss of Rs. 8.

A loss of £ (−5) means a gain of £5.

A fall of −5° temperature means a rise of 5° temperature.

A prize of Rs. (−10) means a fine of Rs. 10.

In general, we may say that a statement involving an algebraical quantity remains unchanged if the sign of the quantity and its character are both reversed. Thus we have :—

A gain of Rs. (+8)=A loss of Rs. (−8)

=A gain of Rs. (−−8)

=A loss of Rs. (−−−8)

=A gain of Rs. (−−−−8)

etc., etc.

From the above and similar examples we conclude that :—

$$+8=--8=----8=\ldots\ldots\ldots=8 \text{ preceeded by}$$

even number of negative signs.

$$-8=---8=------8=\ldots\ldots\ldots=8 \text{ preceded by}$$

odd number of negative signs.

9. Addition of Positive and Negative Numbers.

Study the following examples carefully :—

(*i*) A gain of Rs. 8 + A gain of Rs. 5 = A gain of Rs. 13
    $\therefore (+8)+(+5)=+13.$ [Art. 7]
    or $+8+5=+13.$

(*ii*) A loss of Rs. 8 + A loss of Rs. 5 = A loss of Rs. 13
    $\therefore (-8)+(-5)=-13.$ [Art. 7]
    or $-8-5=-13.$

(*iii*) A gain of Rs. 8 + A loss of Rs. 5 = A gain of Rs. 3
    $\therefore (+8)+(-5)=+3.$ [Art. 7]
    or $+8-5=+3.$

(*iv*) A loss of Rs. 8 + A gain of Rs. 5 = A loss of Rs. 3
    $\therefore (-8)+(+5)=-3.$ [Art. 7]
    or $-8+5=-3.$

From these and similar examples we deduce the following **Rules for the addition of two numbers** :—

Rule 1. *If both the numbers are positive or negative,* **add** *the absolute values and attach the sign of the given numbers to this sum.* [See Examples (*i*) and (*ii*) above.]

Rule 2. *If one number is positive and the other negative find the difference of their absolute values and attach to it the sign of the larger.* [See Examples (*iii*) and (*iv*) above.]

We can easily generalise the above article and obtain the following **rules for adding more than two numbers** :—

Rule 3 *If all the numbers are positive or negative,* **add** *the absolute values and attach the sign of the given numbers to this sum.* For example :—

$$+5+6+3+1=15 ;-5-6-3-1=-15.$$

Rule 4. *If some numbers are positive and others, negative, add the positive and negative numbers separately by Rule 3 given above and then add the two results by Rule 2.* For example :—

$$+5-6-3+8=+5+8-6-3=+13-9=+4$$

$1-3-7-9=1-19=-18.$ [No sign is given with 1 ; therefore sign $+$ is understood with it.]

Note 1. The sum of positive and negative quantities is called their **Algebraical sum**

Note 2. The sum of two numbers having the same absolute values but opposite signs is 0 ; *e g*, $-5+5=0.$

Like Terms ; Their Addition.

When terms do not differ, or when they differ only in their numerical co-efficients, they are called *like terms*, otherwise they are called *unlike terms.* For example, $5x$, $8x$; ab, $3ab$, $-2ab$; $8x^2y^3$, $-x^2y^3$, $-10y^3x^2$ are three different groups of like terms, while $3x$, $4y$; 5, $5a$, $5ab$ are two different groups of unlike terms.

A careful study of the following progressive examples will explain clearly how like terms are added :—

(*i*) $8+5=13$

    8 two's$+5$ two's$=13$ two's [8 two's means 8×2]
    8 five's$+5$ five's$=13$ five's
    8 six's $+5$ six's $=13$ six's
    $8x$ $+5x$ $=13x$
    $8ab$ $+5ab$ $=13ab$
    $8x^2y$ $+5x^2y$ $=13x^2y$
    etc., etc.

Note. Some authors advance the following argument :—

    8 horses$+5$ horses$=13$ horses ;
    Similarly, $8x+5x=13x.$

This is wrong and apt to give a wrong notion to the students, because x and other letters do not stand for denominations, but for numbers.

Similarly we have :—

(ii) $-8x-5x=-13x$ $[\because -8-5=-13]$

       $-8ab^3-5ab^3=-13ab^3$

(iii) $8x-5x=3x$ $[\because 8-5=3]$

       $8lmn-5lmn=3lmn$

(iv) $-8x+5x=-3x$ $[\because -8+5=-3]$

       $-8pq^2r^3+5pq^2r^3=-3pq^2r^3$

(v) $3a+4a+a+6a=14a$ $[\because a=1a$ and $3+4+1+6=14]$

(vi) $-3xy-4yx-xy-6yx=-14xy$

$$\left[\begin{array}{l}\because xy \text{ and } yx \text{ are the same}\\ \text{and}-3-4-1-6=-14\end{array}\right]$$

(vii) $-x^3+5x^3-3x^3+4x^3$

      $=5x^3+4x^3-x^3-3x^3$ [collecting positive and negative terms separately]

      $=9x^3-4x^3=5x^3.$

(viii) $a^2b-8a^2b+3a^2b-2a^2b$

      $=a^2b+3a^2b-8a^2b-2a^2b$

      $=4a^2b-10a^2b=-6a^2b.$

etc., etc.

The rule for adding like terms is now quite clear :—

The literal part (i e., the part consisting of letters) of the required sum is the same as the literal part of each given term and its numerical co-efficient is the algebraical sum of the numerical co-efficients of the given terms.

Note. The sum of two like terms whose numerical co-efficients are equal in absolute value but opposite in sign is zero, e g., $-5x+5x=0$. Hence, while adding, such pairs of terms may be cancelled. For example :—

     $3a-4a-6a+2a+4a=3a+2a-6a=5a-6a=-a.$

11. Addition of Unlike Terms.

'*Unlike terms*' cannot be added. All we can do is to put the terms together with their proper signs. For example, the sum of $5a, -3b$ and $2c$ is $5a-3b+2c$.

12. Addition of Like and unlike Terms.

Collect the different groups of like terms and the group of unlike terms separately. Replace these groups by their sums.

Example   Add together : $12a$, $-5b$, $-a$, $6c$, b, $-3c$, $5d$, $-11a$ and $-3e$.

Solution.

Reqd. sum $=12a-a-11a-5b+b+6c-3c+5d-3e$

[grouping a's b's, c's, and other terms separately]

Now, $12a-a-11a=12a-12a=0$

$$-5b+b=-4b$$
$$6c-3c=3c$$

$\therefore$ Reqd. sum $=0-4b+3c+5d-3e$

$$=-4b+3c+5d-3e.$$

13. The Beginner's Stumbling Blocks.

Very often a careful study of the principles explained in the fore-going articles is not done by the beginner, which leads to several misconceptions and blunders in very simple questions on addition.

Note the following statements very carefully and give the correct result in each case :—

(i) $2a+3b$ is not equal to $5ab$

(ii) $3x+4xy$,,   ,, $7xy$

(iii) $x+x$,,   ,, x^2

(iv) a^2b+a^2b ,,   ,, a^4b^2

(v) $2x+5x^2$,,   ,, $7x^2$

(vi) $3a+2$,,   ,, $5a$

(vii) $5x+4x+x$,,   ,, $9x$

(viii) $3x^2y+5y^2x$,,   ,, $8x^2y$ or $8y^2x$

(ix) $-5x-6x$,,   ,, $+11x$

(x) a^2+a^3 ,,   ,, a^5

(xi) $3a-3$,,   ,, a

(xii) x^5-x^3 ,,   ,, x^2.

The correct results are as follows :—

    (*i*) $2a+3b$ [∵ the terms are unlike and no further simplification is possible.]

    (*ii*) $3x+4xy$ [∵ the terms are unlike and no further simplification is possible.]

    (*iii*) $2x$ [$x+x$ means $1x+1x$.]

    (*iv*) $2a^2b$ [a^2b+a^2b means $1a^2b+1a^2b$.]

    (*v*) $2x+5x^2$ [The terms are unlike.]

    (*vi*) $3a+2$ [,,   ,,   ,,   .]

    (*vii*) $10x$ [$5x+4x+x=5x+4x+1x$]

   (*viii*) $3x^2y+5y^2x$ [The terms are unlike.]

    (*ix*) $-11x$ [*See* Art. 9, Ex. (*ii*)]

    (*x*) a^2+a^3 [The terms are unlike ; the *indices* are not to be added.]

    (*xi*) $3a-3$ [The terms are unlike ; also note that the co-efficient of *a* and the term "-3" cannot be combined.]

    (*xii*) x^5-x^3 [The terms are unlike ; The *indices are* not to be subtracted.]

EXERCISE 4

Simplify :—

1. $3-5$.
2. $1-8$.
3. $-4-5$.
4. $-1-2-3$.
5. $1-2+3$.
6. $2-3+1$.
7. $3-1-2$.
8. $1-2-3-4$. [*Solved*]
9. $3-4+6-5$.
10. $-5+7+5-1-7$.
11. $1-3+5-7+9$.
12. $-3-4-5-6+10$.

———

13. A trader gains Rs. 20, loses Rs. 55, and then gains Rs. 10. Express algebraically the result of his three transactions, and give its arithmetical meaning.

                                                [*Solved*]

14. A merchant loses Rs. 250 in January, gains Rs. 600 in February and loses Rs. 300 in March. Find the net result of his trading during this quarter by employing algebraical symbols.

15.   A cyclist rides 15 miles east, then 25 miles west, then 5 miles east and again 8 miles west.   How far and in which direction is he from the starting point ?   Use algebraical symbols to get your result.

16.   A snail starts from a particular point in a wall .   it climbs 6 feet vertically upwards, slips down 14 feet, then climbs 5 feet upwards and again slips down 1 foot.   Express algebraically its final position from the given point.

_____

Find the sum of :—

17. $3a$ and $5b$.   [Hint]
18. $2a$ and $5ab$
19. a and a.
20. a^2 and a^2.
21. a^3, a^3 and a^3.
22. $2ab$ and $-5ba$.
23. $8a$ and -2.
24. x^3y and y^3x.
25. $-x^4$ and $-x^4$.
26. x^3 and x^4.
27. t and $-t$.
28. $5k$, $3k$ and k.
29. $-l$, $-l^2$ and $-l^3$.
30. $2lm$ and $3mn$.
31. a, ab and abc.
32. $3xy$ and $-5x$.
33. xyz, $-x$, $-y$ and $-z$.
34. b^2, $-b$ and $-b$.   [Hint]
35. a^3, $-a$, $-a$ and $-a$.
36. abc, bca and cab.
37. x^2y^2, $-x^2$ and $-y^2$.
38. a^2b, $-ab^2$, $-ab$, $-a$ and $-b$
39.   1, $2a$, $3b$, $4c$ and $5d$.
40. $-a$, $2a^2$, $3a^3$ and $4a^4$.

_____

Simplify :—

41. $x+x+x$.   [Hint]
42. $a^2+a^3+a^3$
43. $5b-6b+4b$.   [Solved]
44. $8l-l+3l$.
45. $p-10p+3p$.
46. $q-3q-5q$.
47. $ab+6ab-7ab$.
48. $6a^2-a^2-2a^2-5a^2$.
49. $8l^3-9l^3+l^3-10l^3$.
50. $a^2b-3a^2b-6a^2b-2a^2b$.

_____

51. $\frac{1}{2}x^2-\frac{3}{4}x^2-x^2+\frac{1}{4}x^2$.   [Solved]
52. $\frac{2}{3}ab^2+\frac{5}{6}ab^2-\frac{5}{9}ab^2-\frac{1}{12}ab^2$
53. $\frac{1}{2}xy-\frac{2}{3}xy+\frac{3}{4}xy-\frac{1}{8}xy$.
54. $-5a^3+\frac{1}{4}a^3-\frac{3}{4}a^3+2a^3-\frac{1}{4}a^3+\frac{1}{4}a^3$.
55. $-\frac{3}{8}k^3-2k^3-\frac{2}{3}k^3+k^3+\frac{1}{2}k^3+\frac{11}{6}k^3$.
56. $-t^4-\frac{1}{2}t^4-\frac{1}{3}t^4-\frac{1}{4}t^4-\frac{1}{6}t^4+t^4+\frac{5}{12}t^4$.

_____

57. $3a - 7a - ab + 5ab + a.$ [Solved]
58. $5a^2 - 3a - 6a^2 - a^2 - a + 6a.$
59. $x + xy + 3x - 6xy - 6x - 2xy.$
60 $ab - ba - 3ab - 5ba + 6ab.$
61. $5l - 5 - l - 1 + 9l + 9.$ 62. $t + 3t^2 + 4t^3 - 6t - 7t^2 - t^3.$
63. $2 + 2a + 2ab - 3 - 4a - 5ab - a - b.$
64. $1 - x - x^2 - x^3 - 3x - 4x^2 - 5 - 6x^3.$

SOLUTIONS & HINTS—EXERCISE 4

8. $1 - 2 - 3 - 4 = 1 - 9 = -8.$

13. $+ Rs.\ 20 - Rs.\ 55 + Rs.\ 10$

$$= + Rs.\ 30 - Rs.\ 55$$
$$= - Rs.\ 25.$$

This means a loss of Rs. 25.

17. The given terms are *unlike*.
34. The last two terms are *like* and can be added.
41. The co-efficient of each term is 1.

43. $5b - 6b + 4b$
$$= 5b + 4b - 6b$$
$$= 9b - 6b$$
$$= 3b.$$

51. Sum of co-efficients $= \frac{1}{2} - \frac{3}{4} - 1 + \frac{1}{4}$
$$= \frac{1}{2} + \frac{1}{4} - \frac{3}{4} - 1$$
$$= \frac{3}{4} - \frac{3}{4} - 1 = -1.$$

$\therefore$ Reqd. Sum $= -1x^2$ or $-x^2.$

57. $3a - 7a - ab + 5ab + a$
$$= 3a + a - 7a - ab + 5ab$$
$$= -3a + 4ab. \quad \text{[Adding the first three like terms and}$$
$$\text{the last two like terms separately.]}$$

14. Addition of Compound Expressions.

No new principle is involved. We may write down the terms of all the expressions, with their proper signs, in any order and simplify the result as explained in the fore-going articles (especially Art. 12). Or, we may arrange the work so that like terms appear in the same column and then add each column separately, beginning with the left. We shall illustrate both the methods by the following :

Example.   Add $2a-3b+4c$, $b-5c-3a$, $c-a+7b$.

Solution :—

Method 1.

Reqd. Sum $= 2a-3b+4c+b-5c-3a+c-a+7b$

$\quad = 2a-3a-a-3b+b+7b+4c-5c+c$

[re-arranging]

$\quad = -2a+5b+0c$

$\quad = -2a+5b.$

Method 2. $\quad 2a-3b+4c$

$\quad\quad\quad\quad\quad -3a+\ b-5c$

$\quad\quad\quad\quad\quad -\ a+7b+\ c$

$\quad\quad\quad\quad\quad \overline{-2a+5b}$

EXERCISE 5

Add together :—

1. $a-2b$ and $3b$.   [*Solved*].
2. $2a-3b$ and $-5a$.
3. $8a-4b$ and $-a+b$.
4. $x-y$ and $y-z$.
5. $2c+z$ and $x-3c$.
6. $a-b$ and $c-3b$.
7. $p-q$ and $q-p-r$.
8. $x-5y$ and $y-4$.
9. $2p-q$ and $8r-s$.
10. $5a-b$ and $8ab-bc$.
11   $l-2m-3n$ and $m-n-p$.
12. $a-b$, $b-c$ and $c-a$.
13. $xy-2yz$; $yz-2zx$; $zx-2xy$.
14. $2l-2$; $8k-8$; $4l-4$.
15. $a+b$; $ab+bc$; a^2+b^2.

———— —

Work out the following sums on addition :—

16. $2a-3b$ 17. $-3a+b$

$\quad\quad -3a+\ b+c$ $\quad\quad\quad -3b+c$

$\quad\quad -\ a+2b$ $\quad\quad\quad -\ b-c$

18. $-3a^2+4a-4$ 19. $x^2-\ xy$

$\quad\quad -\ a^2-5a$ $\quad\quad\quad\quad -3xy+y^2$

$\quad\quad\quad\quad -6a+5$ $\quad\quad -5x^2+2xy-5y^2$

20.

$$ab- bc- ca$$
$$-2ab-3bc+4ca$$
$$- ab+5bc+2ca$$

21.

$$a^3-a^2b$$
$$+a^2b-ab^3$$
$$-ab^2-b^3$$
$$-ab^3-b^3$$

Add the following expressions after arranging like terms in columns :—

22. $x+2y-3z$; $-3x+y+2z$; $2x-3y+z$. [*Solved.*]

23. $-3a+2b+c$; $a-3b+2c$; $2a+b-3c$

24. $4x+3y+5z$; $3y-2x-8z$; $x-y+z$.

25. $24x-15y+z$; $4z-10y+14x$; $20y-z+x$.

26. $5ab+ca-7bc$; $ab+2bc-ca$; $3ca-3ab+2bc$.

27. $-5xy+6yz$; $8xy-7zx$; $-4yz+3zx$; $-2xy-2yz+4zx$.

28. $5a^2+b^2$; $-3c^2+a^2$; $-b^2+c^2$; $-a^2+2c^2+b^2$.

29. $a^3+b^3+c^3$; $2a^3-2c^3$; $3b^3+3a^3$; c^3-4b^3

30. $4ab-9bc$; $2ca-25ab$; $24bc-ca$; $23ab-15bc$, ca.

Find the sum of :—

31. a^2+ab-b^2 ; $-c^2+bc+b^2$; $-a^2+ac+c^2$.

32. a^3-a^2+a-1 ; $2a^2-2a+2$; $-3a^3+5a+1$.

33. $9a^3-7a+5$; $-14a^2+15a-6$; $20a^2-40a-17$.

34. $10a^3+5a+8$; $3a^3-4a^2-6$; $2a^3-2a-3$.

35. $2x^3-2xy+2yz$; $2xy+2y^3-2zx$; $2zx-2yz+2z^3$.

36. $10x^3-6z^3+2t^3$; $2y^3-4x^3+6t^3$; $8z^3-4x^3-6t^3$.

37. $12a^3-4a+2$; $4a^3+2a+12$; $2a^2-14a^3$; $4a-8$.

38. $2a^2+2b^2$; $-4ab+4c^2$; $-8bc+4a^2$; $-4c^2-6ca-6b^2$.

39. $3a^3-6b^3+3a$; $3b^3-6a^3$; $3b+3a^2$; $6b^2-3a+3b^3$.

40. $3a^2b+4ab^2$; $-3a^2b-6ab^2$; $-a^3+3ab^2$; a^3+3a^2b.

41. x^3+y^3 ; $5xy^2+y^3$; $-10xy^2-x^3$; $2x^2y-2y^3+5xy^2$.

42. $2a^3-8a^2b$; $12ab^2+4a^2b$; $-6ab^2+4b^3$; $2b^3+6a^2b+8ab^2$.

Add together the following expressions :—

43. $x-\frac{1}{2}y$; $y-\frac{2}{3}x$; $\frac{1}{2}x-\frac{1}{3}y$. [*Solved.*]

44. $2a-\frac{1}{3}b$; $b-\frac{1}{4}c$; $c-a$.

45. $a^2+\frac{1}{2}ab$; $b^2+\frac{1}{2}bc$; $\frac{1}{3}a^2+\frac{1}{3}b^2-ab-bc$.

46. $x^2-\frac{1}{2}xy+\frac{1}{3}y^2$; $\frac{1}{3}x^2+xy-\frac{2}{3}y^2$; $-\frac{3}{2}x^2-\frac{1}{2}xy$.

47. $\frac{1}{2}a^3-a^2b$; $\frac{1}{3}ab^2-b^3$; $\frac{1}{3}a^2b-a^3$; $\frac{1}{2}ab^2+\frac{1}{3}b^3$.

48. $\frac{1}{4}x^3-x^2y-\frac{3}{4}y^3$; $\frac{3}{4}x^2y-\frac{3}{8}xy^2+y^3$; $-\frac{3}{4}x^3+\frac{1}{2}xy^2+\frac{1}{4}y^3$

SOLUTIONS & HINTS—EXERCISE 5

1. Reqd. Sum $= a - 2b + 3b$

$\qquad = a + b.$ **Ans.**

22.

$\begin{aligned} & x + 2y - 3z \\ & -3x + y + 2z \\ & \underline{2x - 3y + 3z} \\ & \overline{0x + 0y + 2z = 2z.} \end{aligned}$ **Ans.**

$\begin{cases} \text{Co-eff. of } x = & 1 - 3 + 2 = 0 \\ \quad,, \quad y = & 2 + 1 - 3 = 0 \\ \quad,, \quad z = & -3 + 2 + 3 = 2 \end{cases}$

43.

$\begin{aligned} & x - \tfrac{1}{2}y \\ & -\tfrac{2}{3}x + y \\ & \underline{\tfrac{1}{2}x - \tfrac{1}{3}y} \\ & \overline{\tfrac{5}{6}x + \tfrac{1}{6}y.} \end{aligned}$ **Ans.**

$\begin{cases} \text{Co-eff. of } x = & 1 - \tfrac{2}{3} + \tfrac{1}{2} \\ & = \dfrac{6 - 4 + 3}{6} = \tfrac{5}{6} \\ \quad,, \quad y = & -\tfrac{1}{2} + 1 - \tfrac{1}{3} \\ & = \dfrac{-3 + 6 - 2}{6} = \tfrac{1}{6}. \end{cases}$

CHAPTER III

SUBTRACTION ; SIMPLE BRACKETS

15. **The Meaning of Subtraction** is quite clear from article 8 :

Subtracting b from a means adding $-b$ to a

$\qquad,,\qquad -b\qquad,,\qquad,,\qquad +b$ to a.

Thus we have only to change the sign of "the quantity to be subtracted" and add it to the other quantity.

Now, let the quantity to be subtracted be *compound, $b + c$,* say, and let the other quantity be a. Because $b + c$ is composed of two quantities b and c, we have to subtract b as well as c, and this means that we have to add $-b$ as well as $-c$. Thus the result is $a - b - c$. Arguing exactly in the same manner we conclude that the result of subtracting $b - c$ from a is $a - b + c$. This leads us to the following —

General Rule :—*Change the sign of every term in the expression to be subtracted, and add to the other expression.*

16. Any pair of brackets, the simplest form of which is ( ), are used to indicate that the terms enclosed within them are to be considered as one quantity.

Now, to subtract the quantity $(b+c)$ from a we may add the quantity $-(b+c)$ to a, so that the result is $a-(b+c)$. But, by the last article, this result is also equal to $a-b-c$.

$$\therefore \quad a-(b+c)=a-b-c.$$

Similarly $a-(b-c)=a-b+c.$

This leads us to the following **Rule** :—

When an expression within a pair of brackets is preceded by the sign —, *the brackets may be removed, if the sign of every term within the brackets be changed.*

Conversely : *Any part of an expression may be enclosed within a pair of brackets and the sign* — *prefixed, provided the sign of every term within the brackets be changed.*

We may also state, without any elaborate explanation, the following **Rule** :—

When an expression within a pair of brackets is preceded by the sign +, *the brackets can be removed without making any change in the expression.*

Conversely : *Any part of an expression may be enclosed within a pair of brackets and the sign* + *prefixed, the sign of every term within the brackets remaining unaltered.*

17. The converse rules given in the last article need special mention. We shall illustrate them by a few examples :—

(i) *Insertion of a pair of brackets preceded by the sign* +.

The expression $a-b+c-d$ may be written in any one of the following ways :—

$$a+(-b+c-d)$$
$$a-b+(c-d)$$
$$a-b+c+(-d)$$
$$a+c+(-b-d)$$
$$a-d+(-b+c), \quad \text{etc., etc.}$$

(ii) *Insertion of a pair of brackets preceded by the sign* —.

The expression $a-b+c-d$ may be written in any one of the following ways :—

$$a-(b-c+d)$$
$$a-b-(-c+d)$$

$$a-b+c-(d)$$
$$a+c-(b+d)$$
$$a-d-(b-c). \quad \text{etc., etc.}$$

Note. We shall revert to this topic in a later chapter.

18 From the foregoing articles of this chapter we get two methods for **Subtraction.** We shall illustrate them by an

Example. Subtract $a-2b+3c$ from $2a-3b-c$.

Solution :—

Method 1. $\quad 2a-3b-\ c$
$\qquad\quad a-2b+3c \quad$ (*to be subtracted*)
$\qquad\qquad$ ——————

$\qquad\qquad$ OR $\quad$ (changing signs in the lower exp.)

$\quad 2a-3b-\ c$
$-a+2b-3c \quad$ (*to be added*)
$\qquad$ ——————

$\qquad a-\ b-4c \qquad$ **Ans.**

Note.—The work may be briefly presented as follows :—

$\quad 2a-3b-\ c$
$\quad a-2b+3c$
$\quad -\ +\ -$
$\quad$ ——————
$\quad a-\ b-4c \qquad$ **Ans.**

Or, still better, the change of signs may be done mentally, thus :—

$\quad 2a-3b-\ c$
$\quad a-2b+3c$
$\quad$ ——————
$\quad a-\ b-4c \qquad$ **Ans.**

Method 2.

Reqd. remainder $=(2a-3b-c)-(a-2b+3c)$
$\qquad\qquad\qquad =2a-3b-c-a+2b-3c$
$\qquad\qquad\qquad =a-b-4c. \qquad$ **Ans.**

EXERCISE 6

1. Subtract 2 from 5.
2. Subtract 2 from -5
3. ,, -2 ,, 5.
4. ,, -2 ,, -5.
5. ,, $3a$,, $7a$.
6. ,, $4x$,, $-8x$.
7. ,, $-5k$,, $2k$.
8. ,, $-9t$,, $-8t$.

9. Subtract 0 from $-5b$.   10. Subtract $-6c$ from 0 [*Solved*]
11.   ,, $3xy$,,   0.   12.   ,, $3d$,, $5de$ [*Hint*]
13.   ,, $-3f$,, $6fg$.   14.   ,, $-ab$,, $-abc$.
15.   ,, $2a$,, $-5a^2$.   16.   ,, $-6x^2$,, $9x^3$.
17.   ,, $-2xy$,, $-10yx$.   [*Hint*]
18.   ,, $8a$,,   8.   19.   ,, a^2 ,, $3a$.
20.   ,, $-4a^2b$,, $-5ab^2$.

21. From $2x+3y$ subtract $-3x$.   [*Solved*]
22.   ,, $-3a+5b$,, $-4b$.
23.   ,, $-3c$,, $c-d$.
24.   ,, $5xy$,, $2x+2y$.
25.   ,, $3x-5y$,, $-x-y$.
26.   ,, $5a^2+a$,, a^2-a.
27.   ,,   0   ,, $2a-3b$.
28.   ,, a^3+a^2 ,, $a+1$.
29.   ,, $2ab+2xy$,, $x+y$.
30.   ,, a^3 ,, a^3+a^2.

31. Take $a-b+c$ from $-a+b+c$.   [*Solved*]
32.   ,, $2x-y+z$,, $x-y-z$.
33.   ,, $2p-3q-4r$,, $-p+q-2r$.
34.   ,, $3x^2-2x-5$,, x^2-3x-4.
35.   ,, $-5x^3-2x^2+6$,, $-x^2-x-1$.
36.   ,, $a^3+b^3+c^3-3abc$,, $a^3-b^3-c^3$.
37.   ,, $2a^2b^2+3a^2b-5ab^2$,, ab^2-a^2b.   [*Solved*]
38.   ,, $a^4+2a^3+3a^2+4a$,, $a-a^2-a^4$.
39.   ,, $2a-3b+4$,, $3ab+b+1$.
40.   ,, $5x^2-xy-y^2$,, $3x^4-4y^4$.

41. From $\frac{1}{2}x-\frac{1}{2}y$ take $\frac{1}{3}y-\frac{1}{3}x$.   [*Solved*]
42.   ,, $\frac{1}{3}a+ab-\frac{1}{2}a$,, $a-ab$.
43.   ,, $\frac{3}{4}a^2-\frac{1}{3}a$,, $a^2-\frac{1}{4}a$.
44.   ,, $a^4-a^2b^2-\frac{1}{2}$,, $\frac{3}{4}a^2+\frac{1}{2}$.
45.   ,, $x^2+\frac{3}{2}y^2$,, $\frac{1}{2}x+\frac{1}{2}y$.

Simplify :—

46. $b-a-(2a-3b)$.   [*Solved*]
47. $2xy-y-(3x-y-1)$.
48. $x^2-x-(2x-3)-4$.
49. $2abc-(ab-bc)-b+c$.
50. $1-(2-3a)+(-a)+(3-2a)$.

Find the algebraic difference between :—

51. $2x^2-3x-4$ and $3x^2-2x-1$. [*Solved*]
52. $5ab-1$ ..... and $a-b+1$.
53. 0 ..... and $3a^2-a-2$.
54. 2 ..... and $2a^2+2a+2$.
55. $abcd$ ..... and $a+b+c+d$.

___

56. What must be added to $2ab+b+a$ to get $ab-b-a$? [*Hint*]

57. What must be subtracted from 0 to get $2a+3b-4c$?

58. What expression taken from 1 will leave a remainder $1+a+b$?

59. What expression together with $ab-a-b$ will give $a-b-3$?

60. An expression was subtracted from $x-y-z$ and the remainder obtained was $-2x-2y-2z$. Find the expression.

61. Subtract x^2+2x+3 from the sum of $x-4$ and $1-x-x^2$. [*Solved*]

62. Subtract the sum of a^2-ab-b and $2ab-3b$ from b^2+a^2.

63. From the sum of $3a^2-ab-4b^2$ and b^2-a^2 subtract the sum of $ab-2b^2$ and $3a^2-2b^2$.

64. Add the sum of $3y-4y^2$ and $2-y$ to the remainder left when $1+2y-y^2$ is subtracted from 5.

___

SOLUTIONS & HINTS—EXERCISE 6

10. Reqd. Remainder $=0-(-6c)$
$$=+6c. \qquad \textbf{Ans.}$$

12. The terms are not like.

17. The terms are like.

21. Reqd. Remainder $=(2x+3y)-(-3x)$
$$=2x+3y+3x$$
$$=5x+3y.$$

31. $$\begin{array}{r} -a+b+c \\ a-b+c \\ \underline{-\ +\ -} \\ -2a+2b. \end{array}$$

37. $ab^2 - a^2b$

$\dfrac{-5ab^2 + 3a^2b + 2a^2b^2}{6ab^2 - 4a^2b - 2a^2b^2}$

41. Reqd. remainder $= (\frac{1}{2}x - \frac{1}{2}y) - (\frac{1}{3}y - \frac{1}{3}x)$

$= \frac{1}{2}x - \frac{1}{2}y - \frac{1}{3}y + \frac{1}{3}x$

$= \frac{5}{6}x - \frac{5}{6}y.$ **Ans.**

46. $b - a - (2a - 3b)$

$= b - a - 2a + 3b$

$= 4b - 3a.$ **Ans.**

51. Algebraic difference $= (2x^2 - 3x - 4) - (3x^3 - 2x - 1)$

$= 2x^2 - 3x - 4 - 3x^3 + 2x + 1$

$= -x^3 - x - 3.$ **Ans.**

Note. The Algebraic difference between two given quantities is the result of subtracting the second from the first.

56. Subtract the first expression from the second.

61. Sum of the last two expressions $= x - 4 + 1 - x - x^3$

$= -3 - x^3$

∴   Reqd. remainder $= -3 - x^3 - (x^2 + 2x + 3)$

$= -3 - x^3 - x^2 - 2x - 3$

$= -2x^3 - 2x - 6.$ **Ans.**

CHAPTER IV

MULTIPLICATION

19. When the quantities multiplied are positive integers, multiplication is only a short method for adding.

Thus $3 \times 4 = 3$ taken 4 times

$= 3 + 3 + 3 + 3.$

But in Algebra we have to deal with quantities of all kinds—*integral, fractional, positive, negative.* Therefore it becomes necessary to have a more comprehensive definition for multiplication, which we proceed to obtain in the next article.

20. If a, b, c be any positive integers, we have, from our knowledge of Arithmetic :—

(i) $a \times b = b \times a$

$a \times b \times c = a \times c \times b = b \times c \times a = b \times a \times c$, etc. etc.

That is to say, *the factors of a product may be taken in any order.*

This is the **Commutative Law of Multiplication.**

(ii) $a \times b \times c = a \times (b \times c) = b \times (c \times a)$, etc. etc.

That is to say, *the factors of a product may be grouped together in any manner.*

This is the **Associative Law of Multiplication.**

(iii) $(a+b)c = ac + bc$

$(a-b)c = ac + (-b)c$

etc. etc.

That is to say, *the product of a composite expression by a monomial is found by multiplying each of its terms by the monomial and adding the partial products so obtained.*

This is the **Distributive Law of Multiplication.**

Now we are in a position to give the following *definition* :—

Algebraic Multiplication is an operation concerning which the three laws stated above, which are demonstrably true when the quantities involved (a, b, c, etc.) are positive integers, remain true also *in form* whether these quantities are integers or fractions, positive or negative.

21. The Law of Signs.

If a and b are any two algebraic quantities, we have :—

$$(+a) \times (+b) = +ab \qquad \text{...... (i)}$$
$$(+a) \times (-b) = -ab \qquad \text{...... (ii)}$$
$$(-a) \times (-b) = +ab \qquad \text{......(iii)}$$
$$(-a) \times (+b) = -ab \qquad \text{...... (iv)}$$

The above four cases are included in the following *concise statement regarding the product of two quantities :* —

"Like signs give $+$, unlike signs give $-$"

Proofs :—

Case I. The product of two positive whole numbers is evidently positive. Therefore the result follows from our extended definition of multiplication.

Case II. We have :—

$$+a \times (+b-b) = (+a) \times (+b) + (+a) \times (-b)$$

[Distributive Law]

But $+a \times (+b-b) = (+a) \times 0 = 0$

$\therefore \quad (+a) \times (+b) + (+a) \times (-b) = 0$

$\therefore \quad +ab \quad\quad +(+a) \times (-b) = 0$ [By Case I]

$\therefore \quad (+a) \times (-b)$ must be equal and opposite in character to $+ab$.

$\therefore \quad (+a) \times (-b) = -ab.$

Case III. We have :—

$$(+b) \times (-a) = -ba. \quad\quad \text{[By Case II]}$$

$\therefore \quad (-a) \times (+b) = -ab. \quad\quad \text{[Commutative Law]}.$

Case IV. We have :—

$$(-a)(+b-b) = (-a) \times (+b) + (-a) \times (-b)$$

[Distributive Law]

$$= -ab + (-a) \times (-b). \quad\quad \text{[By Case III]}$$

But $(-a) \times (+b-b) = (-a) \times 0 = 0$

$\therefore \quad -ab + (-a) \times (-b) = 0$

$\therefore \quad (-a) \times (-b)$ is equal and opposite in character to $-ab$.

$\therefore \quad (-a) \times (-b) = +ab.$

22. Consider the form of the product of three negative factors. The product of any two of them is positive and [Article 21, Case IV], and when the product is multiplied by the third factor, we get a negative result [Article 21, Case II].

If the number of factors be four, we shall have to multiply the last result (which is negative) by another negative quantity, and therefore the result will be positive. [Article 21, Case IV].

And so on.

Hence we have the following general statement :—

"The product of any number of negative factors is positive or negative according as the number of factors is even or odd".

Thus, $(-a)(-b)(-c)=-abc,$
$\quad (-a)(-b)(-c)(-d)=+aocd,$
$\quad (-a)(-b)(-c)(-d)(-e)= \cdot abcde,$
$\quad$ etc, etc.

Also we have :—

$(-a)^2=(-a)(-a)=+a^2$

$(-a)^3=(-a)(-a)(-a)=-a^3$

$(-a)^4=(-a)(-a)(-a)(-a)=+a^4$

....

$(-a)^n=+a^n$ if n is even

$\quad$ or $-a^n$ if n is odd.

23. The Index Law.

By the definition of a power we have :—

$x^2 \times x^3=(x \times x) \times (x \times x \times x)$
$\qquad =x \times x \times x \times x \times x$ [Associative Law]
$\qquad =x^5$ or x^{2+3}.

Again, $x^3 \times x^4=(x \times x \times x) \times (x \times x \times x \times x)$
$\qquad\qquad =x \times x \times x \times x \times x \times x \times x$ [Associative Law]
$\qquad\qquad =x^7$ or x^{3+4}.

More generelly, if m and n are positive integers, we have
$\qquad x^m \times x^n=(x \times x \times x \times \dots\dots$ to m factors$)$

$\qquad\qquad\qquad\qquad \times (x \times x \times x \times \dots\dots$ to n factors$)$
$\qquad\qquad =x \times x \times x \times x \times \dots\dots$ to $m+n$ factors

$\therefore \quad x^m \times x^n=x^{m+n}$.

This is called the **Index Law** and may be stated thus :—

"The powers of the same quantity are multiplied together by adding the indices "

Note. The Index Law may be easily extended to the case where three or more factors are multiplied together. Thus we have :—
$\qquad\qquad x^m \times x^n \times x^p=x^{m+n+p}$. $\qquad$ etc., etc.

24. The five laws discussed in the fore-going articles [*viz.*, (*i*) The Commutative Law, (*ii*) The Associative Law, (*iii*) The Distributive Law, (*iv*) The Law of Signs, and (*v*) The Index Law] enable us to multiply together any algebraic

expressions. 'We shall discuss the method for multiplication under three heads :—

(i) Multiplication of two or more monomials,

(ii) Multiplication of a multinomial by a monomial, and

(iii) Multiplication of two multinomials.

25. Multiplication of two or more Monomials.

The procedure is to write down the monomials one after another, with the signs of multiplication between, and then to simplify the result as much as possible by the application of the five laws mentioned in the last article. From this procedure we can easily obtain the following **Working Rule** :—

First put down the sign by the law of signs, then the product of the numerical factors, and lastly the letters in any order, each with an index equal to the sum of its indices in the different factors.

Example 1. Multiply $3a^2b^3c$ by $-4b^5c^3d$.

Solution :—

The signs are unlike ($+$ and $-$) ; therefore the sign of the product is —.
$$3 \times 4 = 12$$

a^2 remains unaltered, because the letter a does not occur anywhere else.
$$b^3 \times b^5 = b^{3+5} = b^8$$
$$c \times c^3 = c^1 \times c^3 = c^{1+3} = c^4$$
d remains unaltered.

∴   Reqd. product $= -12a^2b^8c^4d$.

Example 2. Find the continued product of $2x^2y$, $-3xy^3z$ and $-4y^2z^2$.

Solution :

The signs $+$, $-$ and $-$ give $+$ and $2 \times 3 \times 4 = 24$.
$$x^2 \times x \text{ gives } x^{2+1} \text{ or } x^3$$
$$y \times y^3 \times y^2 \text{ gives } y^{1+3+2} \text{ or } y^6$$
$$z \times z^2 \text{ gives } z^{1+2} \text{ or } z^3$$

Reqd. product $= +24x^3y^6z^3$.

Note. In practice, the explanations given in the above solutions are omitted and all operations are done mentally. Thus :—

$$3a^2b^3c \times (-4b^5c^2d) = -12a^2b^8c^4d.$$
$$2x^2y \times (-3xy^3z) \times (-4y^2z^2) = +24x^3y^6z^3.$$

EXERCISE 7

Multiply :—

1. 5 by —8.
2. —6 by 7.
3. —5 by —9.
4. —3a by 2b.
5. —5x by —7y.
6. xy by —8z.
7. a^2 by a^4.
8. x^3 by x^7.
9. x^3y by xy^3.
10. ab^2c^3 by a^3b^2c.
11. $-3a^3b$ by $4ab^2$.
12. $5a^2b^3$ by $-6ab^2c$.
13. $-6a^2bc$ by $-7bc$.
14. —1 by $10a^3b^3c^3$.
15. $-8ab^2c$ by 0.
16. $-8x^8y^{10}$ by $-8xy^2z^3$.

Find the continued product of :—

17. $2ab$, $3bc$ and $4ca$.
18. a^2b, $-3b^2c$ and $5c^2a$.
19. $-3ab^2c^3$, $-4ab$ and -1.
20. $-2x^3y^3$, $-3x^2y^2$ and xyz.
21. $3x^2z^4$, $-4z^2y$ and $-5x^3z^3$.
22. $-2l^2m^2n^2$, $-3l^3m^3n^3$ and $-4l^4m^4n^4$.

Simplify :—

23. $(-2abx) \times (-4bxy)$.
24. $(-6a^2bc) \times (\frac{3}{4}ab^2c)$.
25. $(-\frac{5}{8}xy^3) \times (\frac{4}{15}xz)$.
26. $(\frac{3}{4}xy^2z^3) \times (-\frac{2}{3}y^3x)$.

Simplify :—

27. $(-ab)(-bc)(-ca)$. [*Hint.*]
28. $(2xy)(-3xy^2)(-4yz^3)$.
29. $(-3xy^2)(4yz^2)(-5zx^2)$.
30. $(-xyz)(-yzt)(-ztx)(-txy)$.

Simplify :—

31. $(2ab)^3$. [*Solved*]
32. $(-8x^2y)^3$.
33. $(-4xyz^2)^4$.
34. $(-a^3b^5)^5$.

Simplify :—

35 $(-2)^2 \times (-a)^3$ [*Solved*] **36.** $(-3)^3 \times (-x^2)^3$.

37. $(-4)(-2x)^2(-3y)^3$　**38** $(-2)(-3)^2(-3a^2)^3-(-1)^4$.

39 $(-5)(-1)^3(-ab)^2(-bc)^2(\cdot ca)^2$.

40 $(-1)^9(-2)^3(-x)^2(-y^2)^3(-z^3)^4$.

If $a=1$, $b=-2$, $c=3$, $d=-4$, *find the value of* :—

41. $5b^3$　[*Solved*]　　　**42.** $2cd^2$.

43. $-3abcd$　　　　　　　**44.** $4a^3b^2cd$.

45. $-a^4b^3c^2d$　　　　　**46.** $ab+bc+cd$.

47. $(a-b)(b-c)(c-d)$.

48. $a^2(b-c)+b^2(c-a)+c^2(a-b)$.　[*Solved*]

49. $a^3+3a^2b+3ab^2+b^3$.

50. $(c+d)(c^2-cd+d^2)$.

51. $a^2+b^2+c^2-ab-bc-ca$.

52. $a^3+b^3+c^3-3abc$.

SOLUTION & HINTS—EXERCISE 7

27. When no sign is given between two pairs of brackets, the sign $\times$ is understood.

31. $(2ab)^3=2ab \times 2ab \times 2ab=8a^3b^3$.　**Ans.**

35. $(-2)^2 \times (-a)^3=(-2)(-2) \times (-a)(-a)(-a)$
$$=4 \times (-a^3)=-4a^3. \quad \textbf{Ans.}$$

41. $5b^3=5 \times (-2)^3=5 \times (-2)(-2)(-2)$
$$=5 \times (-8)=-40. \quad \textbf{Ans.}$$

48. $a^2(b-c)+b^2(c-a)+c^2(a-b)$
$$=(1)^2\{(-2)-(3)\}+(-2)^2\{(3)-(1)\}+(3)^2\{(1)-(-2)\}$$
$$=1(-5)+4(2)+9(3)$$
$$=-5+8+27$$
$$=30. \quad \textbf{Ans.}$$

26. Multiplication of a Multinomial and a Monomial.

From the *Distributive Law of Multiplication* we at once deduce the following **Rule** :—

Multiply each term of the multimonial by the given monomial and add together the partial products.

Example. Multiply $3x-4y+5x^2y^2$ by $-2xy^2$.

Solution. $(3x - 4y + 5x^2 y^2)(-2xy^2)$
$$= (3x)(-2xy^2) + (-4y)(-2xy^2) - (5x^2y^2)(-2xy^2)$$
$$= -6x^2 y^2 + 8xy^3 - 10x^3 y^4 \quad \text{Ans}$$

EXERCISE 8

Multiply :—

1. $a - 2b$ by 5.
2. $2a - 5y$ by -4
3. $-3x + 4xy$ by $2x$.
4. $3x^2 - 4x$ by $-3x$
5. $a - b - c$ by abc.
6. $3a^2 - 4a - 5$ by $-5a^2$
7. $2x^2 - 3xy + y^2$ by $3x^2 y$.
8. $-4x^3 + x^2 y - y^3$ by $-4x^2 y$
9. $ax^3 + by^3 - cz^2$ by $3a^2 b^2 z^2$.
10. $a^2 x^3 - 3b^3 y^2 - 4xy$ by $-a^2 b^3 x^4 y^5$
11. $ax^3 - 3bx^2 + 3cx - d$ by $2bcdx^2$
12. $ax^2 - 2hxy + by^2 - 2gx - 2fy + c$ by $-3x^2 y^2$.

Find the product of :—

13. $4x - 5y + 6z$ and $-\frac{3}{2}xy$.
14. $6x^2 - 4xy - y^3$ and $\frac{3}{4}x^3 y^2$
15. $\frac{1}{y}x^4 + \frac{1}{4}x^2 y^2 - 8y^4$ and $-\frac{3}{2}x^4 y^6$.
16. $-\frac{2}{3}a^3 b^3$ and $-\frac{1}{2}a^4 + 5a^3 - \frac{3}{2}a - 1$

Simplify by removing brackets and collecting like terms :—

17. $1 - 2(3a - b + 1)$.
18. $3xy - 4x(y - 3) + 2x$.
19. $4a^2 b^2 - 2ab^2(2a - b - 3) - 6ab^2$
20. $5a^2(a - b) - 6b(a^2 - b)$.
21. $-2(a^3 - 2) - 4(a^4 - 4) - a(2a^2 - 1)$
22. $2a^2(b^2 - c^2) + 2b^2(c^2 - a^2) + 2c^2(a^2 - b^2)$.
23. $3xy(4x^2 - xy - y^2) - 12y(x^3 + 1) + 3xy^3$
24. $3a^2(a^2 b^2 + c^4) - 3c^2(b^4 + c^2 a^2) - 3b^2(a^4 - b^2 c^2)$

27. (a) Dimension and Degree ; Ascending and Descending Powers.

Each letter of a *term* is called a *dimension* of the term and the number of letters or dimensions is called the *degree* of the term. For example, the term *abc* is of *three dimensions* or *of the third degree*, similarly ax^4 is of *five dimensions*

(being composed of a, x, x, x and x) or *of the figth degree.* A numerical coefficient is not counted. Thus $7x^3y^4$ and $-8z^7$ are each of *seven* dimensions or of the seventh degree.

The degree of an expression is the degree of the term of highest dimensions in it. Thus $3x^5-4x^2+x-9$ is *an expression of the fifth degree* and $3a^4x-6a^2x^5-x^6-a^6$ is *an expression of the seventh degree.* But sometimes we speak of the dimensions of an expression with regard to only one of the letters occuring in it. For example the last expression given above is of six dimensions in x.

When an algebraic expression is so arranged that the highest power of a certain letter is on the left (*i.e.*, in the first term) and in all the following terms the power of the same letter becomes smaller and smaller, it is said to be arranged according to *descending powers* of that letter. If the order of the terms be reversed, the expression will be said to be arranged according to *ascending powers of x.* For example, the expression $3x^2-4x^4+8-3x$ is neither in ascending nor in descending powers of x. Arranged according to *descending powers of x*, it will be $-4x^4+3x^2-3x+8$, and according to *ascending powers* it will be $8-3x+3x^2-4x^4$.

Note. It should be carefully noted that when we are arranging an expression according to the powers of a particular letter, we do not pay any attention to the powers of other letters occuring in it. Beginners often miss this point. For example, $x^5-a^4+2x^3$ is *not* in descending powers of x. The correct arrangement is $x^5+2x^3-a^4$. The beginner is sometimes misled by the coefficients as well. For example $8x-3x^2+1$ is *not* in descending powers of x, for 8 is the *coefficient* and not the *index.* The proper arrangement is $-3x^2+8x+1$.

27. (*b*) Homogeneous Expressions.

A compound expression is said to be homogeneous when all its terms are of the same dimensions. Thus $a^4-3a^3b+2a^2b^2-8b^4$ is a homogeneous expression of four dimensions, for each term in it is of four dimensions or of the fourth degree.

28. The arrangement of expressions according to ascending or descending powers of a letter is of immense use in multiplication and division of multinomials. Students are advised to do the following exercise on it before they proceed further

EXERCISE 9

Arrange according to descending powers of x :—

1. $6x - 5x^2 - 2x^3 + 7$.
2. $10x^2 + x^3 - x^4 - 7x + 20$.
3. $100x - 9x^2 + 50 + x^3$.
4. $x^4 - 5x^3 - 7 - x^2$.
5. $6x^2 - 8 + 5x^3 - 10x + 4x^4$.

Arrange according to ascending powers of (i) x, (ii) a :—

6. $ax^3 - a^4 + 5a^2x^3 - 6x^4 + a^3x$.
7. $a^4 + x^4 - a^3x - ax^3 + 20a^2x^2$.
8. $6ax^4 - x^5 - 6a^5 + 100a^2x^3 + xa^4$.
9. $xa^5 + 30x^6 - a^6 + 5a^2x^4 - x^3a^3$.
10. $x^4 + a^4 - 4a^2x^2 - 5ax^3 - 6a^3x$.

Arrange according to descending powers of a :—

11. $a^2 + b^2 + c^2 - ab - bc - ca$. [*Solved*].
12. $a^3 + b^3 + c^3 - 3abc$.
13. $a^3 - b^3 - 3ab^2 + 3a^2b - c^3$.

Arrange according to descending powers of z :—

14. $x^2 - 4y^2 - 3z^2 - 4xy + 6yz - 3zx$. [*Hint*].
15. $8x^3 - 27z^3 + y^3 + 18xyz$.
16. $x^2y^2 - x^2z^2 - y^2z^2 + 4xyz - y^3 - z^3$.

SOLUTIONS & HINTS—EXERCISE 9

11 The highest power of a is a^2.

The next power of a is found in two terms, $-ab$ and $-ca$. These two terms must be combined and written as a single term, *viz.* $+(-b-c)a$ or $-(b+c)a$. [Art. 17]. Note that in this term the co-efficient of a is $(-b-c)$ or $-(b+c)$.

The remaining terms (b^2, c^2 and $-bc$) do not contain a. They should also be combined into a single term, viz. (b^2+c^2-bc), which will be the absolute term of the expression in descending powers of a

Hence the reqd. arrangement is :—

$$a^2+(-b-c)a+(b^2+c^2-bc)$$

or $a^2-(b+c)a+(b^2+c^2-bc)$.

14. First combine the terms, which **do not contain** z, into a single term : then the terms which **contain** first power of z, and so on.

29. Multiplication of two Multinomials.

The law of distribution gives the method for this case as well. The underlying principle will be explained by the following simple

Example. Multiply $a+b+c$ and $x+y+z$.

Solution. We have to *distribute the product* :—

$$(a+b+c)(x+y+z).$$

To begin with, we regard the second expression as a monomial (*i.e.*, a single quantity) and multiply by the method of article 26. This gives :—

$$a(x+y+z)+b(x+y+z)+c(x+y+z).$$

Applying the same method to each of the three parts thus obtained, we get :—

$ax+ay+az+bx+by+bz+cx+cy+cz$, which is the required product.

Thus we get the following

Rule. *Multiply each term of the multiplicand by each term of the multiplier, with their proper signs, and add together the partial products.*

Note 1. When the number of expressions is more than two, multiplication of any two of them will reduce the number of expressions by one, and repetitions of the same process will ultimately give the required continued product.

EXERCISE 10

Multiply :—

1. $x-3$ and $x+2$. [*Solved*]   2. $x+4$ and $x-5$.
3. $x-6$ and $x-7$.   4. $a+7$ and $a+10$.
5. $a+8$ and $a-8$.   6. $k-8$ and $k-9$.

———

7. $x-5$ and $6-x$. [*Solved*]   8. $a+4$ and $10-a$.
9. $k-12$ and $12-k$.   10. $1-t$ and $t+11$.

———

11. $3x-4$ and $5-2x$. [*Solved*] 12. $5t+4$ and $4t-5$.
13. $6m-5$ and $-4m+7$.   14. $5-9a$ and $6-7a$. [*Hint*]
15. $8w+1$ and $1-8w$.   16. $-6-11v$ and $8-11v$.

———

17. $8x-3y$ and $2y-7x$. [*Solved*].
18. $3x-9y$ and $8y-9x$.   19. $3a-7b$ and $-b+5a$.
20. $-5v+7u$ and $-8u-3v$.

———

21. $2x^2-5$ and $6-5x^2$. [*Solved*].
22. $3x^2-7$ and $5+6x^2$.   23. $3x^2-2a$ and $5a-6x^2$.
24. $-7y^2+z$ and $7y^2+z$.

———

25. $5xy-3a$ and $4a-6xy$. [*Solved*].
26. $2xy-6ab$ and $-xy+2ab$.
27. $5pq-rs$ and $-2pq+3rs$.   28. $6x^2-2x$ and $x-7x^2$.

———

29. $4x^2+6x+9$ and $2x-3$. [*Solved*].
30. x^2-2x-5 and $3x-4$.
31. $2x-5-5x^2$ and $1-6x$.
32. $25-10a+4a^2$ and $2a+5$.

———

33. $4x^2-3xy-y^2$ and $3x-2y$. [*Solved*].
34. x^2-3ax and $x+3a$.
35. $2b^2+3ab-a^2$ and $7a-5b$.
36. $4x^2-6xy+9y^2$ and $2x+3y$
37. $16a^2+12ab+9b^2$ and $5b-4a$.
38. $x^4-x^2y^2+y^4$ and x^2+y^2.

———

39. x^3+8-2x^2 and $x+2$.
40. x^3+x^2+x-1 and $x-1$.
41. $8x+2x^3+16+4x^2$ and $3x-6$.
42. $a^2x-ax^2+x^3-a^3$ and $x+a$.
43. $x^5-x^4y-y^5+xy^4$ and $x+y$. [Hint].
44. $-a^5-a^3b^2+a^4b$ and $-a-b$.
45. $27x^3-36ax^2-64a^3+48a^2x$ and $4a+3x$.

- - -

46. $1+4a-10a^2$ and $1-6a+3a^2$. [Solved]
47. a^2-2+a and a^2-6+a.
48. $x^2-3xy-y^2$ and $-x^2+xy+y^2$.
49. $a^2-2ab+b^2$ and $a^2+2ab+b^2$.
50. $x^2-5xy-y^2$ and x^2+y^2+5xy.
51. $a^2-2ax+4x^2$ and $4x^2+2ax+a^2$.

- - -

52. $1+2x+x^4+3x^2$ and x^3+2+2x;   also deduce the
     product of 10321 and 1022 by giving a suitable value
     to x. [Solved]
53. $x^3-4x^2+11x-24$ and x^2+5+4x.
54. $a^3+5a-24+4a^2$ and $11-4a+a^2$.
55. a^3+1-7a^2+5a and $1+2a^2-4a$.
56. $2a^3+2a+3a^2$ and $3a+2+2a^2$; also deduce the product
     of 2320 and 232 by giving a suitable value to x.
57. $5-7a+a^3$ and $3-2a+a^2$.
58. $x^3+2xy^2+2x^2y$ and $x^2-2xy+2y^2$.
59. $x^3+24xy^2+60y^3+6x^2y$ and $12y^3-6x^2y+x^3+12xy^2$.
60. $x^3-4y^3-2x^2y+3xy^2$ and $4x^3+y^3+3x^2y+2xy^2$.

- - -

61. $x^2+4y^2+9z^2-2xy+6yz+3zx$ and $x+2y-3z$. [Solved]
62. $x^2+y^2+z^2-xy-yz-zx$ by $x+y+z$.
63. $a^2-ax+bx+b^2$ by $a+b+x$.
64. $x^2+y^2-xy+x+y-1$ and $x+y-1$.
65. $a^2+1-ab+a$ and $b+a-1$.
66. $x^2+9y^2-3xy+3y+x+1$ by $x+3y-1$.
67. $a^2-2ab+b^2+c^2$ by $a^2+2ab+b^2-c^2$.
68. $a^2(b-c)+b^2(c-a)+c^2(a-b)$ by $a+b+c$. [Hint]

Find the continued product of :—

69. $x-a)$ $x+a$ and x^2+a^2. [Hint]
70. $2x+3a$, $2x-3a$ and $4x^2+9a^2$.

71. $x^4 - ax + a^2$, $x^2 + ax + a^2$ and $x^4 - a^2x^2 + a^4$.
72. $x - 2a$, $x - a$, $x + a$ and, $x + 2a$. [*Hint*]
73. $a^2 + 3a + 2$, $a^2 - 5a + 6$, and $a^2 + 2a - 3$.
74. $a^2 - 2ab + 3b^2$, $a^2 + 2ab + 2b^2$ and $a - 3b$.

Multiply :—

75. $\frac{1}{2}x^2 + \frac{1}{3}x + \frac{1}{4}$ by $\frac{1}{2}x - \frac{1}{3}$. [*Solved*]
76. $\frac{1}{2}a^2 - 2a + \frac{2}{3}$ by $\frac{1}{2}a + \frac{1}{2}$.
77. $\frac{2}{3}x^2 - \frac{3}{4}xy + \frac{1}{2}y^2$ by $\frac{1}{2}x - \frac{1}{3}y$.
78. $\frac{1}{2}x^2 - \frac{2}{3}xy - \frac{3}{4}y^2$ by $\frac{1}{2}x^2 - \frac{3}{4}y^2 + \frac{2}{3}xy$.

———

79. Find the coefficient of x^4 in the product :—
$(a^3 - 3x^2 + x - 6)(2x^3 - 4x^2 - 3x + 7)$. [*Solved*]
80. Find the coefficient of x^2 in the product :—
$(x^3 - x^2 - x - 1)(3x^2 + x + 2)$.
81. Find the coefficient of a^3 in the expression :—
$3a^3(a - 2) - 2a^2(3a + 1) + a(a - 1)$.
82. What is the coefficient of a^2 in the expansion of :—
$(2a^2 - 3a + 1)^2$.

———

SOLUTIONS & HINTS—EXERCISE 10

1.
$x - 3$
$x + 2$
——————
$x^2 - 3x$ ......Product of $x - 3$ by x
$\quad + 2x - 6$ ...... ,, $x - 3$ by 2
——————
$x^2 - x - 6$ ....Sum of the two partial products.
Ans.

7. $x - 5$ is in descending powers of x and $6 - x$ in ascending powers. Both must be in the same order. We change the order of the second.

$x - 5$
$-x + 6$
——————
$-x^2 + 5x$
$\quad + 6x - 30$
——————
$-x^2 + 11x - 30$ **Ans**

11.

$$3x - 4$$
$$-2x + 5 \quad \text{(arranging in descending powers of } x)$$
$$\overline{-6x^2 + 8x}$$
$$ +15x-20$$
$$\overline{-6x^2 + 23x - 20} \quad \textbf{Ans.}$$

14. The two expressions are already in the same order, *viz.*, ascending powers of x, therefore no re-arrangement is necessary.

17.

$$8x - 3y$$
$$- 7x + 2y \quad \text{(arranging in descending powers of } x)$$
$$\overline{-56x^2 + 21xy}$$
$$ +16xy-6y^2$$
$$\overline{-56x^2 + 37xy - 6y^2} \quad \textbf{Ans.}$$

21.

$$2x^2 - 5$$
$$- 5x^2 + 6 \quad \text{(Re-arranging)}$$
$$\overline{-10x^4 + 25x^2}$$
$$ +12x^2-30$$
$$\overline{-10x^4 + 37x^2 - 30} \quad \textbf{Ans.}$$

25.

$$5xy - 3a$$
$$-6xy + 4a \quad \text{(Re-arranging)}$$
$$\overline{-30x^2y^2 + 18axy}$$
$$ +20axy-12a^2$$
$$\overline{-30x^2y^2 + 38axy - 12a^2} \quad \textbf{Ans.}$$

29.

$$4x^2 + 6x + 9$$
$$2x - 3$$
$$\overline{8x^3 + 12x^2 + 18x}$$
$$ -12x^2-18x-27$$
$$\overline{8x^3 -27}$$

Reqd. product is $8x^3 - 27$.   **Ans.**

33.

$$4x^2 - 3x y - y^2$$
$$3x - 2y$$
$$\overline{12x^3 - 9x^2y - 3xy^2}$$
$$ - 8x^2y+6xy^2+2y^3$$
$$\overline{12x^3 - 17x^2y + 3xy^2 + 2y^3} \quad \textbf{Ans.}$$

39. The first expression, arranged according to descending powers of x is $x^3 - 2x^2 + 8$ The first power of x is

missing, and if we keep the expression in this very form, *a difficulty arises in the work : the same powers of x do not appear in the same column.* This defect can be easily removed by writing a zero in place of the missing term. *This is an important device and must be noted carefully.*

$$x^3 - 2x^2 + \ 0 \ + 8$$
$$x + 2$$
$$\overline{x^4 - 2x^3 + \ 0 \ + 8x}$$
$$+2x^3 - 4x^2 + 0 \ + 16$$
$$\overline{x^4 \qquad\quad -4x^2 + 8x + 16} \qquad \text{Ans.}$$

43. The first exp. should be written as $x^5 - x^4y + 0 + 0 + xy^4 - y^5$, because the terms in x^3 and x^2 are missing.

46. No re-arrangement is necessary in this sum, because both the expressions are already in the same order *viz.*, ascending powers of a.

$$1 + 4a - 10a^2$$
$$1 - 6a + \ 3a^2$$
$$\overline{1 + 4a - 10a^2}$$
$$-6a - 24a^2 + 60a^3$$
$$+ \ 3a^2 + 12a^3 - 30a^4$$
$$\overline{1 - 2a - 31a^2 + 72a^3 - 30a^4} \qquad \text{Ans.}$$

52.
$$x^4 + 0 + 3x^2 + 2x + 1 \qquad \text{[Re-arranging and writing}$$
$$x^3 + 0 + 2x + 2 \qquad\qquad \text{zeros for the missing terms.]}$$
$$\overline{x^7 + 0 + 3x^5 + 2x^4 + \ x^3}$$
$$+ 0 + 0 \ \ + 0 \ \ + 0 \ \ + \ 0$$
$$+ 2x^5 + 0 \ \ + 6x^3 + \ 4x^2 + 2x$$
$$+ 2x^4 + 0 \ \ + \ 6x^2 + 4x + 2$$
$$\overline{x^7 \qquad + 5x^5 + 4x^4 + 7x^3 + 10x^2 + 6x + 2} \qquad \text{Ans.}$$

If we put $x = 10$, the multiplicand becomes equal to $(10)^4 + 0 + 3(10)^2 + 2(10) + 1 = 10000 + 0 + 300 + 20 + 1 = 10321$. Similarly the multiplier equals 1022.

[Note that these values can be obtained simply by writing the coefficients successively ; but it should be remembered that for this purpose *no coefficient should be greater than* 9.]

Hence the required product is obtained by putting $x=10$ in the product found above. This will be seen equal to 10548062 [How can this be obtained without actual substitution ?]

61. The expressions contain three letters : x, y and z. We arrange the expressions in descending powers of x, and the literal coefficients thus obtained, in descending powers of y.

$$x^2 - (2y-3z)x + (4y^2 + 6yz + 9z^2)$$
$$x + (2y - 3z)$$
$$\overline{x^3 - (2y-3z)x^2 + (4y^2 + 6yz + 9z^2)x}$$
$$+ (2y - 3z)x^2 - (4y^2 - 12yz + 9z^2)x + (8y^3 - 27z^3)$$
$$\overline{x^3 \qquad + \qquad (18yz)x \qquad + (8y^3 - 27z^3)}$$

The product may be written as $x^3 + 8y^3 - 27z^3 + 18xyz$. **Ans.**

Important Note. Without proper arrangement the work would present many difficulties

It may also be noted that to get some of the *literal coefficients* in the above work we have to perform the following multiplications separately :—

$$(2y - 3z)(2y - 3z) \text{ and } (2y - 3z)(4y^2 + 6yz + 9z^2)$$

The products are $4y^2 - 12yz + 9z^2$ and $8y^3 - 27z^3$ respectively and can be easily obtained by previous knowledge.

68. Open the brackets and arrange as in question 61.

69. First find the product of $x-a$ and $x+a$, which is $x^2 - a^2$.

Then find the product of $x^2 - a^2$ and the third expression, viz, $x^2 + a^2$.

72. Multiply (i) $x - 2a$ and $x + 2a$, (ii) $x - a$ and $x + a$. Then multiply the two products.

75.
$$\tfrac{1}{2}x^2 + \tfrac{1}{3}x + \tfrac{1}{4}$$
$$\tfrac{1}{2}x - \tfrac{1}{3}$$
$$\overline{\tfrac{1}{4}x^3 + \tfrac{1}{6}x^2 + \tfrac{1}{8}x}$$
$$- \tfrac{1}{6}x^2 - \tfrac{1}{9}x - \tfrac{1}{12}$$
$$\tfrac{1}{4}x^3 \qquad + \tfrac{1}{72}x - \tfrac{1}{12}$$

79.
$$x^3 - 3x^2 + \quad x - 6$$
$$2x^3 - 4x^2 - \quad 3x + 7$$
$$\ldots\ldots + 2x^3 \ldots\ldots$$
$$\ldots\ldots + 12x^4 \ldots\ldots$$
$$- 3x^4 \ldots\ldots$$
$$\ldots\ldots + 11x^4 \ldots\ldots$$

Coefficient of x^4 in the product $= 11$. **Ans.**

Note that all unnecessary work has been omitted

30. Multiplication by the method of detached co-efficients.

(a) When two compound expressions contain powers of one letter only, the labour of multiplication is much lessened by using **detached co-efficients**, that is, by writing down the coefficients only, multiplying them together in the ordinary way, and then inserting the successive powers of the letter at the end of the operation. In using this method, the expressions should, of course, be arranged according to ascending or descending powers of the common letter and zero coefficients used with the missing powers of that letter.

Example. Multiply $3a^3 - 5 - 4a^2$ by $4a - 2 + 4a^2$.

Solution. Arranging in descending powers of a and using zero coefficients with the missing powers, the expressions become $3a^3 - 4a^2 + 0a - 5$ and $4a^2 + 4a - 2$. Hence we have the following work :—

$$3 - 4 + 0 - 5$$
$$4 + 4 - 2$$
$$\overline{12 - 16 + 0 - 20}$$
$$\quad 12 - 16 + 0 - 20$$
$$\quad\quad - 6 + 8 - 0 + 10$$
$$\overline{12 - 4 - 22 - 12 - 20 + 10}$$

It can be easily seen that the first coefficient in the product (viz., 12) is the coefficient of a^5. Hence we insert the successive powers (a^5, a^4, a^3, etc.,) and get the reqd. product to be $12a^5 - 4a^4 - 22a^3 - 12a^2 - 20a + 10$.

(b) When powers of two letters are involved, but the expressions are *homogeneous*, the method of detached coefficients may still be used with advantage.

[For the definition of a **Homogeneous** Expression see Article 27 (*b*).]

Example. Multiply $3a^4 - 2b^4 + ab^3 + 2a^3b$ by $2a^2 - b^2$.

Solution. Arranging in descending powers of a and using zero coefficients with the missing powers, the expressions become $3a^4 + 2a^3b - 0a^2b^2 + ab^3 - 2b^4$ and $2a^2 + 0ab - b^2$. Hence we have the following work :—

$$3 + 2 - 0 + 1 - 2$$
$$2 - 0 - 1$$
$$\overline{6 + 4 + 0 + 2 - 4}$$
$$0 + 0 + 0 + 0 - 0$$
$$-3 - 2 - 0 - 1 + 2$$
$$\overline{6 + 4 - 3 + 0 - 4 - 1 + 2}$$

Clearly, 6 is the coefficient of a^6. Hence, inserting the successive powers of a and b (a^6, a^5b, a^4b^2, a^3b^3, etc.) We get the reqd. product to be

$$6a^6 + 4a^5b - 3a^4b^2 + 0a^3b^3 - 4a^2b^4 - ab^5 + 2b^6$$

or $6a^6 + 4a^5b - 3a^4b^2 - 4a^2b^4 - ab^5 + 2b^6$. (Leaving out the term with zero coefficient).

Note. In the last example we have tacitly assumed that the product of two homogeneous expressions is also a homogeneous expression.

EXERCISE 11

Apply the method of detached coefficients to the following questions of Exercise 10 :—

No. 29 to 60   [32 questions in all]

————

CHAPTER V

DIVISION

31. Definitions and Fundamental Notions.

When a quantity a is divided by another quantity b, the *quotient* is defined to be that quantity which when multiplied by b produces a, and is written as $a - b$, $\dfrac{a}{b}$ or a/b. Also, the expression a is called the *dividend* and b the *divisor*.

Division is thus the inverse of multiplication, and we have :—

$$(a \div b) \times b = a$$

or $$\frac{a}{b} \times b = a$$

or Quotient × Divisor = Dividend.

Since Division is the inverse of Multiplication, it follows that the Laws of *Commutation*, *Association* and *Distribution*, which have been discussed for Multiplication, hold for Division also.

32. The Rule of Signs.

We have $ab \div a = \dfrac{ab}{a} = \dfrac{a \times b}{a} = b$

$$-ab \div a = \frac{-ab}{a} = \frac{a \times (-b)}{a} = -b$$

$$ab \div -a = \frac{ab}{-a} = \frac{(-a) \times (-b)}{(-a)} = -b$$

$$-ab \div -a = \frac{-ab}{-a} = \frac{(-a) \times b}{(-a)} = b$$

Hence in division as well as multiplication,

like signs produce +

unlike signs produce −

33. The Index Law for Division.

We have $x^5 \div x^3 = \dfrac{x.x.x.x.x}{x.x.x} = x.x. = x^2 = x^{5-3}$.

In general, when *m* and *n* are positive integers and $m > n$, we have :—

$$x^m \div x^n = \frac{x.x.x\ldots\ldots\text{to } m \text{ factors}}{x.x.x\ldots\ldots\text{to } n \text{ factors}}$$

$$= x.x.x\ldots\ldots\text{to } (m-n) \text{ factors}$$

$$= x^{m-n}$$

That is, *a power of a quantity is divided by a lower power of the same quantity by subtracting the index of the divisor from the index of the dividend.*

Note If $m < n$, it can be easily seen that

$$x^m \div x^n = \frac{1}{x^{n-m}}$$

Example $x^3 \div x^5$ or $\dfrac{x^3}{x^5} = \dfrac{1}{x^{5-3}} = \dfrac{1}{x^2}$

34   To prove that $x^0 = 1$.

Proof Clearly $\dfrac{x^m}{x^m} = 1$

Also by the Index Law, $\dfrac{x^m}{x^m} = x^{m-m} = x^0$

$$x^0 = 1$$

35.   Division of one monomial by another

From the corresponding rule for multiplication [Art. 25] we easily get the following **Rule** :—

First put down the sign by the law of signs, next divide the the numerical coefficient of the dividend by that of the divisor, and lastly put down the letters in the dividend each having an index obtained by subtracting the index of that letter in the divisor from that in the dividend.

Example.   Divide $35x^7y^5z^2$ by $-7x^2y^4$.

Solution. Quotient $= -\dfrac{35}{7} \cdot \dfrac{x^7}{x^2} \cdot \dfrac{y^5}{y^4} \cdot z^2$

[unlike signs give $-$]

$$= -5x^{7-2}\,y^{5-4}\,z^2$$
$$= -5x^5y^1z^2 \text{ or } -5x^5yz^2. \quad \textbf{Ans.}$$

Note. In practice, each of the processes involved is done mentally. Thus :—

$$\frac{35x^7y^5z^2}{-7x^2y^4} = -5x^5yz^2.$$

Exercise 12

Divide :—[*See solved example of the last article (No. 35.)*]

1. -40 by 5.
2. -42 by -6.
3. 45 by -5.
4. $-6ab$ by $-3a$.
5. $35xy$ by $-5x$.
6. $-8xyz$ by xy.

7. a^6 by a^2.
8. $-x^{10}$ by x^3
9. x^3y^3 by $-x^2y$.
10. $-a^4b^4c^4$ by ab^2c^3.
11. $-12a^4b^3$ by $-3a^3b$.
12. $-30a^3b^5c$ by $5a^2b^5$
13. $42a^2b^2c^2$ by $-6a^2bc$.
14. $-10a^3b^3c^3$ by -1.
15. $-48x^6y^7z^8$ by $12x^5y^5z^5$.
16. $64x^9y^{12}z^9$ by $-8x^8y^{10}$.

Simplify :--

17. $8ab^2x^2y \div -12abx$.
18. $-4a^3b^3c^2 \div \frac{2}{3}ab^2c$.
19. $-\frac{1}{6}x^2y^3z \div -\frac{5}{8}xy^3$
20. $-\frac{1}{2}x^2y^5z^3 \div \frac{3}{4}xy^2z^4$

---

36. Division of a Multinomial by a Monomial.

From the corresponding rule for multiplication [Art. 26] we get the following.

Rule :—*Divide each term of the multinomial by the monomial and add together the partial quotients.*

Example :—Divide $20a^2bc - 35a^3b^3 \div 40a^4b$ by $-5ab$.

Solution :—Quotient $= \dfrac{20a^2bc}{-5ab} + \dfrac{-35a^3b^3}{-5ab} + \dfrac{40a^4b}{-5ab}$

$$= -4ac + 7a^2b^2 - 8a^3. \qquad \textbf{Ans.}$$

Exercise 13

Divide :—*[See solved example of the last article (No. 36.)]*

1. $5a - 10b$ by 5.
2. $-8x + 20y$ by -4.
3. $-6x^2 + 8x^2y$ by $2x$.
4. $-9x^3 + 12x^2$ by $-3x$.
5. $a^2bc - ab^2c - abc^2$ by abc.
6. $-15a^4 + 20a^3 + 25a^2$ by $-5a^2$.
7. $6x^4y - 9x^3y^2 + 3x^2y^3$ by $3x^2y$.
8. $16x^5y^3 - 4x^4y^4 + 4x^2y^6$ by $-4x^2y^3$.
9. $3a^3b^2x^3z^2 + 3a^2b^3y^3z^2 - 3a^2b^2cz^5$ by $3a^2b^2z^2$.
10. $-a^4b^3x^7y^5 + 3a^2b^6x^4y^7 + 4a^2b^3x^5y^6$ by $-a^2b^3x^4y^5$.
11. $2abcdx^5 - 6b^2cdx^4 + 6bc^2dx^3 - 2bcd^2x^2$ by $2bcdx^2$.
12. $-3ax^4y^3 + 6hx^3y^4 - 3bx^2y^5 + 6gx^3y^3 + 6fx^2y^4 - 3cx^2y^3$ by $-3x^2y^3$.

Simplify :—

13. $(-3x^2y + \frac{15}{5}xy^2 - 9xyz) \div (-\frac{3}{2}xy)$
14 $(\frac{9}{2}x^5y^3 - 3x^4y^3 - \frac{3}{4}x^3y^4) \div (\frac{3}{4}x^3y^2)$.
15. $(-\frac{5}{6}x^5y^8 - x^6y^8 + 12x^4y^{10}) \div (-\frac{1}{3}x^4y^6)$.
16. $(8a^7b^4 - 4a^6b^3 + \frac{7}{2}a^5b^3 + \frac{4}{3}a^3b^3) \div (-\frac{4}{3}a^3b^2)$

37. Division of one multinomial by another

In principle the process is to express the dividend as the sum of a number of parts, called *partial dividends*, each exactly divisible by the divisor, and then to obtain the quotient by case (ii)

Example   Divide $2x^2 + 6 + 7x$ by $x + 2$

Solution.   Quotient $= \dfrac{2x^2 + 7x + 6}{x + 2} = \dfrac{(2x^2 + 4x) + (3x + 6)}{(x + 2)}$

$$= \frac{2x^2 + 4x}{x + 2} + \frac{3x + 6}{x + 2} = \frac{2x(x + 2)}{(x + 2)} + \frac{3(x + 2)}{(x + 2)} = 2x + 3$$

Comparing the above work with the corresponding work in multiplication given in the margin we easily get the following

$$\begin{array}{r} x + 2 \\ 2x + 3 \\ \hline 2x^2 + 4x \\ 3x + 6 \\ \hline 2x^2 + 7x + 6. \end{array}$$

Working Rule

Arrange the dividend and the divisor according to the power (both ascending or both descending) *of some common letter*

Divide the first term of the dividend by the first term of the divisor . this gives the first term of the quotient.

Multiply the whole divisor by the first term of the quotient and subtract the product from the dividend

Treat this remainder as a new dividend and proceed as before till no remainder is left

Thus, the work of the above example may be conveniently presented as follows :—

$$x+2 \;) \; 2x^2+7x+6 \; (\; 2x+3$$
$$\underline{2x^2+4x}$$
$$3x+6 \quad \text{Reqd. quotient}=2x+3. \text{ Ans.}$$
$$\underline{3x+6}$$
$$\times$$

38. Inexact Division.

When the dividend and [the divisor are both arranged *according to the descending powers* of some common. letter, it may happen that at some stage of the process described in the last article, the highest power of this letter in the new dividend is lower than its highest power in the divisor. Such a division is said to be *inexact*; the portion of the quotient already obtained is called the *integral quotient*, and the last partial dividend is called the *remainder*. For example, if the dividend in the example of the last article be $2x^2+7x+10$, we get,

> integral quotient $=2x+3$
>
> and remainder $=4$

Also, complete quotient (as in Arithmetic)

$$=2x+3+\frac{4}{x+2}$$

EXERCISE 14

Divide :—

1. x^2-x-6 by $x+2$.   [*Solved*]
2. x^2-x-20 by $x-5$.   3. $x^2-13x+42$ by $x-7$.
4. $a^2+17a+70$ by $a+10$.   5. a^3-64 by $a-8$. [*Hint*]
6. $k^2-12k+27$ by $k-9$.   7. $-x^2+11x-30$ by $6-x$.
8. $-a^2+6a+40$ by $10-a$. 9. $-k^2+24k-144$ by $12-k$.
10. $-t^2-10t+11$ by $1-t$.

---

11. $23x-6x^2-20$ by $5-2x$.   [*Solved*]
12. $-9t-20+20t^2$ by $4t-5$.
13. $62m-24m^2-85$ by $-4m+7$.

14. $30-89a+63a^2$ by $6-7a$.   [Hint]
15. $1-64w^2$ by $1-8w$.
16. $-18+33v+121v^2$ by $-6-11v$.

———

17. $37xy-56x^2-6y^2$ by $2y-7x$.   [Solved]
18. $-27x^2-27y^2+90xy$ by $3y-9x$.
19. $15a^2+7b^2-38ab$ by $-b+5a$.
20. $15v^2-56u^2+19uv$ by $-8u-3v$.
21. $37x^2-10x^4-30$ by $6-5x^2$.   [Hint]
22. $18x^4-35-27x^3$ by $5+6x^2$.
23. $27x^2a-10a^2-18x^4$ by $5a-6x^2$.
24. z^2-49y^4 by $-7y^2+z$.

———

25. $3x^3-10x^2-7x+20$ by $3x-4$.   [Solved]
26. $18x^3-15x^2+32x-5$ by $1-6x$.
27. $12x^3+2y^3+3xy^2-17x^2y$ by $3x-2y$.
28. $26a^2b-ab^2-10b^3-7a^3$ by $-5b+7a$.

———

29. $x^6-x^4y^2+x^2y^4-y^6$ by $x+y$.   [Solved]
30. $8x^3-27$ by $2x-3$.   [Hint]
31. $8a^3+125$ by $2a+5$.   32. x^3-9a^2x by $x+3a$.
33. $8x^3+27y^3$ by $2x+3y$.   34. $27b^3-64a^3$ by $3b-4a$.
35. x^6+y^6 by x^2+y^2. [Hint]   36. $x^4-4x^2+8x+16$ by $x+2$.
37. x^4-2x+1 by $x-1$.   38. $6x^4-96$ by $3x-6$.
39. x^4-a^4 by $x+a$.   40. $a^6+a^3b^3$ by $-a-b$.
41. $81x^4-256a^4$ by $4a+3x$.

———

42. $12-8a-7a^2+2a^3+a^4$ by $6-a-a^2$.   [Solved]
43. $1-2a-31a^2+72a^3-30a^4$ by $1-6a+3a^2$.
44. $-x^4-y^4+4x^3y-x^2y^2-4xy^3$ by $-x^2+xy+y^2$.
45. $a^4-2a^2b^2+b^4$ by $a^2+2ab+b^2$.
46. $x^4-25x^2y^2+10xy^3-y^4$ by x^2+y^2+5xy.
47. $a^4+16x^4+4a^2x^2$ by $4x^2+2ax+a^2$.

———

48. $x^7+7x^3-6x-2+2x^2-5x^5$ by x^3-2-2x.   [Solved]
49. $2a^5-25a^2+a+1+39a^3-18a^4$ by $1+2a^2-4a$.
50. $15-31a+19a^2-4a^3-2a^4+a^5$ by a^2-2a+3.

51. $x^3 - 120 - 41x$ by $x^2 + 5 + 4x$.
52. $a^3 - 264 + 151a$ by $11 - 4a + a^2$
53. $4a^5 + 4a - a^3$ by $2a^2 + 8a + 2$
54. $x^5 + 4xy^5$ by $x^2 + 2y^2 - 2xy$.
55. $x^4 + 720y^6 + 1008xy^5$ by $x^3 + 12y^3 - 6x^2y + 12xy^2$.

---

56. $x^3 + 8y^3 - 27z^3 + 18xyz$ by $x + 2y - 3z$. $\quad$ [Solved]
57. $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.
58. $a^3 + a^2b + ab^2 + b^3 + 2b^2x - ax^2 + bx^2$ by $a + b + x$
59. $x^3 + y^3 + 3xy - 2x - 2y + 1$ by $x + y - 1$.
60. $a^3 - ab^2 + 2ab + b - 1$ by $a + b - 1$.
61. $x^3 + 27y^3 - 1 + 9xy$ by $x + 3y - 1$.
62. $a^4 - 2a^2b^2 + b^4 + 4abc^2 - c^4$ by $a^2 + 2ab + b^2 - c^2$
63. $a^3b - ab^3 + b^3c - bc^3 + c^3a - ca^3$ by $a + b + c$.

---

64. $\frac{1}{4}x^3 - \frac{1}{12} + \frac{1}{12}x$ by $\frac{1}{2}x - \frac{1}{3}$. $\quad$ [Solved]
65. $\frac{1}{4}a^3 + \frac{1}{2} + \frac{1}{12}a - \frac{3}{8}a^2$ by $\frac{1}{2}a + \frac{1}{3}$.
66. $\frac{1}{3}x^3 - \frac{1}{6}y^3 + \frac{1}{2}xy^2 - \frac{2}{3}x^2y$ by $\frac{2}{3}x^2 + \frac{1}{2}y^2 - \frac{1}{3}xy$.
67. $\frac{1}{4}x^4 + \frac{9}{16}y^4 - \frac{13}{34}x^2y^2$ by $\frac{1}{2}x^2 - \frac{3}{4}y^2 + \frac{7}{8}xy$.

---

68. Divide $2x^4 - x^3 + 4x^2 + 7x + 1$ by $x^2 - x + 3$ and give the complete quotient. $\quad$ [Hint]

69. Give the complete quotient when $80x^4 + 11x^3 - 82x^2 - 5x + 3$ is divided by $3x^2 + 2x - 4$.

70. What must be subtracted from $x^3 - x^2 - 8x - 10$, so that the result may be exactly divisible by $x^2 + 3x + 3$? (The answer should not contain any power of x higher than the first). $\quad$ [Hint]

71. What must be subtracted from $2y^3 - 8y^2 - 4y$ to make it exactly divisible by $2y^2 - 5y - 1$? (Answer not to contain power of y higher than the first).

72. What must be added to $6m^3 - m^2 - 16m + 6$ to make it exactly divisible by $8m^2 + 4m - 1$? (Answer not to contain power of m higher than the first). $\quad$ [Hint]

73. What must be added to $x^4 + x^3 + 7x^2 + x + 1$ to make it a multiple of $x^2 + 2x + 3$? (Answer not to contain power of x higher than the first)

74. For what value of a will a^2+2a-3 divide $a^4-a^3-5a^2+12a-6$ exactly ? (Verify your answer) [Hint]

75. For what value of a will $3a^3-2a^2-a$ be a factor of $6a^5-13a^4+4a^3+3a^2+a-2$? Verify your answer.

76. What should be the value of k so that a^2+3a+2 may divide $a^4+6a^3+13a^2+12a+k$ without any remainder ? [Hint]

77 What value should a have so that the expression $x^5-5x^4+9x^3-6x^2+ax+2$ may be exactly divisible by x^2-3x+2 ?

78. What expression multiplied by $a^3-2a^2b+2ab^2-b^3$ will give a^6-b^6 as the product ?

79. Divide $x^4+4x^3+5x^2+4x+1$ by x^2+3x+1 and by giving a suitable value to x show that 14541 is divisible by 131, and also find the other factor of 14541 without actual division. [Hint]

80. Divide $x^4+3x^3+8x^2+9x+9$ by x^2+2x+3 and by giving a suitable value to x deduce that 123 is a factor of 13899.

SOLUTIONS & HINTS—EXERCISE 14

1.
$$x+2 \overline{)\ x^2-x-6\ (\ x-3}$$
$$\underline{x^2+2x}$$
$$-3x-6$$
$$\underline{-3x-6}$$
$$\times$$

5. a^2-64 should be written as a^2+0-64.

11. Arranging the expression in descending powers of x we have :—

$$-2x+5 \overline{)\ -6x^2+23x-20\ (\ 3x-4}$$
$$\underline{-6x^2+15x}$$
$$8x-20$$
$$\underline{8x-20}$$
$$\times$$

Note.—The quotient does not change if the given expressions are multiplied by the same number. In the above

example let us multiply each of the given expressions by -1, in other words, let us change the signs of both. Then we have to divide $6x^2-23x+20$ by $2x-5$ Since the first term of the divisor is now positive, the work will be evidently much easier.

14. Both the expressions are already in ascending powers of a, therefore no change in them is necessary before division

17. Arranging in descending powers of x and changing signs of both the given expressions, we have the following work :—

$$7x-2y \,\big)\; 56x^2-37xy+6y^2 \,\big(\; 8x-3y$$
$$\underline{56x^2-16xy}$$
$$-21xy+6y^2$$
$$\underline{-21xy+6y^2}$$
$$\times$$

21. There is no need of inserting zeros for the missing powers in the question, for, we may regard the expressions to be in powers of x^2 (and not x). No power of x^2 is missing in the two expressions. [Note that x^2 is the first power of x^2 and x^4 is the second power of x^2].

Or

We may write y for x^2. Then the question reduces to $(37y-10y^2-30)\div(6-5y)$. After getting the quotient we should replace y by x^2.

25.
$$3x-4 \,\big)\; 3x^3-10x^2-7x+20 \,\big(\; x^2-2x-5$$
$$\underline{3x^3-4x^2}$$
$$-6x^2-7x$$
$$\underline{-6x^2+8x}$$
$$-15x+20$$
$$\underline{-15x+20}$$
$$\times$$

29      $x+y)x^6+0-x^4y^2+0+x^2y^4+0-y^6(x^5-x^4y+xy^4-y^5$

$$\frac{x^6+x^5y}{-x^5y-x^4y^2}$$
$$\frac{-x^5y-x^4y^3}{}$$

$$x^2y^4+0$$
$$\frac{x^2y^4+xy^5}{-xy^5-y^6}$$
$$\frac{-xy^5-y^6}{\times}$$

30. $8x^3-27$ will be written as $8x^3+0+0-27$.

35.   *Either* write a for x^2 and b for y^2, so that the question reduces to $(a^3+b^3)\div(a+b)$;

or insert zeros for the missing *even powers of x only*, i.e., write the dividend as $x^6+0+0+y^6$ and the divisor as it is.

42       $6-a-a^2$) $12-8a-7a^2+2a^3+a^4$ ($2-a-a^2$
$$12-2a-2a^2$$
$$\frac{}{-6a-5a^2+2a^3}$$
$$-6a+\ a^2+a^3$$
$$\frac{}{-6a^2+a^3+a^4}$$
$$-6a^2+a^3+a^4$$
$$\frac{}{\times}$$

48.

$x^3+0-2x-2)x^7+0-5x^5+0+7x^3+2x^2-6x-2(x^4-3x^2+2x+1$
$$\frac{x^7+0-2x^5-2x^4}{}$$
$$-3x^5+2x^4+7x^3+2x^2$$
$$-3x^3-0\ \ +6x^3+6x^2$$
$$\frac{}{2x^4+\ x^3-4x^2-6x}$$
$$2x^4+\ 0\ -4x^2-4x$$
$$\frac{}{x^3+0\ \ -2x-2}$$
$$x^3+0\ \ -2x-2$$
$$\frac{}{\times}$$

56.   Arranging each expression in descending powers of a

and the literal co-efficients thus obtained in descending powers of y we have the following work :—

$$x+(2y-3z)\overline{)x^3+\quad 0\quad +(18yz)x+(8y^3-27z^3)}\left(\begin{array}{l}x^2-(2y-3z)x+\\ (4y^2+6yz+9z^2)\end{array}\right.$$

$$\underline{x^3+(2y-3z)x^2}$$
$$-(2y-3z)x^2+(18yz)x$$
$$\underline{-(2y-3z)x^2-(4y^2-12yz+9z^2)x}$$
$$(4y^2+6yz+9z^2)x+(8y^3-27z^3)$$
$$\underline{(4y^2+6yz+9z^2)x+(8y^3-27z^3)}$$
$$\times$$

64.

$$\tfrac{1}{2}x-\tfrac{1}{3}\overline{)\tfrac{1}{4}x^3+0+\tfrac{1}{12}x-\tfrac{1}{12}}\left(\tfrac{1}{2}x^2+\tfrac{1}{3}x+\tfrac{1}{4}\right.$$
$$\underline{\tfrac{1}{4}x^3-\tfrac{1}{6}x^2}$$
$$\tfrac{1}{6}x^2+\tfrac{1}{12}x$$
$$\underline{\tfrac{1}{6}x^2-\tfrac{1}{9}x}$$
$$\tfrac{1}{8}x-\tfrac{1}{12}$$
$$\underline{\tfrac{1}{8}x-\tfrac{1}{12}}$$
$$\times$$

Or Multiply the given expressions by 72 (the L.C.M. of the denominators of all the fractions) and the question reduces to $(18x^3+0+x-6)\div(36x-24)$ and we have :—

$$36x-24\overline{)18x^3+0+x-6}\left(\tfrac{1}{2}x^2+\tfrac{1}{3}x+\tfrac{1}{4}\right.$$
$$\underline{18x^3-12x^2}$$
$$12x^2+x$$
$$\underline{12x^2-8x}$$
$$9x-6$$
$$\underline{9x-6}$$
$$\times$$

68. See Article 38.

70. Divide the first expression by the second, keeping them in descending powers of x. The remainder obtained is the required answer.

72. Proceed as in the last two questions. We know that the remainder must be *subtracted*. Hence, if we change the sign of the remainder we get the expression which must be *added*.

74. The remainder obtained is $a-3$. We require that this remainder should be zero. **Hence,** what should be the value of a ?

[In practice, we put the remainder equal to zero and solve the equation thus obtained for the letter whose value is to be found.]

Verification :—When $a=3$, the dividend will be found $=12$ and the divisor also $=12$. Hence the dividend is exactly divisible by the divisor.

76. Proceed as in question 74.

79. If we put $x=10$, the dividend becomes $=14541$, [this is easily obtained by writing down the co-efficients in the expression one after the other, after arranging it in descending powers of x and inserting zeros for the missing powers] and the divisor $=131$. Since the original division is exact, therefore the division of 14541 by 131 is also exact. [For, the division must be true for all values of x.]

The other factor of 14541 is obtained by putting $x=10$ in the original quotient.

39. Division by the method of Detached Co-efficients can be performed exactly on the lines of Article 30.

Example 1. Divide $2x^5+4-3x^4-11x+8x^2$ by x^3+1-2x.

Solution. Arranging in descending powers of x and using zero coefficients with the missing powers, the expressions become :—

$$2x^5-3x^4+0x^3+8x^2-11x+4 \text{ and } x^3+0-2x+1.$$

Hence we have the following work :—

```
1+0−2+1 ) 2−3+0+8−11+4 ( 2−3+4
             2+0−4+2
            ───────────
            −3+4+6−11
            −3−0+6− 3
            ───────────
              4+0− 8+4
              4+0− 8+4
            ───────────
                 ×
```

In the above quotient, 2 is the coefficient of x^2 ($\because 2x^5 \div x^3 = 2x^2$). Hence the reqd. quotient is $2x^2-3x+4$.

Example 2. Divide $x^4 + 4a^4$ by $x^2 + 2ax + 2a^2$.

Solution. The expressions are homogeneous, and the first expression may be written as $x^4 + 0x^3a + 0x^2a^2 + 0xa^3 + 4a^4$

Hence we have the following work :—

$$
\begin{array}{r}
1+2+2\ )\ 1+0+0+0+4\ (\ 1-2+2 \\
\underline{1+2+2} \\
-2-2+0 \\
\underline{-2-4-4} \\
2+4+4 \\
\underline{2+4+4} \\
\times
\end{array}
$$

Reqd. quotient is $x^2 - 2xa + 2a^2$.

EXERCISE 15

Apply the method of Detached Co-efficients to the following questions of Exercise 14 :—

No. 25 to 55 (31 questions in all)

CHAPTER VI

BRACKETS

40. An expression enclosed within a pair of brackets is to be regarded as a single quantity, so that the terms of which it is composed should be operated upon in the same way.

The brackets in common use are :—

( )............called Circular Brackets or parenthesis

{ }......... „ Curly or Crooked Brackets

[ ]........... „ Square Brackets or crochets

———— .... „ Vinculum or Bar (which is placed *above* an expression)

41. Removal of Brackets.

Students are already familiar with the rules used in opening brackets in the following examples :—

$$a + (b-c+d) = a+b-c+d$$
$$a - (b-c+d) = a-b+c-d$$
$$a + x(b-c+d) = a+xb-xc+xd$$
$$a - x(b-c+d) = a-xb+xc-xd.$$

We have only to discuss the case when there are *brackets within brackets*. In fact, no new principle is to be taught.: we may either begin with the outermost pair or with the innermost one and go on removing them in succession according to the rules already taught. For example :—

$$a - \{ b-(c-d)+e \} = a-b+(c-d)-e$$
$$= a-b+c-d-e \qquad \text{[Method 1]}$$

Or

$$a - \{ b-(c-d)+e \} = a- \{ b-c+d+e \}$$
$$= a-b+c-d-e. \qquad \text{[Method 2]}$$

But a beginner is apt to commit mistakes in the use of Method 1. Therefore Method 2 is always to be preferred. That is to say :—

In opening brackets within brackets we should begin with the innermost pair and proceed in succession to the outermost pair which should be removed last of all.

42. Another form of the Vinculum or Bar.

The line between the numerator and denominator of a fraction is a kind of Vinculum or Bar.

Thus $\dfrac{3x-4}{2}$ may be written as $\frac{1}{2}(3x-4)$.

EXERCISE 16

Simplify the following expressions by removing the brackets and collecting like terms :—

1. $6x-y-(4x-y)$.
2. $2x-y+z-(2x-y-z)$.
3. $2x-(y-z)+2x+(y-z)+y-(z+2x)$.

4. $2x+y+(14x-y)-(4x-3y)-(10x+6y)$.

5. $2x-y+z-(y-2x+z)+(z-x+y)-(2x-z+y)$.

6. $4a-3b-3c-(2a-b+2c)+(2a+4b+5c)-(c-2a-b)$.

---

7. $x-\{2y-z-(3x-y-z)\}$.   *[Solved]*

8. $2x-(4y-t)-\{x-2y-(2z-2t)\}$.

9. $x-\{4y-(3z+4y-x)\}$.

10. $x-[2y+\{x-(2y+x)\}]$.

11. $x-[2x-\{6y-(4z-2x)\}]$.

12. $\{x-(2y-z)\}+\{2y-(z-x)\}-\{z-(x-2y)\}$

13. $4x-(5y+[3z-2x])-(10x-[y+z])$.

14. $-[2x-\{y-(z-2x)\}]-[y-\{z-(2x-y)\}]$.

---

15. $-\{-[-(2x-\overline{y-z})]\}$.   *[Solved]*

16. $-[-\{-(y+\overline{z-2x})\}]+[-\{-(z+2x-y)\}]$.

17. $-2-[6a+\{3-(4a-\overline{a-6})-4\}-2a]$.

18. $2x-[5y-\{2x-(5z-2z-\overline{y}-4y)+4x-(2x-\overline{2y}\cdot z)\}]$.

---

19. $a-\{2b(3-4a)-a(4-b)\}$.   *[Solved]*

20. $1-2(3-4x)-[5\{x-(3+2x)\}]$.

21. $2-8[4-\{5-6(7-\overline{8-9})\}]$.

22. $x^4-x[-x\{-x(1-\overline{x\div3})+x\}]$.

23. $(2a-\overline{a\div b})\{a(a+2b)-b(a-b)\}$.

24. $(a-\overline{b-c})[\{a(a-b)+2b(a+c)-c(a+b)\}+b^2+c^2]$.

SOLUTIONS & HINTS—EXERCISE 16

7.   Given Exp. $=x-\{2y-z-(3x-y-z)\}$

$\qquad\qquad =x-\{2y-z-3x+y+z\}$

$\qquad\qquad =x-2y+z+3x-y-z$

$\qquad\qquad =x-3y$.

15.   Given Exp. $=-\{-[-(2x-\overline{y-z})]\}$

$\qquad\qquad =-\{-[-(2x-y+z)]\}$

$\qquad\qquad =-\{-[-2x+y-z]\}$

$\qquad\qquad =-\{2x-y+z\}$

$\qquad\qquad =-2x+y-z$.

19   Given Exp.$=a-\{\,2b(3-4a)-a(4-b)\,\}$
$\qquad\qquad =a-\{\,6b-8ab-4a+ab\,\}$
$\qquad\qquad =a-6b+8ab+4a-ab$
$\qquad\qquad =5a-6b+7ab.$

---

43   **Insertion of Brackets** has already been discussed in Art 17   However, one thing more may be noted here :—

Opening the bracket of the exp. $a-b(x-y)$ we get $a-bx+by.$

Now let us perform the reverse process, *i.e.*, insert the last two terms of $a-bx+by$ in a pair of brackets with negative sign outside.

According to Art. 17, we get $a-(bx-by).$

Now the student should also note that whenever a factor is common to every term within a pair of brackets, it may be removed and placed outside as a multiplier of the expression within the brackets.   In the last result b is such a common factor.   Hence we finally get our result in the form $a-b(x-y).$

EXERCISE 17

Bracket the last two terms with positive sign outside :—

1 $x+y-z.$ **2.** $x-y+z.$
3. $x-y-z.$ **4.** $x+y-z.$
5 $2x+3y+3z.$ **6.** $4x-ay+az.$
7 $5a-b^2c-b^2d.$ **8.** $3l+5abm-5abn.$

Bracket the last two terms with negative sign outside :—

9 $ax-by+bz.$ **10.** $3x^2-4a^2y-4a^2z.$
11. $abc+bcd-cda.$ **12.** $6a^2+15ab+20b^2.$

Enclose the first and the third terms within one pair of brackets and the remaining within another pair, with positive sign before both :—

13. $x^2-3x+3y+y^2.$ **14.** $-2xy-y^2+x^2-x+1.$
15. $abc+bca+cad-dab.$ **16.** $a^2b-b^2c-c^2a-a^2c-c^2b$

Enclose the first and the fourth terms within one pair of

brackets and the remaining within another, with positive sign before the former and negative sign before the latter :—

17. $a^3 - b^3 - c^3 - d^3$.
18. $a^2bc - b^2ca + c^2ab + a^2b^2c^2$.
19. $3x^3 + 4a^2y - 6c^2 - 18xy^2$
20. $20 - 10a^2 - 5a^3 - a^4$.

Bracket the same powers of x with positive sign before each pair of brackets :—

21. $ax^4 + 3x^3 + 2x^4 - bx^3 + x^2 - cx^2$. [*Solved.*]
22. $3x - x^4 - x^3 + ax - bx^4 - cx^3$.
23. $1 - 2x - 3x^2 - 4x^3 - 2ax^3 - 3bx^2 - 4cx - 5d$. [*Hint*]
24. $a^2c^2 - 2ac + bx^3 - 4c + x + x^2 + 1 - x^3$

Bracket the same powers of a with negative sign before each pair of brackets :—

25. $ax^3 - a^2x^4 + 3a^3 - xa + a - a^2 + a^3x$.
26. $2abc - 3a^2c + 4a^3b - a - a^2 - a^3 - ax$.

SOLUTIONS & HINTS—EXERCISE 17

21. $ax^4 + 3x^3 + 2x^4 - bx^3 + x^2 - cx^2$
$= ax^4 + 2x^4 + 3x^3 - bx^3 + x^2 - cx^2$
$= (ax^4 + 2x^4) + (3x^3 - bx^3) + (x^2 - cx^2)$
$= x^4(a + 2) + x^3(3 - b) + x^2(1 - c)$.

23. The terms which do not contain x should be grouped in a separate pair of brckets: thus : $(1 - 5d)$.

Note that this part also contains a power of x, *viz.* x^0 for we may write the term as $(1 - 5d)x^0$. [The student will later on learn that $x^0 = 1$.]

CHAPTER VII

LITERAL SUBSTITUTIONS

44. Numerical substitutions have already been dealt with in Chapter I, and also, incidentally, in Chapter IV [Ex. 7, questions 41 to 52]. Here we propose to discuss *Literal Substitutions. i.e.,* substitutions of expressions for single letters. The process is evidently quite simple and will not present any difficulty if only one instruction is carefully borne in mind, *viz* :—

While substituting an expression for a letter always close the expression within a pair of brackets. The importance of the topic, however, should not be underestimated. Though simple, it is of fundamental importance.

EXERCISE 18

1. If $a=2x+3y-4z$ and $b=-x-5y+z$, find the value of $a-2b$ in terms of x, y, z. [*Solved.*]

2. If $a=4x-y-z$ and $b=2x+3y-z$, find the value of $2a-3b$ in terms of x, y, z.

3. If $x=a+2b$, $y=b-2c$, $z=c-2a$, find the value of $2x-3y-4z$ in terms of a, b, c.

4. If $x=2a+3b$, $y=3a+4b$ and $z=3b-4a$, find the value of $-x-3y+5z$ in terms of a, b, c.

5. If $a=2l-4m+6n$, $b=-3l+6m-9n$, $c=4l+3$; find the value of $\dfrac{a}{2}+\dfrac{b}{3}-\dfrac{c}{4}$ in terms of l, m, n.

6. If $a=5p-10q$, $b=7q-14r$ and $c=6r-12p$, find the value of $\dfrac{2a}{-5}+\dfrac{3b}{7}-\dfrac{5c}{6}$ in terms of p, q, r.

If $x=2a+3b$ and $y=2a-3b$, find the values of the following expressions in terms of a and b :—

7.	x^2 [*Solved.*]	8.	y^3.
9	xy	10.	$x^2+2xy+y^2$
11	$x^2-2xy+y^3$.	12.	x^3 [*Solved.*]
13	y^3	14	$3x^2y$.
15	$3xy^2$	16	$x^3+3x^2y+3xy^2+y^3$
17	$x^3-3x^2y+3xy^2-y^3$	18	$(x+y)(x^2-xy+y^2)$.
			[*Solved.*]
19	$(x-y)(x^2+xy+y^2)$	20	$\frac{1}{3}(x^3+y^3)(x+y)(x-y)$.

If $a=2x^2$, $b=-3xy$, $c=-4y^2$, find the values of the follow-ing expressions in terms of x and y :—

21	ac	22	b^2.
23	abc	24	$-a^2$
25	-4^3	26	$-3a^2b$. [*Solved.*]
27	$4b^2c$	28	$-5c^2a$

29. $(a+b+c)^2$. [*Hint.*]   **30.** $a^2b^2c^2$.

31. $\dfrac{6ac}{b}$.   **32.** $\dfrac{16b^2}{ac}$.

33. $-\dfrac{a^2b^2}{9c}$.   **34.** $-\dfrac{2b^2c^2}{3a}$

35. $\dfrac{3a^3}{4b}$.   **36.** $\dfrac{4b^3}{3a^2c}$.

SOLUTIONS & HINTS—EXERCISES 18

1. $a-2b=(2x+3y-4z)-2(-x-5y+z)$
$=2x+3y-4z+2x+10y-2z$
$=4x+13y-6z.$

7. $x^2=(2a+3b)^2=(2a+3b)\times(2a+3b)=4a^2+12ab+9b^2$.

12. $x^3=(2a+3b)^3=(2a+3b)(2a+3b)(2a+3b)$
$=(4a^2+12ab+9b^2)(2a+3b)=8a^3+36a^2b+54ab^2+27b^3$.

18. First factor
$=x+y=(2a+3b)+(2a-3b)=2a+3b+2a+3b=4a.$

Second factor
$=x^2-xy+y^2=(2a+3b)^2-(2a+3b)(2a-3b)+(2a-3b)^2$
$=(4a^2+12ab+9b^2)-(4a^2-9b^2)+(4a^2-12ab+9b^2)$
$=4a^2+12ab+9b^2-4a^2+9b^2+4a^2-12ab+9b^2$
$=4a^2+27b^2$
$\therefore$ Given exp.$=4a(4a^2+27b^2)=16a^3+108ab^2.$

25. $-3a^2b=-3(2x^2)^2(-3xy)$
$=-3(4x^4)(-3xy)$
$=36x^5y.$

29. $a+b+c=(2x^2)+(-3xy)+(-4y^2)$
$=2x^2-3xy-4y^2$

$(a+b+c)^2=(2x^2-3xy-4y^2)^2=(2x^2-3xy-4y^2)\times$
$(2x^2-3xy-4y^2).$ Multiply the two factors.

TEST PAPERS—SET 1
(CHAPTERS I TO VII)
Paper 1 (Ex. 19)

1. If $a=1$, $b=2$, $c=8$, $d=-5$ and $e=8$, evaluate
   $[(e-d)(b+c)-(d-c)(c+a)](a+d)$.

2. Subtract $8x-7x^3+5x^2$ from the sum of $4+8x^2-11x^3$ and $12x^3-8x^2+x-4$.

3. Simplify $4x-y-(6x-2y)+(4x-3y)-(2x-2y)$.

4. Multiply x^3+2x+1 by x^2+2x+3, and hence deduce the product of 1021 and 123 by giving a suitable value to x.

5. Divide $2-4a^4+a^5+8(a^3+a^2-a)$ by a^2-2-a by the method of detached co-efficients.

6. A tradesman loses Rs. $(a-2)$; what does he gain? If by another transaction he gains Rs. $(2a-3)$ find his total (i) gain, (ii) loss.

Paper 2 (Ex. 20)

1. If $x=5$ and $y=3$, evaluate $\sqrt[3]{x^3-3x^2y+3xy^2-y^3}$.

2. Subtract $6x^3+3x-1$ from $2x^3+x^2$ and add the result to $8x^3+4x-1$.

3. If $x=12a-8b+2c$, $y=2a+b+c$ and $z=20a+b-7c$, find the value of $x+4y-z$ in terms of a, b and c.

4. Multiply a^2-3+2a by a^2+1-3a by the method of detached coefficients.

5. Divide $-17+70x+28x^4-71x^3-35x^2$ by $6-13x+4x^2$, giving the complete quotient.

6. Simplify $\frac{1}{3}(x+1)+\frac{1}{4}(x+3)-\frac{1}{5}(x+4)-16$ by adding like terms and find its value for $x=41$.

Paper 3 (Ex. 21)

1. If $a=2$, $b=-3$, $c=-1$, evaluate
   $$\frac{a^3+b^3+c^3-3abc}{a^2+b^2+c^2-ab-bc-ca}$$

2. Add the sum of $2y-5y^3$ and $1-3y^2+3y$ to the remainder left when $2-2y^2+y$ is subtracted from $1+5y^3$.

3. Simplify $14x - 12y - \{ 10x - 3[3y - 2(2x - 3y)] \}$

4. Multiply by the method of detached coefficients :—
   $1 + 3k^2 - k$ and $2 + 2k^2 - 5k$.

5. Divide the sum of $2x^4 + 12x^2 + 8$ and $1 - x^4 - 22x^3$ by the product of $x - 3$ and $x + 1$.

6. If $x = 2a - 3$ and $y = 3a - 4$, find the value of $x^2 - 2xy + y^2$ in terms of a.

Paper 4 (Ex. 22)

1. Evaluate $x^4 y^2 z^2 \sqrt{y^2 - z^2}$ when $x = 10$, $y = \frac{1}{2}$, $z = \frac{1}{3}$.

2. Take $x^2 - 2y^2$ from $3xy - 5y^2$ and add the remainder to the sum of $5xy - x^2 - 3y^2$ and $3x^2 + 6y^2$.

3. Simplify $4x - [6y + (4y - z) - 4z + \{ 4x - (6y - \overline{z - 4y}) \}]$.

4. Multiply $x^3 - y + x$ by $x^2 - y^2 + xy$.

5. Divide $4a^5 + 6a^4 + 4a^3 + 5a^2 + 4a + 1$ by $2a^3 + a + 1$ and hence show that 2011 is a factor of 464541. Deduce also the other factor of 464541.

6. If $a = x^2 - 2x + 3$ and $b = 2x^3 + x - 4$, find the value of $\frac{a^2 - b^2}{a - b}$ in terms of x.

Paper 5 (Ex. 23)

1. If $a = 3$, $b = -1$, $c = -2$, find the value of $a^2 + 4b^2 + 9c^2 - 4ab - 12bc + 6ac$.

2. To what expression must $8a^3 - 6a^2 - 5a$ be added so as to get 0 as our result ?

3. Add together $2x + 6y - (2z + t)$, $6x - (3y - 2z) + 2t$, and $4x - [3y - (2z - 3t)]$.

4. Multiply $1 + a + a^2 + 2x - 2ax + 4x^2$ by $1 - a - 2x$.

5. Divide $a^8 + 3a^4 + 2a^6 + 2a^2 + 1$ by $a^4 - 2a^3 + 1 - 2a + 3a^2$ by the method of detached coefficients.

6. Express the product of 8 and a in 5 different ways. Distinguish between 4·6 and 4.6.

Paper 6 (Ex. 24)

1. If $x = 3a + b$ and $y = 3a - b$, find the value of $(x + y) \times (x^2 - xy + y^2)$ in terms of a and b.

2. Find the sum of $a+b+c$, $5a+c-7b$ and $3b-9a$, and subtract the result from $2c-3b+a$.

3. Simplify $\dfrac{2x}{15}+\dfrac{x-6}{12}-\dfrac{3}{10}\left(\dfrac{x}{2}-5\right)$ by adding like

terms and find its value for $x=-15$.

4. Multiply $x^3+4y^2+9z^2+2xy+3xz-6yz$ by $x-2y-3z$.

5. Divide $x^4+2x^2y^3+9y^4$ by x^2+3y^3-2xy.

6. What is the coefficient of x^2 in $2x^2y^2$? How will you write the fifth power of a ? Multiply x^{20} by x^{15}. Divide x^{200} by x^{100}.

CHAPTER VIII

SIMPLE EQUATIONS

45. Definitions.

(a) An **equation** is a statement that two algebraical expressions are equal. For example,

(i) $3(x-2)=3x-6$ (ii) $3x-1=5$ are equations.

The parts of an equation separated by the sign of equality are called *members* or *sides* of the equation. The first part is called the *left hand member* or *left hand side* and the second, *right hand member* or *right hand side*.

(b) If the two members of an equation are *always* equal, that is, equal for all values of the letters involved, the equation is called an *identical equation* or simply an *identity*. Thus equation (i) above is an identity, for, the relation is true for *any* value of x. This can also be seen by opening the bracket on the left hand side and noting that the two sides are exactly **the same.**

If the two members of an equation are only equal for one or more *particular* values of the letters involved, it is called an *equation of condition*, or simply *an equation*. Thus relation (ii) of the last article, *viz.*, $3x-1=5$ is an *equation* in the ordinary sense of the term, which is true only when $x=2$. The value 2 is said to *satisfy* the equation.

(c) In any equation, the letter whose value it is required to find, is called the *unknown quantity*. The process of finding its value is called *solving the equation* and the value so found is called the *root* or *solution* of the equation.

(d) An equation which, in its simplest form, contains no power of the unknown quantity higher than the first, is called a *simple equation*. The unknown quantity is generally denoted by x.

46. The solution of simple equations depends only on the following axioms :—

(i) If equals be added to equals, the sums are equal.

(ii) If equals be taken from equals, the remainders are equal.

(iii) If equals be multiplied by equals, the products are equal.

(iv) If equals be divided by equals, the quotients are equal.

The first two axioms, however, may be replaced by a single rule, called the **Rule of Transposition**, explained below :—

Suppose $x+a=b$ ....(i)

Subtracting a from both sides we get .—
$$x+a-a=b-a$$
or $\qquad x=b-a$ ....(ii)

Similarly, from the relation
$$x-a=b$$ ....(iii)

we easily get the relation
$$x=b+a$$ ....(iv)

[by adding a to both sides]

Relation (ii) can be obtained from relation (i) by taking the term $+a$ to the other side and changing it into $-a$.

Relation (iv) can be obtained from relation (iii) by taking the term $-a$ to the other side and changing it into $+a$.

Thus we have the following important

Rule. *Any term may be transposed from one side of an equation to the other by simply writing it down on the opposite side with its sign changed.*

47. The following two rules, which are corollaries of axioms (*iii*) and (*iv*) of Art. 46, are also extremely useful :—

Rule 1. *We may change the sign of every term in an equation.* For, this is equivalent to multiplying both sides by -1.

Rule 2. *If each side of an equation is expressed in the form of a single fraction, we may transfer any factor of the numerator of one side to the denominator of the other side, and any factor of the denominator of one side to the numerator of the other side.*

For example, from the equation

$$\frac{ax}{b} = \frac{c}{d},$$

we have, by transferring b to the other side,

$$ax = \frac{bc}{d},$$

which, in fact, amounts to multiplying both side by b.

[Axiom (*iii*)]

Again, in the last result, we may transfer a to the other side and get :—

$$x = \frac{bc}{ad},$$

which amounts to dividing both sides by a.   [Axiom (*iv*)]

Caution. Beginners are apt to confuse the Rule of Transposition with Rule 2 of this article. For example, in the above process, they might reason thus :   "While, taking b to the other side of the equation why should not its sign be changed ?" They forget that' they are not *transposing* a "*term*" *of one side to the other*, but *transferring a factor of its denominator to the numerator of the other side*, which is equivalent to multiplying both sides by b

48. **Rules for solving a simple equation.**

We may now solve any simple equation by employing the rules explained in the last two articles. For the sake of convenience they are re-stated here briefly in one place :—

Rule 1. (*Rule of Transposition*). We may transpose any term from one side to the other by changing its sign.

Rule 2. We may multiply both sides by the same number.

Rule 3. We may divide both sides by the same number.

Rule 4. We may change signs of all the terms (on both sides).

Rule 5. We may transfer any factor of the numerator of one side to the denominator of the other side and any factor of the denominator of one side to the numerator of the other side.

- EXERCISE 25

Solve the following equations, giving reason for each step :—

1. $x+4=6$. [*Solved*]
2. $x+5=9$.
3. $x-5=3$.
4. $x-3=-7$.
5. $x-6=-6$.
6. $-x+1=4$. ; [*Hints*]
7. $-x-8=2$.
8. $-x-6=-6$.

9. $2x=6$. [*Solved*]
10. $7x=35$.
11. $5x=-15$.
12. $-8x=18$.
13. $-6x=-6$.
14. $4x=0$. [*Solved*]
15. $-5x=0$.
16. $-10x=0$.

17. $\dfrac{x}{3}=9$. [*Solved*]
18. $\dfrac{x}{4}=-8$.

19. $-\dfrac{x}{5}=\dfrac{2}{5}$.
20. $-\dfrac{x}{2}=-\dfrac{1}{4}$.

21. $-\dfrac{3x}{4}=\dfrac{9}{20}$. [*Solved*]
22. $\dfrac{4x}{3}=-\dfrac{2}{3}$.

23. $-\dfrac{5x}{6}-\dfrac{1}{3}=0$. [*Hint*]
24. $-\dfrac{7x}{8}+1=0$.

25. $\dfrac{2x}{3}+\dfrac{8}{9}=0$.
26. $-\dfrac{5x}{16}=0$.

27. $-3x+4=16$. [*Solved*]   28. $2x-6=4$.
29. $4x+41=21$.   30. $5x-8=-8$.
31. $4x-3=-6$.   32. $-8x+11=8$.

———

33. $2x-3=4x-5$. [*Solved*]   34. $3x+4=x-2$.
35. $4x-3=-2x+1$.   36. $-x-8=5x+1$.
37. $5x-2=2(x-2)$.   38. $3x-1=2(x-\frac{1}{2})$.

———

39. $\frac{1}{2}x-\frac{1}{4}x=2x-7$.   [*Solved*]

40. $\dfrac{x}{6}-\frac{1}{4}=\dfrac{x}{12}-\frac{1}{4}$.   41. $\dfrac{x}{3}-\frac{3}{2}=\dfrac{x}{5}+\frac{1}{2}$.

42. $\dfrac{x}{6}=1\frac{1}{4}+\dfrac{2x}{9}-\dfrac{x}{3}$.

———

SOLUTIONS & HINTS—EXERCISE 25

1. $x+4=6$

    or $x=6-4$ [transposing 4 to the other side]
    or $x=2$.

6.   Transpose 1 to the other side ; simplify right hand side ; then change signs of both sides.

9. $2x=6$

    or $\dfrac{2x}{2}=\dfrac{6}{2}$ [Dividing both sides by 2]

    or $x=3$.

Or thus :—

         $2x=6$

Transferring the factor 2 to the other side we have :—

        $x=\frac{6}{2}=3$.   [Rule 5 of Art. 48]

14. $4x=0$

   $\therefore$ $x=\frac{0}{4}$ [Transferring the factor 4]

        $=0$

Note.   Remember that 0 divided by *any finite number other than 0* is always equal to 0.

17. $$\frac{x}{3}=9$$

or $\quad\frac{x}{3}\times 3=9\times 3$ [Multiplying both sides by 3]

or $\quad x=27$

Or thus :—

$$\frac{x}{3}=9$$

or $\quad x=9\times 3$ [Transferring the factor 3]

$\quad\quad =27.$

21. $$-\frac{3x}{4}=\frac{9}{20}$$

or $\quad -3x=\frac{9}{20}\times 4=\frac{9}{5}$ [Transferring the factor 4]

or $\quad -x=\frac{9}{5\times 3}=\frac{3}{5}$ [Transferring the factor 3]

or $\quad x=-\frac{3}{5}.$ [Changing signs of both sides]

Or thus :—

$$-\frac{3x}{4}=\frac{9}{20}$$

$\therefore\quad -\frac{3x}{4}\times 20=\frac{9}{20}\times 20$ [Multiplying both sides by 20]

or $\quad -15x=9$

or $\quad x=\frac{9}{-15}=-\frac{9}{15}$ [Transferring the factor -15]

$\quad\quad =-\frac{3}{5}.$

The Method of Cross-multiplication. In the last solution, we have multiplied by 20, which is the L. C. M. of the denominators 4 and 20. This has cleared the equation of fraction. If we multiply by the product of 4 and 20 (*i.e.*, by 80) we get :—

$$-\frac{3x}{4}\times 80=\frac{9}{20}\times 80$$

or $\quad -3x\times 20=9\times 4.............(i)$

i.e., Numerator of L. H. S. × Denominator of R. H. S.
= Numerator of R. H. S. × Denominator of L. H. S.

The statement is generally true and is called the *Method of Cross Multiplication*.

From (i)　$-60x=36$

$\therefore \qquad\qquad x=-\tfrac{36}{60}=-\tfrac{3}{5}.$

23. First transpose $-\tfrac{1}{3}$ to the other side. Then apply Rule 5 or the Method of Cross-multiplication.

27.　　　$-3x+4=16$

or　　　　$-3x=16-4=12$　　　[Transposing 4]

or　　　'　$x=\tfrac{12}{-3}=-4.$　　　[Transferring the factor -3]

33.　　　$2x-3=4x-5$

or　　$2x-4x=-5+3$　　　[Transposing 3 and $4x$]

or　　　$-2x=-2$　　　[Simplifying]

or　　　　$x=\dfrac{-2}{-2}$　　　[Transferring the factor -2]

　　　　　$=1.$　 Ans.

39.　　$\tfrac{1}{2}x-\tfrac{1}{4}x=2x-7$

or　$2x-x=8x-28$　　　[Multiplying both sides by 4, the L. C. M. of 2 and 4]

or　$2x-x-8x=-28$　　　[Transposing $8x$]

or　　　$-7x=-28$　　　[Simplifying].

or　　　　$x=\dfrac{-28}{-7}=4$　　　[Transferring the factor -7]

49. After doing the above exercise, the following steps of procedure for solving a simple equation must be quite evident :—

(i) *If necessary, clear the equation of fractions and brackets.*

(ii) *Transpose all the terms which contain the unknown quantity to one side of the equation and the known quantities to the other side.*

(iii) *Collect the terms on each side.*

(iv) *Divide both sides by the coefficient of the unknown quantity (or say, transfer the coefficient of the unknown quantity to the other side) and the root required is obtained.*

EXERCISE 26

Solve the following equations and test your answer in each case :—

1. $2x-5(x-1)=x+1.$ [*Solved.*]
2. $4(x-3)-2x=x-9.$ 3. $4-3(x+3)=2x+7.$
4. $6-5(x-4)=2-(3x-14).$ 5. $x-2(x+1)-3(x+2)=8.$
6. $2x+3(x+3)+7(x+9)=0.$
7. $10+3(x-7)-5(x-5)=0.$
8. $x-2(x-8)+6(x-3)=3(x+2)+8.$
9. $2(x+9)-3(x+10)-4(x+11)+11=0.$
10. $3(x+10)+4(x+8)-5(x+6)=12.$

Solve the following equations :—

11. $x-[2x-\{6-(4-2x)\}]=\frac{5}{2}.$ [*Solved*].
12. $x+4-[2+\{x-(2+x)\}]=\frac{1}{2}.$
13. $-[2x-\{1-(1-2x)\}]-[1-\{1-(2x-1)\}]=0.$
14. $-[-\{-(8+1-2x)\}]+[-\{-(2x-7)\}]=0.$
15. $-10-[6x+\{3-(4x-x-6)-4\}-2x]=0.$
16. $1-2(3-4x)-[5\{x-(3+2x)\}]=0.$

Solve the following equations, testing your result in each case :—

17. $4x(x-2)-(2x-3)^2=1-x.$ [*Solved.*]
18. $(x-7)(x-8)-(x-6)^2+x=4.$
19. $(x+1)(2x+1)-2(x+1)^2-1=0.$
20. $4(x-4)(x+1)-(2x-5)^2+9=0.$
21. $5(x+4)(2x+1)-10x(x-1)=20.$
22. $3(x-5)(x-4)-(3x-1)(x-6)=14.$
23. $(x-1)(x-2)+(x-2)(x-3)-2(x+1)^2+18=0.$
24. $3(x+1)(x-1)-4(x-1)(x-2)=(x+3)^2-16.$

Solve the following equations and verify the result by substitution in each case :—

25. $\dfrac{x+4}{2}-\dfrac{x+6}{6}=\dfrac{5x}{6}.$ [*Solved.*]

26. $\dfrac{x-8}{3}-\dfrac{x-4}{2}=x-10.$ 27. $\dfrac{x+1}{5}-\dfrac{x+11}{10}+1=0.$

28. $3-\dfrac{x-3}{4}-\dfrac{x+3}{2}=0.$ 29. $11-\dfrac{x}{10}-\dfrac{x-10}{5}=x.$

30. $x + \dfrac{x+5}{5} - \dfrac{x+1}{4} + 4 = 0.$

31. $\dfrac{2x}{3} - \dfrac{3x+2}{2} - \dfrac{x+3}{6} + \dfrac{3}{2} = 0.$

32. $1 - x - \dfrac{x+1}{2} + \dfrac{x+2}{3} - \dfrac{5(x-1)}{6} = 0.$

33. $1 + x + \dfrac{x+4}{3} - \dfrac{x+5}{4} - \dfrac{3(x+1)}{8} = 0.$

34. $\frac{1}{2}(x-2) - \frac{2}{5}(x+1) + 2(x-1) = 0.$ [Hint.]

35. $-\frac{2}{3}(x-3) - \frac{3}{4}(x+1) - \frac{1}{6}(x+3) + 4 = 0.$

36. $\frac{5}{6}(x-4) - \frac{3}{8}(3x-4) - \dfrac{3x+4}{8} + 5 = 0.$

37   $7x - \frac{1}{14}(5x-11) = 1\frac{2}{7}(x-5) + 34.$

38   $3 + \dfrac{x}{2} = \frac{1}{2}\left(\cdot 4 - \dfrac{2x}{3}\right) - \frac{5}{6} + \frac{1}{3}(11-x).$

39   $\frac{2}{3}(x-4) + \dfrac{2+x}{2} = 7 - \dfrac{23-2x}{\cdot 5} - \dfrac{2x-1}{7}$

40   $\frac{1}{3}\left(\dfrac{3x}{4} - 3\right) + \dfrac{5x}{2} - \dfrac{15x}{4} = \dfrac{3(x-4)}{5} - (x+1)$

41   $5x - \left(15x - \dfrac{2x-1}{2}\right) - \frac{1}{6}(10x-57) + \frac{5}{8} = 0$ [Hint.]

42   $\dfrac{5x}{4} - (x+2) + 4\frac{3}{4} = 5x - \left(\cdot 1 + \dfrac{5x-2}{3}\right)$

43   $\dfrac{7x-1}{4} - \frac{1}{3}\left[2x - \dfrac{1-x}{2}\right] = 6\frac{1}{3}.$ [P.U   1927]

44. $\dfrac{5x}{3} - \dfrac{x-2}{4} = 2\frac{1}{4} - \frac{1}{2}\left(x - \dfrac{2x-1}{3}\right)$ [P.U   1937]

45. $\dfrac{7x+5}{6} - \dfrac{\frac{5}{12}x + \frac{1}{2}}{\frac{1}{3}} - \dfrac{\frac{4}{3} - \frac{5}{6}x}{2} = 0$ [Hint.]

46   $\dfrac{\frac{1}{12}x + \frac{1}{3}}{\frac{1}{4}} - \dfrac{x-4}{5} = 2 + \dfrac{x - \frac{1}{3}}{5}$

47   $\dfrac{x-3}{2} - \dfrac{1 - \frac{1}{7}x}{\frac{2}{3}} = \dfrac{x-1}{3} + 7 - x.$

48. $\dfrac{\frac{2}{3}x-\frac{1}{3}}{\frac{2}{3}}=5x-17\frac{1}{2}-\dfrac{\frac{1}{2}x+\frac{1}{4}}{\frac{1}{3}}.$

49. $\dfrac{\cdot 2x+\cdot 15}{\cdot 025}-\dfrac{\cdot 2x-\cdot 05}{\cdot 05}-15=0.$ [Hint.]

50. $x-\dfrac{x-\cdot 15}{\cdot 35}-\dfrac{1-\cdot 2x}{\cdot 07}=0.$

51. $\dfrac{x}{2}-\dfrac{\cdot 01x-1\cdot 5}{\cdot 12}=\dfrac{\cdot 05x+\cdot 76}{\cdot 06}.$

52. $\cdot 15x+\dfrac{\cdot 027x-\cdot 045}{\cdot 12}=\dfrac{\cdot 72}{\cdot 4}-\dfrac{\cdot 01x-\cdot 02}{\cdot 1}.$

SOLUTIONS & HINTS—EXERCISES 26

1. $2x-5(x-1)=x+1$

or $2x-5x+5\ =x+1$ [by opening the bracket]

or $2x-5x-x\ =1-5$ [by transposition]

or $-4x=-4$ [simplifying both sides]

or $x=\dfrac{-4}{-4}$ [dividing both sides by -4]

         $=1.$ **Ans.**

Test. If $x=1$,   L.H.S.$=2-5(1-1)=2-5\times 0=2-0=2$ and R.H.S.$=1+1=2$. Thus the two sides are equal ; therefore the answer is correct.

11. $x-[2x-\{\ 6-(4-2x)\ \}]=\frac{5}{2}.$

    L.H.S.$=x-[2x-\{\ 6-(4-2x)\ \}]$

         $=x-[2x-6+4-2x]$

         $=x-[-2]$

         $=x+2$

   $\therefore$ The equation takes the form —

         $x+2=\frac{5}{2}$

      $\therefore$ $x=\frac{5}{2}-2=\frac{1}{2}$. **Ans.**

17     $4x(x-2)-(2x-8)^2=1-x$

or $(4x^2-8x)-(4x^2-12x+9)=1-x$

                   [Performing multiplications]

or $4x^2-8x-4x^2+12x-9=1-x$ [Opening brackets]

or $4x^2-8x-4x^2+12x+x=1+9$ [by transposition]

or $\qquad 5x=10$ [simplifying both sides]

or $\qquad x=\frac{10}{5}$ [dividing both sides by 5]

$\qquad\qquad =2.$ **Ans.**

Test. If $x=2$, we have :—

L.H.S. $=8(2-2)-(4-3)^2=8\times0-(1)^2=0-1=-1$

and R.H.S. $=1-2=-1$

$\because$ The two sides are equal,

$\therefore$ The solution is correct.

Important Note. In the first step of the work we have found the product $(2x-3)^2$ or $(2x-3)(2x-3)$ but *the brackets have been retained* and have been opened in the next step. Students often commit a mistake here. They write down the product as found, changing the sign of the first term only. They forget that the minus sign is before the whole product. To avoid this mistake, they should form the habit of always writing down a product within a pair of brackets and opening the brackets in the next step.

25. $\qquad \dfrac{x+4}{2}-\dfrac{x+6}{3}=\dfrac{5x}{6}$

or $3(x+4)-2(x+6)=5x$

$\qquad$ [Multiplying both sides by 12, the L.C.M. of 2, 3 and 6]

or $3x+12-2x-12=5x$ [Opening brackets]

or $3x-2x-5x=-12+12$ [by transposition]

or $-4x=0$ [simplifying both sides]

or $\qquad x=\dfrac{0}{-4}$ [Dividing both sides by -4].

$\qquad\qquad =0.$ **Ans.**

Verification :—

$\qquad$ If $x=0$,

$\qquad$ L.H.S. $=\frac{4}{2}-\frac{6}{3}=2-2=0$.

and R.H.S. $=\frac{0}{6}=0$

$\because$ L.H.S. = R.H.S.

$\therefore$ The solution is correct.

Important Note. While multiplying by 6 in the first step of the work, *we have replaced the bars of the compound fractions by pairs of brackets.* This is a very important device to avoid mistakes in signs. A careless student will perform this multiplication thus :—

$$3x + 12 - 2x + 12 = 5x$$

forgetting that the whole numerator of the second fraction is to be multiplied by -2 so that $6 \times (-2)$ gives -12.

34. Multiply both sides by 6 (the L.C.M. of 2 and 3) we have :—

$$3(x-2) - 2(x+1) + 12(x-1) = 0.$$

The remaining process is quite easy.

Note. While multiplying by 6 a student might multiply the outer factor as well as the expression within bracket by 6. This is not an uncommon mistake and must be avoided carefully.

41. Opening the first bracket we shall have $-15x + \dfrac{2x-1}{2}$

The possibility of mistake here is to change the sign of -1.

It should be noted that $-\dfrac{2x-1}{2}$ is to be regarded a single

term (on account of the bar, which has the force of a pair of brackets) and the sign of this term has been changed from $-$ to $+$. Changing -1 will mean changing the term itself, which should not be done.

45. To simplify $\dfrac{\frac{5}{12}x + \frac{1}{2}}{\frac{1}{3}}$, multiply the numerator and de-

nominator by 12 (the L.C.M. of 12, 2 and 3). The fraction

then takes the form $\dfrac{5x+6}{4}$. Similarly multiply the numerator

and denominator of the next fraction by 6 (the L.C.M of 3 and 6). The students know that multiplying the numerator and denominator of a fraction by the same number does not change its value.

49. Change each decimal into a vulgar fraction and the equation takes the form of the previous type (Questions No. 45 to 48.)

CHAPTER IX

PROBLEMS LEADING TO SIMPLE EQUATIONS

50 Symbolical Expressions.

Simple questions in Arithmetic which can be done at once in a single step without any difficulty at all, are apt to puzzle a beginner if the figures are replaced by symbols. It is desirable to give the students some practice in the symbolical expressions arising out of such simple questions, or mere statements, before we proceed to problems.

Study the following statements and questions very carefully. They are given both in figures as well as symbols which will facilitate their understanding.

Ex. 1. (a) Number of annas in 3 rupees $= 3 \times 16 = 48$.

(b) Number of annas in x rupees $= x \times 16 = 16x$.

Ex. 2. (a) If 3 is taken from 5, what is left ?

Ans. $5 - 3 = 2$.

(b) If a is taken from b, what is left ? Ans. $b - a$.

Ex. 3. (a) What must be added to 4 to get 7 ?

Ans. $7 - 4 = 3$.

(b) What must be added to x to get y ?

Ans. $y - x$.

Ex. 4. (a) The difference of two numbers is 3 and the smaller number is 5; find the other

Ans. $5 + 3 = 8$.

(b) The difference of two numbers is x and the smaller number is y, find the other

Ans. $y + x$.

Ex. 5. (a) The three consecutive numbers, the first of which is 4, are 4, 5 and 6.

[*i.e.*, 4, 4 + 1 and 4 + 2.]

(b) The three consecutive numbers, the first of which is a, are a, $a + 1$ and $a + 2$.

Ex. 6. (a) Write down three consecutive even numbers, the middle one of which is 10.

> Ans. 8, 10, 12 [i.e., 10−2, 10, 10+2.]

(b). Write down three consecutive even numbers, the middle one of which is 2n.

> Ans. $2n-2$, $2n$, $2n+2$.

Ex. 7 (a) The cost of 6 books at 2 rupees each

> =Rs. 2×6 =Rs. 12.

(b) The cost of a books at b rupees each

> =Rs. $b \times a$ =Rs. ba or Rs. ab.

Ex. 8. (a) A man is 40 years old now. (i) How old was he 5 years ago ? (ii) How old will he be 10 years hence ?

> Ans. (i) 40 years− 5 years=35 years.
> (ii) 40 years+10 years=50 years

(b) A man is x years old now. (i) How old was he y years ago ? (ii) How old will he be z years hence ?

> Ans. (i) x years−y years=$(x-y)$ years.
> (ii) x years+z years=$(x+z)$ years.

Ex. 9. (a) Simple interest on Rs. 200 for 3 years at 5 p.c.

> =Rs. $\dfrac{200 \times 3 \times 5}{100}$ =Rs. 30.

(b) Simple interest on x rupees for 2 years at y p.c.

> =Rs. $\dfrac{x \times 2 \times y}{100}$ =Rs. $\dfrac{xy}{50}$.

Ex. 10. (a) Distance travelled by a man in 3 hours at 4 miles per hour=4 miles $\times 3$=12 miles.

(b) Distance travelled by a man in a hours at b miles per hour=b miles $\times a$=ab miles.

Ex. 11. (a) Two angles of a triangle are 40 degrees and 80 degrees respectively. Find the third angle.

> Sol. Sum of three angles=180 degrees.
> Sum of the two given angles=40+80
> =120 degrees.
> ∴ The third angle=180−120=60 degrees.

(b) Two angles of a triangle are x degrees and y degrees respectively. Find the third angle.

Sol. Sum of three angles = 180 degrees.

Sum of two angles = $(x+y)$ degrees.

$\therefore$ The third angle = $\{ 180-(x+y) \}$ degrees or $(180-x-y)$ degrees.

An Important Example :—

Ex. 12. (a) A number consists of two digits. The digit in the tens' place is 3, and that in the units' place is 5. Find the number.

Sol. In Arithmetic we have only to write down the digits in their proper places. Hence, clearly, the required number is 35.

(b) A number consists of two digits. The digit in the teh's place is x and that in the units' place is y. Find the number.

Sol. [A beginner is apt to write down the answer as xy. But this is quite wrong, for, whereas 35 does not mean 3×5, xy means $x \times y$.

The students should note that 35 is only a short way of writing 3 tens+5 units, *i.e.*, 3×10 units+5 units *i.e.*, $(3 \times 10+5)$ units.]

The given number $=x$ tens$+y$ units

$=x \times 10$ units$+y$ units

$=(10x+y)$ units, or simply $10x+y$.

EXERCISE 27

1. What must be subtracted from a, so that the remainder may be b ?

2. What must be added to x to make 100 ?

3. By what should x be multiplied to get y as product ?

4. What number divided by a gives b as quotient ?

5. In a division sum the divisor is 5, the quotient x and the remainder y, find the dividend.

6. What is the excess of 100 over x ? [*i.e.,* "By how much does 100 exceed x ?" or "By how much is 100 greater than x ?"]

7. What is the defect of $2a$ from $3b$? [*i.e.*, "**By how much is $2a$ less than $3b$?**"]

8. 50 is divided into two parts. If one part is k, what is the other ?

9. If x is one factor of y, what is the other ?

10. How many pies are there in x rupees ?

11. How many pence are there in a pounds and b shillings ?

_____

12. If x oranges cost one rupee, find the price of n oranges. [*Hint.*]

13. How many rupees will 100 books cost at x annas per book ?

14. x years hence a man will be 42 years old. What is his present age ?

15. m men can do a piece of work in 10 days. How many days will n men take to do it ?

16. How many hours will it take to walk k miles at the rate of 3 miles per hour ?

17. Find the speed of x miles per hour in feet per second. [*Hint.*]

18. A peon's salary is x rupees per mensem. In the month of August he spends y annas daily. Find his savings (in rupees) for that month.

19. In a class of 50 boys, x are Hindus, y Muslims, 10 Sikhs and the remaining Christians. How many Christians are there ?

20. Write down 4 consecutive numbers of which a is the least.

21. Write down three consecutive numbers of which k is greatest.

22. Write down three consecutive even numbers of which $2n$ is the largest.

23. Write down three consecutive odd numbers, the middle one being $2n-1$.

24. The cost price of an article is *a* rupees and selling price *b* rupees. Find gain %. [*Hint.*]

25. The cost price of an article is *x* rupees. For what price should it be sold to gain *y* per cent. ?

26. Find the amount of Rs. 500 in 2 years at *x* per cent per annum simple interest.

27. The area of the floor of a room is *x* yds. If its length be *y* feet, find its breadth in inches.

28. The side of a square room is *x* yds. How many feet of carpet *y* inches wide is required to cover its floor ?

29. The three angles of a quadrilateral are 30 degrees, *a* degrees and *b* degrees respectively. Find the fourth angle.

30. The vertical angle of an isosceles triangle is *x* degrees. What is each base angle equal to ?

31. A number consists of two digits. If the digit in the units' place be 5 and that in the tens' place be *a*, what is the number ?

32. The digits of a number beginning from the left are *x*, *y* and *z*. What is the number ?

HINTS—EX. 27

12. Cost of *x* oranges = Re. 1

$\therefore$,,   ,, 1 orange = Re. $\dfrac{1}{x}$ etc.

17. One hour = 3600 seconds.

x miles = $x \times 1760 \times 3$ ft.

$\therefore$ Distance covered in 3600 seconds = $x \times 1760 \times 3$ ft. etc.

24. Clearly, gain on Rs. a = Rs. $(b-a)$. Find gain on Rs. 100.

51. Problems.

Problems, which cannot be easily solved by arithmetical methods, are extremely simplified by the use of Algebra.

The method of procedure is as follows —

(i) *Denote the unknown* quantity by the letter x.*

(ii) *Express the other quantities of the problems also symbolically (i.e., in terms of x) as explained in the last article.*

(iii) *By the conditions of the question, write down the relation among these quantities in the form of an equation**.*

(iv) *Solve this equation and get the required value.*

Note. Students must carefully note the above four steps in the typical solutions of the next exercise. *Also, they must verify their answer in each case.*

EXERCISE 28

(*Verify your solution in each case*)

1. Find a number such that its third may be greater than its fifth by 16. [*Solved*]

2. A number exceeds its fifth part by 40 : find it.

3. Find a number such that its three-fourths may be less than its four-fifths by 8.

4. A number diminished by 12 is equal to its third part increased by 28. Find the number.

5. Find that number the double of which exceeds 72 as much as the number itself is below 78.

6. From a certain number 4 is taken, and the remainder is divided by 3 ; the quotient is then increased by 8 and divided by 5 and the result is 4 , find the number. [*Hint*]

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* In some cases it will be found more convenient to denote some ther quantity (not that whose value is required to be found) by x, after finding its value, to deduce the required value or values from See solution to Question No. 14 of the next exercise.

** Remember that an equation is a relation between *pure numbers*; therefore, when forming the equation, the denominations of the quantities (viz., rupees, feet, years, etc.,) must be omitted. Hence, also, the value of x obtained is a pure number, therefore the statements $x = 20$ ft. ; $x = 16$ years, etc., are wrong. Notice the absurdity if "20 ft." be substituted for x in the statement "Reqd. length $= x$ ft.", we get Reqd. length $= (20$ ft.$)$ ft.

7   A certain number is increased by 5 and then divided by 4 ; the quotient obtained is subtracted from 12 and the remainder multiplied by 6 ; the final result is 12 ; find the number.

---

8. The sum of two numbers is 74 and difference 12 ; find the numbers. [*Solved*]

9. The sum of two numbers is 100 and their difference is 56 ; find them.

10 Divide 200 into two parts so that half of one part may be less than the other part by 32.

11. Divide 225 into two parts such that three times one part may exceed seven times the other by 45.

12. Divide 120 into two parts such that the difference between the greater and 123 may be equal to twice the difference between the less and 76. [*Hint*]

13 The difference of two numbers is 5 and the difference of their squares is 125 ; find the numbers. [*Hint*]

---

14 Divide 81 into four parts such that if the first part be diminished by 5, the second increased by 1, the third multiplied by 2 and the fourth divided by 3, the result in each case may be the same. [*Solved*]

15 Divide 178 into 4 parts such that if the first be increased by 4, the second diminished by 6, the third multiplied by 3 and the fourth divided by 5, the result in each case is the same.

16. The sum of four numbers is 213 ; the first increased by 12 is the same as the second diminished by 8 or the third multiplied by 6 or the fourth divided by 3. Find the numbers.

---

17. Divide 158 into four parts such that the first may exceed the second by 10. the third by 12 and the fourth by 20.   [*Hint*]

18 Divide Rs. 140 among A. B, C and D so that A may get Rs 18 more than B, B Rs. 4 more than C, and C Rs 2 less than D.

19. A gentleman left Rs. 11000 to be divided among his four sons A, B, C and D; B was to have twice as much as A, C as much as A and B together, and D as much as B and C together. How much had each?

[*Hint*]

20. A sum of money is distributed among three persons A, B and C. A and B together get £108, A and C together £95, and B gets £22 less than twice C's share. Find the share of each. [*Hint*]

21. A sum of Rs. 10000 was divided among four persons, so that the first and second together received Rs. 5600, the first and third together Rs. 5200, and the first and fourth together Rs. 4400; find the share of each.

---

22. Find three consecutive numbers whose sum is 90.

[*Hint*]

23. Find five consecutive even numbers whose sum is 100.

[*Hint*]

24. Find six consecutive odd numbers whose sum is 180.

[*Hint*]

25. Find two consecutive numbers, the squares of which differ by 41.

26. Find two consecutive even numbers whose squares differ by 52.

27. Find two consecutive odd numbers whose squares differ by 80.

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28. A sum of Rs. 4 6as. is made up of 180 coins which are either annas or pies; how many are there of each? [*Hint.*]

29. A sum of Rs. 13 7as. is made up of 433 coins which are either two-anna pieces or pice; how many coins are there of each kind?

30. A sum of £8 10s. was paid in crowns, half crowns and shillings. The number of half crowns was four times the number of crowns and twice the number of shillings; how many were there of each?

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31. A person meeting a company of beggars gave four annas to each and had one rupee left : he found that he should have required twelve annas more to enable him to give the beggars six annas each ; how many beggars were there ? [*Hint*]

32. A labourer is engaged for 30 days on the condition that he receives 2*s*. 6*d*. for each day he works and loses 1*s*. for each day he is idle. He receives £2. 7*s*. in all. For how many days was he idle ? [*Hint.*]

33. A is twice as old as B ; 12 years ago he was five times as old ; what are their present ages ? [*Solved.*]

34. A father is four times as old as his son ; in 20 years he will only be twice as old ; find their ages.

35. The sum of the ages of A and B is 60 years, and 10 years hence A will be three times as old as B ; find their present ages.

36. A father's age is three times the sum of the ages of his two children, but 20 years hence his age will be equal to the sum of their ages ; find his present age. [*Hint.*]

37. A father's age is four times as much as the sum of the ages of his three children, but six years hence his age will be only double the sum of their ages ; find his present age.

38. A and B begin to play each with Rs. 120. If they play till A's money is doule B's, what does A win ? [*Hint.*]

39. Two persons A.and B engage at play ; A has Rs. 144 and B Rs. 104 when they begin ; at the end of the play A has three times as much money as B has ; how much did B lose ?

40. A and B have Rs. 160 between them. If A gives to B one-third of his sum, he will be left with one-third of what B will then have. How much have A and B separately ?

41. A has Rs. 500 and B Rs. 240 ; A spends twice as much as B does ; then B gives Rs. 45 to A ; now A's money is three times B's money ; find the sum spent by each.

42. A number consists of two digits, the sum of which is 10. If 18 be added to the number, the digits are reversed. Find the number. [*Solved.*]

43. A number consists of two digits, the sum of which is 11. If 45 be taken from the number, the digits change their places ; find the number.

44. A number of two digits is subtracted from the number formed by reversing the digits, and the remainder is 18. If the digits differ by 2, find the number.

45. A number consists of three digits which decrease by 1 from left to right. The quotient of the number when divided by the sum of the digits is 48. Find the number.

46. A number consists of two digits, the digit in the units' place being greater than that in the tens' place by 3. If another digit be annexed to its left equal to the digit in the tens' place, the new number of three digits would be 9 times the original number. Find the number.

---

47. A person had Rs. 2000, part of which he lent at 4 per cent and the rest at 5 p.c. ; the whole annual interest received was Rs. 88. How much was lent at 4 p.c. ? [*Solved.*]

48. A sum of Rs. 1000 is lent for 3 years, partly at 4 p.c. and partly at 6 p.c. simple interest ; the total interest received is Rs. 156 ; find the part lent at 6 p.c.

49. I lent a sum of Rs. 2025, partly to A for 2 years at 6 p.c. simple interest, and partly to B for 3 years at 8 p.c. simple interest. If the interest received from A be greater than that received from B by Rs. 18, find the sum lent to each.

50. A person spent Rs. 564 in buying tables and chairs. If each table cost Rs. 7 and each chair Rs. 3, and the

total number of articles bought was 108, how many of each did he buy ?

51. A person bought two cars for Rs. 9000. He sold one at again of 10% and the other at a loss of 8%. If he gained Rs. 216 on the whole, find the cost price of each car.

52. A person bought 102 eggs, partly at 3 for an anna and partly at 2 for an anna ; by selling all at the rate 8 annas per dozen, he gained Rs. 1. 10as. ; find the number of eggs bought in each lot.

53. By investing Rs. 4000 partly in 6 per cent stock at 120 and partly in 4 p.c. stock at 88 I get an annual income of Rs. 187, 8as. How much do I invest in each kind of stock ?

54. A man travelled a distance of 49 miles in 9 hours and 40 minutes, partly on foot at the rate of 3 miles per hour and partly on bicycle at the rate of 7 miles per hour. Find the distance travelled on foot.

55. 280 candidates appeared at an examination ; the pass percentage of boys was 70 and that of girls 35, whlie the general pass percentage was 60 ; find the total. number of girl candidates.

_____

56. How many pounds of tea at 2s. 3d. per lb. must be mixed with 12 lbs. of tea at 2s. 11d. per lb. so that on selling the mixture at 2s. 9d. per lb. there may be a gain of 10 % ?

_____

57. The length of a room is greater than its breadth by 8 feet. If the length be increased by 2 feet and the breadth decreased by 3 feet, the area decreases by 50 sq. ft. .Find the length and breadth.

58. The length of a room exceeds its breadth by 7 ft. If the length be increased by 5 ft. and breadth diminished by 3 ft., the area remains unaltered. Find the dimensions of the room.

_____

59. If a man walks at the rate of 3 miles per hour, he reaches his destination 10 minutes too late : if he walks at the rate of 4 miles per hour, he reaches 15 minutes too soon ; how far is his destination ? [*Hint.*]

60. Each base angle of an isosceles triangle is double the vertical angle ; find the vertical angle. [*Hint.*]

61. Three of the angles of a quadrilateral are proportional to the numbers 2, 3 and 4 and their sum is four times the fourth angle. Find all its angles.

62. ABC is a triangle ; AD is the bisector of angle A and AE the perpendicular to BC. Prove that angle DAE is equal to half the difference of angles B and C. [*Solved.*]

63. In a triangle ABC, the internal bisector of angle B and the external bisector of angle C meet in D. Prove that angle BDC is half of angle A.

64. ABCD is a cyclic quadrilateral ; angle A is twice angle B and four times angle C ; calculate the angles of the quadrilateral. [*Hint.*]

SOLUTIONS & HINTS—EXERCISE 28

1. *Step* (i) Let the required number be x.

Step (ii) Third part of the number $= \dfrac{x}{3}$.

Fifth part of the number $= \dfrac{x}{5}$.

Step (iii) By the condition of the question $\dfrac{x}{3}$ is greater than $\dfrac{x}{5}$ by 16.

Therefore, if we add 16 to $\dfrac{x}{5}$ it will become equal to $\dfrac{x}{3}$. Hence we have :—

$$\frac{x}{3} = \frac{x}{5} + 16.$$

Step (iv) Let us solve the above equation. We have :—

$$5x = 3x + 240 \quad \text{[Multiplying both sides by 15]}$$

or $5x - 3x = 240$ [By suitable transposition]

or $\qquad 2x = 240$ [Simplifying L.H.S.]

or $\qquad x = \dfrac{240}{2}$ [Dividing both sides by 2]

$$= 120$$

∴ The reqd. number = 120. **Ans.**

Verification. One-third of 120 is 40 and one-fifth is 24, and since 40 is greater than 24 by 16, therefore the solution is correct.

6.   Let the reqd. number be x.

Taking 4 from x we get the remainder $x - 4$.

Dividing the remainder by 3 we get the quotient $\dfrac{x-4}{3}$.

Increasing the quotient by 8 we get $\dfrac{x-4}{3} + 8$.

Dividing this result by 5 we get the quotient

$$\frac{1}{5}\left(\frac{x-4}{3} + 8 \right)$$

Put this result equal to 4 and solve the equation thus obtained.

8.   *Step* (i)   Let the smaller number be x.

   Step (ii)   The other number is greater than this by 12.   Therefore it is $x + 12$.
Also, sum of the two numbers $= x + (x + 12)$
$= 2x + 12$.

Step (iii)   By the condition of the question,
$$2x + 12 = 74.$$

Step (iv) $\qquad$ ∴ $\qquad 2x = 74 - 12$ [By transposition]
$$= 62$$

∴ $\qquad x = \dfrac{62}{2}$ [Dividing both sides by 2]

$$= 31.$$

∴   The smaller number$=31$
and greater number$=31+12=43$ } **Ans.**

Verification.   Sum of 43 and 31 is 74, and difference is 12, therefore the solution is correct.

12.   A bit of common sense has to be used in the question. Suppose the greater part is x. Now, what is the difference (Arithmetical difference, of course ; not Algebraical) between x and 128 ? Is it $x-128$ or $128-x$? Clearly, x is less than the whole (120) and therefore it must be less than 128. Hence this difference is $128-x$. Similarly, consider the other difference.

13.   If one number is x, the other must be $x+5$.

14.   *Step* (*i*)   Here it is more convenient to suppose the result in each case to be x. [See Foot Note, Art. 51].

Step (*ii*)   ∴ First   part$-5=x$, whence First part
$$=x+5.$$

Second part$+1=x$, whence Second part
$$=x-1.$$

Third   part$\times 2=x$, whence Third part
$$=\frac{x}{2}.$$

Fourth part$\div 3=x$, whence Fourth part
$$=3x.$$

Step (*iii*)   The sum of the four parts should be equal to 81.

∴ $(x+5)+(x-1)+\dfrac{x}{2}+3x=81.$

Step (*iv*)   or $2x+10+2x-2+x+6x=162$ [Multiply-
ing by 2]

or $2x+2x+x+6x=162-10+2$ [By trans-
position]

or $11x=154$ [Simplifying both sides]

or $x=14$

First   part $=14+5=19$

Second part $=14-1=13$

Third   part $=\dfrac{14}{2}=7$

Fourth part $=3\times14=42$ **Ans.**

[*Verify the solution*].

17. Let the first part be x.

Then, the second part $=x-10$ ($\because$ the second is less than the first by 10)

Similarly, the third part $=x-12$

and   the fourth part $=x-20$.

19. Suppose A got Rs. x

Then,   B ,, Rs. $2x$

$\therefore$ C ,, Rs. $(x+2x)=$ Rs. $3x$

$\therefore$ D ,, Rs. $(2x+3x)=$ Rs. $5x$

etc., etc.

20. Suppose A gets £x

$\therefore$ B ,, £$(108-x)$ [$\because$ A and B get 108]

and   C ,, £$(95-x)$ [$\because$ A and C ,, £95]

$\therefore$ twice C's share $=£2(95-x)$.

B gets £22 less than this. Hence, if we subtract £22 from it, the remainder must be equal to B's share. This gives us an equation.

22. Let the consecutive numbers be x, $x+1$ and $x+2$ [or, better, $x-1$, x and $x+1$. Attempt both ways.]

23. Let the consecutive even numbers be $2x$, $2x+2$, $2x+4$, $2x+6$, $2x+8$ [or, much better, $2x-4$, $2x-2$, $2x$, $2x+2$, $2x+4$], where x is an integer.

Note that each of the numbers supposed must be divisible by 2. Also each number must be greater than the previous one by 2.

24. Let the consecutive odd numbers be $2x+1$, $2x+3$, $2x+5$, $2x+7$, $2x+9$, $2x+11$, [or, much better, $2x-5$, $2x-3$, $2x-1$, $2x+1$, $2x+3$, $2x+5$], where x is an integer.

Note that each of the numbers supposed must leave a remainder 1 when divided by 2. Also each number must be greater than the previous one by 2.

25. Let the number of annas be x

Then, the number of pies $= 180 - x$.

Write down the value of x annas in pies; add this to $180 - x$ pies; the sum obtained is equal to the number of pies in Rs. 4, 6as.

31. Let the number of beggars be x.

∴ The sum given away to beggars $= 4as. \times x = 4x$ as. Sum left $= 16as$.

∴ The sum he had $= (4x + 16)$ annas.

Similarly, find the sum he had, from the other condition of the question.

Equate the two.

32. Suppose he was idle for x days;
therefore he worked for $(30 - x)$ days.
His wages for 1 working day $= 30d$.
∴ „ „ $(30 - x)$ „ $= 30(30 - x)d$ .... (i)
Fine for one day's absence $= 12 \, d$.
∴ „ „ x „ „ $= 12x \, d$. ......(ii)
Subtracting (ii) from (i) we get the sum he received.
Equate this to the given sum.

33. *Step* (i) Let B's present age be x years.

Step (ii) ∴ A's present age $= x \times 2$ or $2x$ years.

∴ A's age 12 years ago $= (2x - 12)$ years.
and B's „ „ „ $= (x - 12)$ years.

Step (iii) By the condition of the question $(2x - 12)$ is five times $(x - 12)$. Therefore if we multiply $x - 12$ by 5 it will become equal to $2x - 12$. Hence we have :—

$$2x - 12 = 5(x - 12)$$

Step (ii)　∴　$2x-12=5x-60$ [Opening the bracket]

　　　　or　$2x-5x=-60+12$　[By　suitable　trans-position]

　　　　or　　$-3x=-48$ [Simplifying both sides]

　　　　or　　　$x=\dfrac{-48}{-3}$ [Dividing both sides by -3]

　　　　　　　　　$=16$

∴　B's present age$=16$ years

and A's present age$=16\times2$ or 32 years. } **Ans.** [*Verify.*]

36.　Suppose the father's present age$=x$ years.

　　∴　The sum of the ages of the two children $=\dfrac{x}{3}$ years.

　　20 years hence, the father's age$=x+20$ years

and the sum of the ages of the **two** children

　　　　　　　　　　　　$=\left(\dfrac{x}{3}+40\right)$ years

etc. etc.　　　$\left[.\text{Not}\left(\dfrac{x}{3}+20\right)\text{years}\right]\left(\text{Why ? }\right)$

38.　Suppose A won x rupees.

　　∴　A's money at the end of the game$=$Rs. $(120+x)$

　　and B's　　,,　　　,,　　,,　　$=$Rs. $120-x)$

　　etc. etc.

42.　*Step*　(i)　Let the digit in the tens' place be x.

　　Step　(ii)　∴　The digit in the unit's place$=10-x$.

　　The value of the number$=x$ tens$+(10-x)$ units

{ For　fuller　explanation }　$=10x$ units$+(10-x)$ units

{　of the step, see Art. 50, }$=(10x+10-x)$ units

{　solved Example No. 12. }$=(9x+10)$ units, *i.e.*, $9x+10$.

When 18 is added to it, the result$=9x+10+18=9x+28$......(a)

Also, if the digits are reversed, that is, if x goes to the unit's place and $10-x$ to the ten's place, the value of the the number$=10(10-x)+x=100-10x+x=100-9x$ ......(b)

Step (iii) By the condition of the question, the results (a) and (b) are equal. Hence we have :—

$$9x + 28 = 100 - 9x.$$

Step (iv) ∴ $9x + 9x = 100 - 28$ [By suitable transposition]

or $\quad 18x = 72$ [Simplifying both sides]

or $\quad x = \dfrac{72}{18}$ [Dividing both sides by 18

$$= 4$$

∴ The digit in the ten's place $= 4$

and $\quad$,, $\quad$,, $\quad$ unit's $\quad$,, $\quad = 10 - 4 = 6$

∴ The reqd. number $\quad = 46$. **Ans.**

[*Verify the solution.*]

47. *Step* (i) Suppose the sum lent at 4 per cent $=$ Rs. x.

Step (ii) ∴ The sum lent at 5 per cent $=$ Rs. $(2000 - x)$

The annual interest received from the former

$$= \text{Rs.} \ \frac{x \times 4 \times 1}{100} = \text{Rs.} \ \frac{x}{25}.$$

The annual interest received from the latter

$$= \text{Rs.} \ \frac{(2000 - x) \times 5 \times 1}{100} = \text{Rs.} \ \frac{2000 - x}{20}$$

∴ Total annual interest received

$$= \text{Rs.} \ \left(\frac{x}{25} + \frac{2000 - x}{20} \right).$$

Step (iii) Clearly, by the condition of the question :—

$$\frac{x}{25} + \frac{2000 - x}{20} = 88.$$

Step (iv) ∴ $4x + 10000 - 5x = 8800$ [Multiplying by 100]

or $\quad 4x - 5x = 8800 - 10000$ [By transposition]

or $\quad -x = -1200$ [By simplifying both sides]

or $\quad x = 1200.$

∴ The sum lent at 4 per cent $=$ Rs. 1200. **Ans.**

[*Verify the solution.*]

59. Let the destination be at a distance of x miles.

Time taken to cover x miles at 3 m. p. h. $= \dfrac{x}{3}$ hours

Similarly   ,,   ,,   4 ,, $= 20x$ minutes
                                               $= 15x$ minutes

The proper time which should have been spent is found either by subtracting 10 minutes from $20x$ minutes or adding 15 minutes to $15x$ minutes. Equate these two values

60. Suppose, the vertical angle $= x$ degrees.

∴ Each base angle $= 2x$ degrees.

The sum of the three angles $= 180$ degrees

The equation is obvious.

62. Let
$\angle DAE = x$ degrees
$\angle B = m$,,
$\angle C = n$,,

[Note that m and n are known quantities]

In △ AED,
$\angle AED = 90$ degrees

∴ $\angle ADE = (90 - x)$ degrees

Similarly, $\angle BAE = (90 - m)$ degrees

Also, side CD of △ ADC is produced,

∴ $\angle DAC = \angle ADE - \angle C$
$= (90 - x - n)$ degrees

Now, ∵ AD is the bisector of $\angle BAC$,

∴ $\angle BAE + \angle EAD = \angle DAC$

∵ $(90 - m) + x = 90 - x - n$

or $x + x = 90 - n - 90 + m$ [By transposition]

or $2x = m - n$ [Simplifying both sides]

or $x = \dfrac{m - n}{2}$ [Dividing both sides by 2]

∴ $\angle DAE =$ half the difference of angles B and C.
[Q.E.D.]

64. Note that $\angle A + \angle C = 180$ degrees.
and $\angle B + \angle D = 180$ degrees.

CHAPTER X

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE WITH TWO UNKNOWN QUANTITIES

52. Given an equation in two unknown quantities, we can find as many pairs of values of those quantities satisfying the equation as we please

Take, for an example, the equation $x + 2y = 5$, or, which is the same thing, $x = 5 - 2y$

If we put $y = 0$, we get $x = 5$; therefore the values $x = 5$, $y = 0$ satisfy the equation.

If we put $y = 1$, we get $x = 3$, therefore the values $x = 3$, $y = 1$ also satisfy the equation.

Similarly, by giving *any* other value to y we can get the corresponding value of x, so that the pair of values thus determined satisfy the given equation

53. Now suppose we are given two equations in the *same* two unknown quantities. Then, the pairs of values of these quantities which satisfy both these equations are *limited in number*, and if the equations are of the first degree in those quantities, there is *only one pair of values* satisfying the two equations.

Consider the equations $x + 2y = 5$ and $7x - 3y = 1$

The only values of x and y which satisfy both these equations are $x = 1$, $y = 2$.

How to obtain such values is explained in the following articles.

This is an example of *simultaneous equations of the first degree with two unknown quantities.*

Similarly, we can have simultaneous equations of the first degree with more than two unknown quantities, which will be dealt with in the next chapter.

Finding the values of the unknown quantities satisfying all the given equations is known as *solving the equations.*

54. There are *four methods* for solving simultaneous equations of the first degree with two unknown quantities.

We shall illustrate them by solving the equations of the last article, viz. :—

$$x + 2y = 5 \qquad\qquad ..\ ....(i)$$
$$7x - 3y = 1 \qquad\qquad ....(ii)$$

Method 1.

From (i) we have $x = 5 - 2y$ [By transposing $2y$]

Substituting this value of x in (ii) we get :—

$$7(5 - 2y) - 3y = 1$$
or $$35 - 14y - 3y = 1$$
or $$-14y - 3y = 1 - 35$$
or $$-17y = -34$$
or $$y = 2$$

Substituting the value of y in [(i) we have :—

$$x + 2 \times 2 = 5$$
or $$x + 4 = 5$$
or $$x = 5 - 4 = 1$$
.. $$\left. \begin{array}{l} x = 1 \\ y = 2 \end{array} \right\} \quad \textbf{Ans.}$$

We may state the above method in words :—

Step (i) *From either equation express one of the unknown quantities in terms of the other.*

Step (ii) *Substitute this value in the other equation.*

Step (iii) *Solve the resulting simple equation. thus getting the value of the unknown quantity involved.*

Step (iv) *Substitute this value in either of the given equations and get the value of the other unknown quantity.*

Method 2

· From (i) $x = 5 - 2y$ ....(iii)

From (ii) $7x = 1 + 3y$

.. $$x = \frac{1 + 3y}{7} \qquad\qquad ..\ .(iv)$$

Equating the two values of x from (iii) and (iv) :—

$$5-2y=\frac{1+8y}{7}$$

or $\quad 35-14y=1+8y \qquad$ [Multiplying by 7]

or $-14y-8y=1-35$

or $\qquad -17y=-34$

or $\qquad\qquad y=2$

The value of x can now be found as in Method 1

This method, stated in words, is as follows :—

Step (i) *Express either of the unknown quantities in terms of the other from each equation.*

Step (ii) *Equate the two expressions thus obtained.*

Steps (iii) and (iv) as in Method 1.

Method 3.

Eq. (i) $\times 7$ gives $7x+14y=35$ $\qquad$ ....(3)

Eq. (ii) is $\qquad 7x-3y=1$ $\qquad$ ....(4)

Eq. (3) $-$Eq. (4) gives $17y=34$

$\qquad\qquad\quad \therefore \qquad y=2$

The value of x can be found, as before, by substituting this value of y in (i) or (ii).

Or thus :—

Eq. (i) $\times 3$ gives $3x+6y=15$ $\qquad$ ....(5)

Eq. (ii) $\times 2$,, $\quad 14x-6y=2$ $\qquad$ ....(6)

Eq. (5) $+$Eq. (6) gives $17x=17$

$\qquad\qquad \therefore \qquad x=1$

Substituting this value of x in (i) we get :—

$$1+2y=5$$

or $\qquad\qquad 2y=5-1=4$

or $\qquad\qquad y=2$

Observation :—In the first solution by this Method, we multiplied Eq. (i) by 7 so that the coefficients of x in the two equations may become equal (each$=7$). Then we subtracted

the two equations so that the terms in x may disappear and we may be left with only one unknown quantity whose value can be found by the methods of the last chapter.

In the second solution we tried to get rid of y [how ?] and proceeded similarly.

Hence Method 3 may be stated as follows :—

Step (i) *Multiply the equations by such numbers as will make the coefficients of one of the unknown quantities in the resulting equations numerically equal.*

Step (ii) *Add or subtract the resulting equations to get rid of of that unknown quantity.*

Step (iii) *and* (iv) *as before.*

Note. It depends upon the nature of the given equations as to which of the above three methods should be preferred. However, Method 3 is generally found most convenient. This point can be understood in detail by a careful study of the typical solutions of the next exercise.

Method 4 will be discussed in the next article.

EXERCISE 29

Solve the following simultaneous equations and verify your solution in each case :—

1. $4x-3y=5$
    $3x-6=0.$ [*Solved*]

2. $3x-4y=1$
    $4x-12=0.$

3. $6x+2y=8$
    $3y=3.$

4. $2x+3y-10=0$
    $\dfrac{y}{2}-1=0.$

5. $2x-3y+3=0$
    $\dfrac{2x}{3}-2=0.$

6. $x+4=0$
    $-2x+3y=14.$

7. $2x-y=7$
    $3x+y=3.$ [*Solved*]

8. $3x+y=10$
    $2x-y=10.$

9. $3x+y=-1$
    $-4x+y=6.$ [*Hint*]

10. $x+2y=3$
     $x+5y=0.$

11. $x-3y+19=0$
     $4y-x-24=0.$ [*Hint*]

12. $3x-y=18$
     $y-2x=-12.$
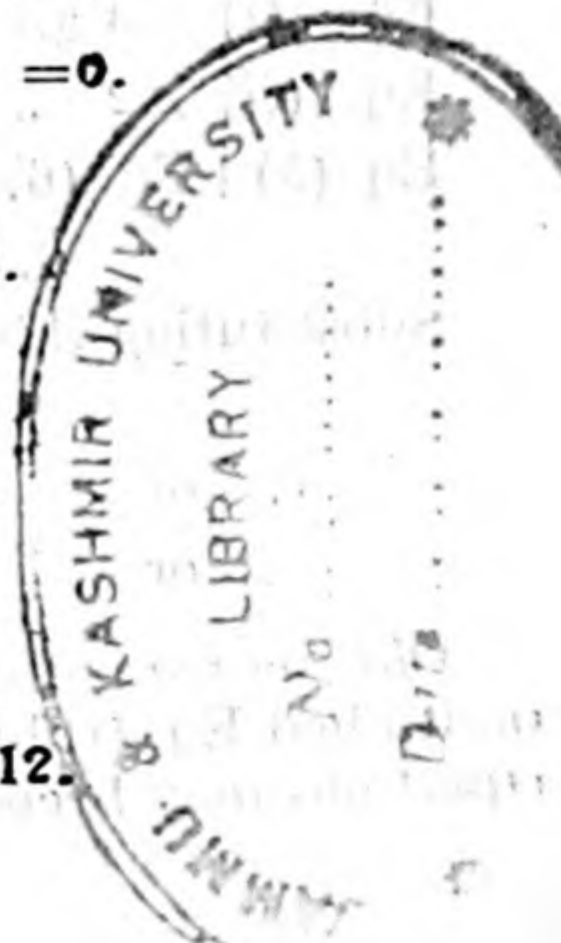
13     $3x-2y=20$
$3y+4x-4=0.$ [Solved]

14. $2x+3y=18$
$5x-3y-7=0.$

15     $6x-5y=7$
$5x-6y-4=0$

16. $3x-2y=19$
$7x+5y-23=0.$

17    $3y-7x+10=0$
$x-2y-3=0$

18. $4x+3y=22$
$5y-3x+2=0.$

19    $3x-2y=7$
$5y+4x=17.$

20. $2x+3y-4=0$
$5y+7x=25.$

21    $1+7y-5x=0$
$1+7x-9y=0.$

22. $7x-3y=x+3y=8$ [Hint]

23    $x+2y+10=7x-5y+13=0.$
24. $3x+y=2x-5y=17.$
25. $6x-10y+9=10x+14y-31=0.$
26. $4x-2y=2x+2y-1=7.$

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27    $43y-15x+2=0$
$25x-53y=22.$ [Hint]

28. $34x+13y=178$
$51x-12y=330.$

29. $33x-17y=32$
$44x-19y=50.$

30. $15y-16x+3=0$
$24x-17y-21=0$

31    $1+17x+18y=0$
$5+24y+19x=0.$

32. $21x-19y-8=0$
$23y-28x+20=0.$

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33. $7x+4y=11$
$5(x+1)=3(1-y).$ [Hint]

34. $4(x-y)=3x+y-7$
$5x+y=6x-1.$

35. $\dfrac{x-5}{3}=\dfrac{3y+11}{2}=\dfrac{x+4y-1}{3}.$ [Hint]

36. $3(x-3)+4=y+3=\dfrac{5x-3y}{3}$

37. $\dfrac{13x-5y}{2}=\dfrac{5(9x-y)}{6}=5(x-y).$

38. $9x-\frac{3}{4}(y-2)=5x-\frac{1}{3}(3-y)=6.$

39   $\frac{1}{4}(4x-3y)+5=6(x-3)$
   $\frac{1}{6}(5x-2y)+8=10(y-3).$

40. $\frac{1}{3}(x-y)-\frac{1}{3}(x+y)=\frac{1}{6}$
$\frac{3}{4}(x+y)+\frac{1}{3}(x-y)+\frac{79}{20}=0$

41   $\frac{1}{3}(3x+y)-\frac{1}{5}(2x+y+1)=0$
    $4-\frac{1}{3}(x-y)=2(x-4y).$

42. $\frac{1}{2}(7x-y)+5=x+\frac{3}{2}y$

$\frac{1}{3}(4x-3y)+\frac{7}{3}=6(5x-2y).$

43. $\frac{1}{3}(x+3y)-\frac{1}{2}(2x+5y)+2\frac{5}{2}=0$

$\frac{1}{5}(x+y)+\frac{1}{3}(y+5)\ \ -6\ =0.$

44. $2x-\frac{1}{3}(2y-1)=3\frac{5}{24}+\frac{1}{4}(3x-2y)$

$4y-\frac{1}{4}(5-2x)=6-\frac{1}{5}(3-2y).$

---

45. $\dfrac{3}{x}+2y=7$

$\dfrac{5}{x}-8y=6.$ [Solved]

46. $\dfrac{5}{x}+3y=8$

$\dfrac{4}{x}-10y=56$

47. $\dfrac{15}{x}-4y=1$

$\dfrac{6}{x}+7y=9.$

48. $2x-\dfrac{5}{y}=28$

$4x+\dfrac{3}{y}=-9$

49. $2x+\dfrac{3}{y}=2$

$5x+\dfrac{10}{y}-5\frac{5}{6}=0.$

50. $\dfrac{2}{3x}+11y=4\frac{1}{3}$

$\dfrac{1}{5x}+6y=2\frac{1}{5}.$ [Hint]

51. $\dfrac{1}{5x}+\dfrac{y}{9}=5$

$\dfrac{1}{3x}+\dfrac{y}{2}=14.$

52. $\dfrac{3x}{4}+\dfrac{4}{5y}-3\frac{1}{2}=0$

$\dfrac{4x}{5}+\dfrac{5}{6y}-1\frac{3}{30}=0.$

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53. $\dfrac{3}{5x}-\dfrac{5}{7y}=3\frac{2}{3}$

$\dfrac{2}{3x}+\dfrac{15}{4y}=3\frac{3}{8}.$ [Solved]

54. $\dfrac{3}{x}+\dfrac{2}{y}=1\frac{3}{8}$

$\dfrac{4}{x}+\dfrac{10}{y}=2$

55. $\dfrac{4}{x}-\dfrac{5}{y}=7$

$\dfrac{5}{x}+\dfrac{9}{y}+1\frac{3}{2}=0.$

56. $\dfrac{2}{x}+\dfrac{7}{2y}=4\frac{1}{2}$

$-\dfrac{1}{y}+\dfrac{1}{3x}=4.$

57 $\quad \dfrac{7}{y} - \dfrac{2}{z} = 29$ $\qquad$ 58. $\quad \dfrac{5}{2x} + \dfrac{8}{5y} = 18$

$\quad \dfrac{3}{4x} - \dfrac{1}{2y} + \dfrac{1}{4} = 0$ $\qquad\qquad \dfrac{8}{x} - \dfrac{3}{10y} = 14\frac{1}{2}.$

SOLUTIONS & HINTS—EXERCISE 29

1 $\qquad 4x - 3y = 5$ $\qquad\qquad$ ...... (i)

$\qquad 3x - 6 = 0$ $\qquad\qquad$ .....(ii)

From (ii) $3x = 6$ [By transposition]

$\qquad \therefore \quad x = 2.$

Substituting this value of x in (i) we have :--

$\qquad 8 - 3y = 5$

or $\qquad -3y = 5 - 8 = -3$

or $\qquad y = 1.$

$\qquad \therefore \quad \begin{matrix} x = 2 \\ y = 1 \end{matrix} \Big\}$ Ans.

Verification :—

When $x = 2$ and $y = 1$,

L.H.S. of Eq. (i) $= 4 \times 2 - 3 \times 1 = 8 - 3 = 5$, which is equal to R.H.S.

$\therefore$ Eq. (i) is satisfied.

Again, L.H.S. of Eq. (ii) $= 3 \times 2 - 6 = 6 - 6 = 0$, which is equal to R.H.S.

$\therefore$ Eq. (ii) is also satisfied.

Hence the solution is correct.

7. $\qquad 2x - y = 7$ $\qquad\qquad$ ...... (i)

$\qquad 3x + y = 8$ $\qquad\qquad$ ......(ii)

Adding (i) and (ii) we have :—

$\qquad 5x = 10$

$\qquad \therefore \quad x = 2.$

Substituting this value of x in Eq. (ii) we get :—

$\qquad 3 \times 2 + y = 8$

or $6+y=3$

or $y=3-6=-3$

$$\left.\begin{array}{l}x=2\\y=-3\end{array}\right\}\text{Ans.}$$

Verification :—

When $x=2$ and $y=-3$,

L.H.S. of Eq. $(i)=2\times2-(-3)=4+3=7$, which is equal to R.H.S.

∴ Eq. (i) is satisfied.

Also, L.H.S. of Eq. $(ii)=3\times2+(-3)=6-3=3$, which is equal to R.H.S.

∴ Eq. (ii) is also satisfied.

∴ The solution is correct.

9. Subtract the equations so that y may cancel out.

11. Arrange the second equation properly, that is, write it as :—

$$-x+4y-24=0.$$

13. $3x-2y=20$ ...... (i)

$4x+3y=4$ ......(ii) [After suitable arrangement]

Eq. $(i)\times3$ gives $9x-6y=60$ ......(iii)

Eq. $(ii)\times2$,, $8x+6y=8$ ......(iv)

Eq. $(iii)+$Eq. (iv) gives $17x=68$

∴ $x=4$

Substituting this value of x in (ii) we get :—

$16+3y=4$

∴ $3y=4-16=-12$

∴ $y=-4$

∴ $\left.\begin{array}{l}x=4\\y=-4\end{array}\right\}$ Ans.

Verification :—

When $x=4$ and $y=-4$,

L.H.S. of $(i)=12-2(-4)=12+8=20$

$=$R.H.S.

$\because$ Eq: (i) is satisfied. .

Also, L.H.S. of $(ii) = 16 - 12 = 4 =$ R.H.S.

$\therefore$ Eq. (ii) is also satisfied.

$\therefore$ The solution is correct.

22. The two given equations are :—

$$7x - 3y = 8$$
$$x + 3y = 8$$

27. The given equations are :—

$$-15x + 48y = -2 \qquad \ldots\ldots (i) \text{ [After suitable arrangement]}$$
$$25x - 53y = 22 \qquad \ldots\ldots (ii)$$

To get rid of y we have to multiply the equations by large numbers, *viz.*, 53 and 48 respectively. Therefore we try to get rid of x. For this purpose we need not multiply by 25 and 15. Take the L.C.M. of 15 and 25. It is 75. Multiply the equations by such numbers as to make 'the coefficient of x in each equation equal to 75 (disregarding the sign.) These numbers are evidently 5 and 3, etc., etc.

33. Simplify the second equation after removing brackets and put it in the form of the first equation.

35. We may take any one of the following pairs of equations :—

Or   *Or*

$$\left. \begin{array}{l} \dfrac{x-5}{3} = \dfrac{3y+11}{2} \\[2mm] \dfrac{x-5}{3} = \dfrac{x+4y-1}{3} \end{array} \right\} \quad \left. \begin{array}{l} \dfrac{x-5}{3} = \dfrac{3y+11}{2} \\[2mm] \dfrac{3y+11}{2} = \dfrac{x+4y-1}{3} \end{array} \right\} \quad \left. \begin{array}{l} \dfrac{x-5}{3} = \dfrac{x+4y-1}{3} \\[2mm] \dfrac{3y+11}{2} = \dfrac{x+4y-1}{3} \end{array} \right\}$$

Suppose we take the first pair.

Multiply the first Eq. by 6 to get rid of fractions, simplify, and arrange suitably.

Similarly deal with the second equation

Then solve as before.

45. $\dfrac{3}{x}+2y=7$ ....(i)

       $\dfrac{5}{x}-8y=6$ ....(ii)

[It is quite apparent that we can get rid of y more conveniently.]

Eq. (i) $\times$ 4 gives $\dfrac{12}{y}+8y=28$ ....(iii)

Eq. (ii) is $\dfrac{5}{x}-8y=6$ ....(iv)

By addition $\dfrac{17}{x}=34$

$\therefore$ $\dfrac{1}{x}=\frac{34}{17}$ [Dividing by 17]

          $=2$

$\therefore$ $x=\frac{1}{2}$ [Inverting both sides]

Or, after addition, we may proceed thus :—

By cross multiplication, $34x=17\times1$

$\therefore$ $x=\frac{17}{34}=\frac{1}{2}$

Substituting this value of x in (i) we get :—

     $\dfrac{3}{\frac{1}{2}}+2y=7$

or $6+2y=7$

or $2y=7-6=1$

or $y=\frac{1}{2}$

$\therefore$ $\left.\begin{array}{l}x=\frac{1}{2}\\y=\frac{1}{2}\end{array}\right\}$ Ans.

Verification :—

When $x=\frac{1}{2}$, $y=\frac{1}{2}$,

L. H. S. of (i) $=\dfrac{3}{\frac{1}{2}}+2\times\frac{1}{2}=6+1=7=$ R. H. S.

∴ Eq. (*i*) is satisfied

Also L. H. S. of (*ii*) $= \dfrac{5}{\frac{1}{2}} - 8 \times \frac{1}{2} = 10 - 4 = 6 = $ R H S

∴ Eq (*ii*) is also satisfied

∴ The solution is correct

50. Multiply Eq. (*i*) by 3 and Eq. (*ii*) by 5 to get rid of numerical fractions. Then proceed as before.

53. $\dfrac{3}{5x} - \dfrac{5}{7y} = \frac{2}{5}$ ....(*i*)

$\dfrac{2}{3x} + \dfrac{15}{4y} = 3\frac{5}{8}$ ....(*ii*)

Eq. (*i*) × 35 gives $\dfrac{21}{x} - \dfrac{25}{y} = 2$ ....(*iii*)

Eq. (*ii*) × 36 „ $\dfrac{24}{x} + \dfrac{135}{y} = 85$ ....(*iv*)

[L. C. M. of 25 and 135 is 675 ; To get 675 we have to multiply 25 by 27 and 135 by 5]

Eq (*iii*) × 27 gives $\dfrac{567}{x} - \dfrac{675}{y} = 54$ ....(*v*)

Eq (*iv*) × 5 „ $\dfrac{120}{x} + \dfrac{675}{y} = 175$ ....(*vi*)

Eq. (*v*) + Eq. (*vi*) gives $\dfrac{687}{x} = 229$

By cross multiplication $229x = 687$

∴ $x = \dfrac{687}{229} = 3$

Substituting this value of x in (*iii*) we get :—

$21 - \dfrac{25}{y} = 2$

or $7 - \dfrac{25}{y} = 2$

or $-\dfrac{25}{y} = 2 - 7 = -5$

or $\qquad \dfrac{25}{y} = 5 \qquad$ [Changing signs of both sides]

or $\qquad 5y = 25 \qquad$ [By cross multiplication]

or $\qquad y = 5$

$$\left.\begin{array}{l} x = 3 \\ y = 5 \end{array}\right\} \quad \text{Ans}$$

Verification —

When $x = 3$ and $y = 5$.

L H S of $(i) = \dfrac{3}{5 \times 3} - \dfrac{5}{7 \times 5} = \frac{1}{5} - \frac{1}{7} = \frac{2}{35} = $ R. H. S.

$\qquad$ Eq. (i) is satisfied

Also, L H S. of $(ii) = \dfrac{2}{3 \times 3} + \dfrac{15}{4 \times 5} = \frac{2}{9} + \frac{3}{4} = \frac{35}{36} = $ R. H. S.

$\therefore$ Eq. (ii) is also satisfied

$\therefore$ The solution is correct.

55. Method of Cross Multiplication.

This is the fourth method of solving the equations of this chapter. We shall explain it by solving the equations :—

$$ax + by + c = 0 \qquad \qquad .(i)$$
$$lx + my + n = 0 \qquad \qquad ..(ii)$$

Where the coefficients a, b, c, l, m, n stand for *known quantities*.

Set down these coefficients as in the scheme below :—

$$\begin{array}{cccc} (3) & (-11) & (2) & (3) \\ (1) & (0) & (8) & (1) \end{array}$$

[Note that in each line we have begun with the second coefficient, which is followed by the third. first. and second again. That is. the coefficients have been written down *in order beginning and ending with the 2nd* This can be easily memorised by the aid of the number 2312 There is no harm *if* the method is nick-named " **2312 Method** "]

Now multiply the coefficients *across* in the way indicated by the arrows, and subtract the product formed *in ascending* from the product formed *in descending*. We get :—

$$bn - \bar{c}m, \quad cl - an, \quad am - bl.$$

Put down these expressions as the denominators of *x*, *y* and 1 and equate the fractions thus obtained ; this gives :—

$$\frac{x}{bn-cm} = \frac{y}{cl-an} = \frac{1}{am-bl}$$

From these we easily get :—

$$\left.\begin{aligned} x &= \frac{bn-cm}{am-bl} \\[2mm] y &= \frac{cl-an}{am-bl} \end{aligned}\right\} \quad \textbf{Ans.}$$

[*Explanation of the last step.* $\because \dfrac{x}{bn-cm} = \dfrac{1}{am-bl}$

$\therefore$ by transferring $bn-cm$ to R. H. S., where it must appear in the numerator, we have $x = \dfrac{bn-cm}{am-bl}$.

Similarly, from $\dfrac{y}{cl-an} = \dfrac{1}{am-bl}$ we have $y = \dfrac{cl-an}{am-bl}$].

Proof :—

Multiplying equation (*i*) by *m* and (*ii*) by *b* we get :—

$$amx + bmy + cm = 0$$
$$blx + bmy + bn = 0$$

By subtraction,

$$amx - blx + cm - bn = 0$$

or $\qquad amx - blx = bn - cm$ [By transposition]

or $\qquad (am-bl)x = (bn-cm)$

$$\frac{x}{bn-cm} = \frac{1}{am-bl}$$ [By transferring $am-bl$ to R.H.S. and $bn-cm$ to L.H.S.

Similarly, $\dfrac{y}{cl-an}=\dfrac{1}{am-bl}$

$$\therefore \quad \dfrac{x}{bn-cm}=\dfrac{y}{cl-an}=\dfrac{1}{am-bl}.$$

56. We shall now illustrate the above method by solving some particular cases.

Example 1. Solve the equations :—

$$4x-5y=22$$
$$4y+3x=1$$

Solution :—

[We must put the equations in the same form as the equations of the last ariticle : that is, *R. H. S. of each should be 0 and like terms should be in the same column.* This step is very important and should be carefully understood.]

Re-arranging the terms we have :—

$$4x-5y-22=0 \qquad (-5) \qquad\qquad (-22) \qquad\quad (4) \qquad\quad (-5)$$

$$3x+4y-1=0 \qquad (4) \qquad\qquad (1-1) \qquad\quad (3) \qquad\quad (4)$$

By cross multiplication.

$$\dfrac{x}{(-5)(-1)-(-22)(4)}=\dfrac{y}{(-22)(3)-(4)(-1)}=\dfrac{1}{(4)(4)-(-5)(3)}$$

or $\dfrac{x}{5+88}=\dfrac{y}{-66+4}=\dfrac{1}{16+15}$

or $\dfrac{x}{93}=\dfrac{y}{-62}=\dfrac{1}{31}$

$$x=\tfrac{93}{31}=3$$
$$y=\tfrac{-62}{31}=-2 \Big\} \quad \textbf{Ans.}$$

Note. If the coefficients are put within brackets along *with their signs* in their arrangement for cross multiplication probable mistakes of signs in the products will be avoided

Example 2. Solve the equations :—

$$\dfrac{2}{x}+3y=11$$

$$\dfrac{1}{x}+\dfrac{y}{8}=0$$

Solution :—

Here we should temporarily regard $\dfrac{1}{x}$ as the first unknown quantity.

Re-arranging the first equation and multiplying the second by 8 we get :—

$$\frac{2}{x}+3y-11=0 \qquad (2) \qquad (-45) \qquad (24) \qquad (2)$$

$$\frac{8}{x}+y+0=0 \qquad (-8) \qquad (-42) \qquad (15) \qquad (-8)$$

We have arranged the coefficients of $\dfrac{1}{x}$ and y.

By cross multiplication,

$$\frac{\frac{1}{x}}{(3)(0)-(-11)(1)}=\frac{y}{(-11)(8)-(2)(0)}=\frac{1}{(2)(1)-(3)(8)}$$

or
$$\frac{\frac{1}{x}}{0+11}=-\frac{y}{-88-0}=\frac{1}{2-24}$$

or
$$\frac{\frac{1}{x}}{11}=\frac{y}{-88}=\frac{1}{-22}$$

$$\therefore \quad \frac{1}{x}=\frac{11}{-22}=-\tfrac{1}{2}, \text{ whence } x=-2$$

and
$$y=\frac{-88}{-22}=4$$

} **Ans.**

[Note that the constant term in the second equation is missing, therefore we take it equal to 0].

Example 3. Solve the equations :—

$$\frac{4}{x}+\frac{1}{3y}=7\tfrac{1}{2} \qquad\qquad \dots\dots(i)$$

$$\frac{3}{2x}-\frac{4}{5y}=4\tfrac{1}{2} \qquad\qquad \dots\dots(ii)$$

Solution :—

Multiplying (*i*) by 6 and (*ii*) by 10 we have, after suitable re-arrangement :—[This step is to make the coefficients integral]

$$\frac{24}{x}+\frac{2}{y}-45=0$$

$$\frac{15}{x}-\frac{8}{y}-42=0$$

We have arranged the coefficients of $\frac{1}{x}$ and $\frac{1}{y}$.

∴ **By** cross multiplication,

$$\frac{\frac{1}{x}}{(2)(-42)-(-45)(-8)}=\frac{\frac{1}{y}}{(-45)(15)-(24)(-42)}$$
$$=\frac{1}{(24)(-8)-(2)(15)}$$

or $$\frac{\frac{1}{x}}{-84-360}=\frac{\frac{1}{y}}{-675+1008}=\frac{1}{-192-80}$$

or $$\frac{\frac{1}{x}}{-444}=\frac{\frac{1}{y}}{333}=\frac{1}{-222}$$

∴ $$\frac{1}{x}=\frac{-444}{-222}=+2, \text{ whence } x=\tfrac{1}{2}$$

·and $$\frac{1}{y}=\frac{333}{-222}=-\tfrac{3}{2}, \text{ whence } y=\tfrac{2}{3}$$ **Ans.**

57. In addition to questions on cross multiplication, the next exercise contains some questions on easy miscellaneous types, wherein the students won't feel any difficulty, because necessary hints and solutions have been provided with

EXERCISE 30

Solve by the method of *cross multiplication* :—

1. $2x+3y=8$
  $5y-4x=6.$ [*Hint*]

2. $5x+2y=8$
  $7x+8y=6.$

3. $3x+8y+12=0$
  $3y+7x \quad =19$

4. $5x+6y=13$
  $7y=6-4x.$

5. $13-5y=3x$
  $15-9y=4x.$

6. $20-3y=2x$
  $30-4y=3x.$

7.—26. as in Ex. 29 (same Nos.)

27.—40. as in Ex. 29 (Nos. 45—58).

Solve the following equations :—

41. $\frac{1}{3}x+4y=1\frac{5}{2}xy$
  $\frac{4}{5}x-\frac{3}{2}y+2\frac{2}{3}1xy=0.$ [*Hint*]

42. $5x+4y=41xy$
  $2x-3y+2xy=0.$

43. $4x+3y-5xy=0$
  $\frac{y}{2}-\frac{2x}{3}=1\frac{3}{6}xy.$

44. $\frac{y}{3}+\frac{x}{4}=0$
  $3y+2x=xy.$

45. $\frac{3y}{2}+10x=\frac{xy}{2}=\frac{9y}{2}-2x.$

46. $2y+x=0=7y-x-36xy.$

47. $\frac{6xy}{3x+4y}=-\frac{6}{8},\quad \frac{5xy}{4x-3y}=\frac{1}{2}.$ [*Hint*]

48. $\frac{5xy}{2x-6y}=1,\quad \frac{2xy}{8x-3y}=-2.$

————

49. $\frac{4}{x-\frac{1}{2}}+\frac{5}{y-\frac{2}{3}}=41$
  $-\frac{3}{x-\frac{1}{2}}+\frac{2}{y-\frac{2}{3}}=-2$ [*Solved*]

50. $2(x+1)+\frac{3}{y+1}=4$
  $3(x+1)+\frac{2}{y+1}=5.$

51. $2(x-1)-\frac{3}{y-1}=3$
  $3(x-1)+\frac{15}{y-1}=-6.$

52. $\frac{3}{x+2}-2(y-2)=11$
  $\frac{5}{x+2}+14(y-2)=1.$

53. $\dfrac{y-2}{2}+\dfrac{2}{x+3}=1\frac{1}{4}$ 54. $3\left(\dfrac{1}{x+2}+\dfrac{1}{y+3}\right)=\frac{5}{2}$

  $\dfrac{y-2}{3}+\dfrac{3}{x+3}=1\frac{2}{3}.$ $3(x+2)+2(y+3)=2(x+2)(y+3).$

_____

55. $148x+231y=527$ 56. $31x+43y=117$
  $231x+148y=610.$ [Solved] $43x+31y=105.$

57 $57x-73y=25$ 58. $10\frac{3}{2}x+9\frac{9}{2}y=301$
  $73x-57y=105.$ $9\frac{9}{2}x+10\frac{3}{2}y=305.$

59. $83x-\dfrac{67}{y}=150=67x-\dfrac{83}{y}.$

60. $\dfrac{137}{x}-\dfrac{73}{y}=557$

  $\dfrac{73}{x}-\dfrac{137}{y}=493.$

_____

SOLUTIONS & HINTS—EXERCISE 30

1. The equations must be written as :—

$$2x+3y-8=0 \atop 4x-5y+6=0 \Big\}$$

41. Dividing each equation by xy we get :—

$$\dfrac{1}{3y}+\dfrac{4}{x}=1\frac{5}{2}$$
$$\dfrac{4}{5y}-\dfrac{3}{2x}+2\frac{1}{5}=0 \Bigg\}$$

and these equations are exactly the same as in **solved** Example 3 of Art. 56.

47. Inverting both sides of the first equation we get :—

$$\dfrac{3x+4y}{6xy}=-\frac{5}{6}$$

or $\dfrac{3x}{6xy}+\dfrac{4y}{6xy}=-\frac{5}{6}$

or $\dfrac{1}{2y}+\dfrac{2}{3x}=-\frac{5}{6}$

Similarly, we get another equation of this form from the second equation.

These two equations can be solved easily.

49. $\dfrac{4}{x-\frac{1}{2}}+\dfrac{5}{y-\frac{2}{5}}=41$ ....(i)

$-\dfrac{3}{x-\frac{1}{2}}+\dfrac{2}{y-\frac{2}{5}}=-2$ ....(ii)

Write m for $x-\frac{1}{2}$ and n for $y-\frac{2}{5}$; the equations become :—

$$\dfrac{4}{m}+\dfrac{5}{n}=41$$

$$-\dfrac{3}{m}+\dfrac{2}{n}=-2$$

Solving these equations we get $m=\frac{1}{4}$, $n=\frac{1}{5}$.

Substituting the values of m and n in these results we have :—

$$x-\tfrac{1}{2}=\tfrac{1}{4}$$
$$y-\tfrac{2}{5}=\tfrac{1}{5}$$

These equations give $x=\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$
and $y=\frac{1}{5}+\frac{2}{5}=\frac{3}{5}$ } **Ans.**

55. $148x+231y=527$ ....(i)
$231x+148y=610$ ....(ii)

Solution :—

[Equalising the coefficients of x or of y here is very inconvenient. But we note that the coefficient of x in (i) is the same as the coefficient of y in (ii) and *vice versa.* In such cases the artifice given here is very useful.]

Eq. (i) +Eq. (ii) gives $379x+379y=1137$

or $x+y=3$ ....(iii) [Dividing by 379]

Eq. (i) —Eq. (ii) gives $-83x+83y=-83$

or $x-y=1$ ....(iv) [Dividing by −83]

Eq. (iii)+Eq. (iv) gives $2x=4$

$\therefore$ $x=2.$

Substituting this value of x in (iii) we have :—

$$2 + y = 3$$
$$y = 3 - 2 = 1$$
$$\left. \begin{array}{l} x = 2 \\ y = 1 \end{array} \right\} \quad \textbf{Ans.}$$

CHAPTER XI

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE WITH MORE THAN TWO UNKNOWN QUANTITIES

58. We have already seen that one equation is enough to give us the value of one unknown quantity, but two equations are required if the number of unknown quantities be two. It must now be remembered that in general, the number of equations required is equal to the number of unknown quantities involved. Thus, we can find the values of five unknown quantities if we are given five equations involving those quantities.

59 The principle underlying the *solution of equations* o this chapter will be explained by supposing that we are given five equations involving five unknown quantities x, y, z, t, u.

Let the five equations*, denoted symbolically, be :—

$$\left. \begin{array}{l} E_1 \; (x, y, z, t, u) = 0 \\ E_2 \; (x, y, z, t, u) = 0 \\ E_3 \; (x, y, z, t, u) = 0 \\ E_4 \; (x, y, z, t, u) = 0 \\ E_5 \; (x, y, z, t, u) = 0 \end{array} \right\} \quad \text{Group 1}$$

*$E_1 \; (x, y, z, t, u) = 0$ means any equation involving x, y, z, t, u. It may be $2x - 3y + 6z + t - 10u + 15 = 0$, or any other such equation with any known coefficients. Some of these coefficients may even be 0, that is, some of the terms may be missing. The other four equations are also of a similar type.

Select any two equations from Group 1, equalise the co-efficients of any unknown quantity (say, u) and by addition or subtraction get rid of this quantity u. Thus we get an equation which involves x, y, z and t only. It may be denoted by $E_6 \; (x, y, z, t) = 0$.

Similarly, select three more pairs of equations from Group 1, all different from one another, and get similar equations in x, y, z, t. Thus we have :—

$$E_6 \ (x, y, z, t)=0$$
$$E_7 \ (x, y, z, t)=0$$
$$E_8 \ (x, y, z, t)=0$$
$$E_9 \ (x, y, z, t)=0$$

Group 2

Similarly, by eliminating another unknown quantity (say t) from Group 2 we can have :—

$$E_{10} \ (x, y, z)=0$$
$$E_{11} \ (x, y, z)=0$$
$$E_{12} \ (x, y, z)=0$$

Group 3

And, from Group 3, we get :—

$$E_{13} \ (x, y)=0$$
$$E_{14} \ (x, y)=0$$

Group 4.

And, from Group 4 :—

$$E_{15} \ (x)=0$$

This last equation contains only one unknown quantity, x, and can be easily solved.

Having obtained the value of x we substitute it in either of the equations of Group 4, and get the value of y.

Having thus obtained the values of x and y, we substitute them in any equation of Group 3 and get the value of z.

The substitution of the values of x, y and z in any equation of Group 2 gives us the value of t.

Last of all, we substitute these values of x, y, z and t in one of the equations of Group 1 and get the value of u.

Thus all the unknown quantities are completely determined.

Note 1. The unknown quantities can be eliminated *in any order*. For example, we may form Group 2 by eliminating t or z or y or x.

Note 2. In the original group (*i.e.*, Group 1) we may have one or more equations in which the same letter is missing. So much the better. These equations can be put

in Group 2 and this group completed by obtaining requisite number of similar equations from Group 1.

Note 3. Group 4 contains two equations in x and y which can be solved by the methods of the previous chapter: but here, too we have given one of the methods of that chapter, *viz.*, Method 3 of Art. 54. In fact, the whole method explained in this article is an extension of Method 3 of Art. 54

60. The particular case of the last article *when the number of unknown quantities is three* (beyond which the Matriculation candidates generally do not go although there is no such limitation set down in the University syllabus) is now quite easy to understand. To begin with we have :—

$$\left.\begin{array}{l} E_1\ (x,\ y,\ z)=0 \\ E_2\ (x,\ y,\ z)=0 \\ E_3\ (x,\ y,\ z)=0 \end{array}\right\} \quad \text{Group 1.}$$

If we eliminate one of the unknown quantities, say z, we have :—

$$\left.\begin{array}{l} E_4\ (x,\ y)=0 \\ E_5\ (x,\ y)=0 \end{array}\right\} \quad \text{Group 2.}$$

This group may now be solved by any method of the last chapter. Suppose we eliminate y, thus :—

$$E_6\ (x)=0.$$

This gives the value of x.

Substituting the value of x in any equation of Group 2, we get the value of y.

Substituting the values of x and y in any equation of Group 1, we get the value of z.

Thus the given equations are completely solved.

Note. Notes 1, 2 and 3 of the last article should be read again very carefully. We wish to illustrate Note 2 in a particular case.

Suppose the equations are :—

$$\left.\begin{array}{l} E_1\ (x,\ y,\ z)=0 \\ E_2\ (x,\ y,\ z)=0 \\ E_3\ (y,\ z)\ \ \ =0 \end{array}\right\} \quad \text{Group 1.}$$

That is to say, x is missing in the third equation.

So much the better. We can put the third equation, as it stands, in Group 2. The other equation for this group can be obtained by eliminating x between the first two equations.

Let us have another illustration. Suppose the equations are :—

$$\left.\begin{array}{l} E_1\ (x,\ y)=0 \\ E_2\ (y,\ z)=0 \\ E_3\ (z,\ x)=0 \end{array}\right\} \quad \text{Group 1.}$$

Here we may put any one of the given equations Group 2, and get the other equation for that group from remaining two equations. Thus we may have :—

Group 2 $\left\{\begin{array}{l} E_1\ (x,\ y)=0 \\ E_4\ (x,\ y)=0,\ \text{obtained by eliminating } z \text{ between} \\ \qquad\qquad\qquad \text{the second and third equations.} \end{array}\right.$

Or

Group 2 $\left\{\begin{array}{l} E_2\ (y,\ z)=0, \\ E_4\ (y,\ z)=0,\ \text{obtained by eliminating } x \text{ between} \\ \qquad\qquad\qquad \text{the first and third equations.} \end{array}\right.$

Or

Group 2 $\left\{\begin{array}{l} E_3\ (z,\ x)=0 \\ E_4\ (z,\ x)=0,\ \text{obtained by eliminating } y \text{ between} \\ \qquad\qquad\qquad \text{the first two equations.} \end{array}\right.$

61. With a little care and attention the students will be able to understand the theory explained in the last two articles. The method could have been explained by starting with easy particular cases involving three unknown quantities But the authors are of opinion that the discussion of pure theory must be introduced at this stage, especially for the sake of those students who intend joining College. This does not mean that the weaklings are going to be handicapped in any way. Typical questions of graded difficulty in the next exercise have been solved, and necessary hints provided, as usual. The students are, however, advised to consult these solutions or hints only when they fail in their independent attempts after the study of the above articles.

EXERCISE 31

Solve the following equations and verify your result in each case :—

1. $x+y+z=6$
    $2x+y-z=1$
    $3x-2y+z=2$. [Solved]

    $2x+y+2z=10$
    $3x-y-2z=0$
    $5x+y-3z=6$.

2. $x-y+z=2$
    $3x+y+z=12$
    $4x+3y-z=17$

4. $x+2y+3z=6$
    $x-3y+4z=2$
    $-x+4y-3z=0$.

-----

5. $3x-y+2z=1$
    $4x+3y-3z=14$
    $2x-4y+5z=-7$. [Solved]

6. $x+3y+2z=11$
    $2x+y+3z=14$
    $3x+2y-z=5$.

7. $2x+y-4z=-14$
    $7x+4y-3z=19$
    $5x-6y+4z=15$.

8. $5x-3y+2z=13$
    $4x-5y+z=6$
    $x+y+3z=12$.

9. $x+y+z=10$
    $2x+3y+4z=33$
    $3x-y+z=8$. [P.U. 1918]

10. $2x-y-z=-3$
    $2x+3y-5z=-7$
    $4x-5y+7z=21$.

11. $2x-y-3z=-13$
    $x-2y+z=0$
    $5z-3x-4y=0$. [Hint]

12. $3x+2y+4z=-1$
    $3z+x-3y=0$
    $10y+4x+z=0$.

13. $3x+2z-4y-17=0$
    $5x-z-3y=0$
    $8x+5y-3z+35=0$.

14. $7x-5y+9z=32$
    $8y-11z+11x=4$
    $15z-17x+2y=19$.

-----

15. $3x+2y+z=8$
    $4x+5y-10z=15$
    $6x+7y-9=0$. [Hint]

16. $6x-5y-3z=8$
    $3x-7y+14=0$
    $4x-5y-3=0$.

17. $3x-4y-5z-22=0$
    $4y-9z=5y-11z=1$.

18. $5x-6y-7z=8$
    $5z-3x+4=6z-5x+9=0$.

19. $3x+4y+3z=20$
    $7x-3y-15=0$
    $7z-4y+1=0$. [Hint]

20. $4x+3y-2z=5x-4y+8$
    $=4z-2=0$.

21.
$$5x-4y+13=0$$
$$7x-8z-37=0$$
$$5y+9z-17=0.$$

22.
$$3y+x-4=5z+2y-2$$
$$=3x-2z-3=0.$$

23.
$$x+\tfrac{1}{4}(y+z)=3\tfrac{3}{4}$$
$$y+\tfrac{1}{3}(x+z)=5$$
$$z+\tfrac{1}{2}(y+x)=6\tfrac{1}{2}. \quad [Hint]$$

24.
$$\tfrac{1}{6}(x-1)+\frac{y}{3}-\frac{z}{2}=0$$
$$\tfrac{1}{4}y=\tfrac{1}{2}(x-1-z)$$
$$\tfrac{1}{10}(z-1)=\frac{y}{2}-\frac{x}{5}.$$

25.
$$\frac{x}{2}-\frac{2y-8}{3}-z=\frac{2x}{3}-\frac{y+z+5}{6}=\frac{x-3y-4z}{6}+2=0.$$

26.
$$\frac{2x+3y}{38-4z}=\frac{3x-2y}{26-5z}=\frac{4x+6y}{21+3z}=1.$$

27.
$$\frac{2}{y}+\frac{15}{z}-\frac{4}{x}=2$$
$$\frac{13}{x}+\frac{4}{y}-\frac{3}{z}=14$$
$$\frac{9}{z}+\frac{9}{x}-\frac{6}{y}=9. \quad [Solved]$$

28.
$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=6$$
$$\frac{2}{x}-\frac{3}{y}+\frac{4}{z}=8$$
$$\frac{3}{x}-\frac{4}{y}+\frac{5}{z}=10.$$

29.
$$\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=2$$
$$\frac{3}{x}-\frac{4}{y}+\frac{5}{z}=18$$
$$\frac{2}{x}+\frac{3}{y}+\frac{4}{z}=38.$$

30.
$$\frac{2}{x}+\frac{3}{y}+\frac{4}{z}=9$$
$$\frac{3}{x}+\frac{5}{y}-\frac{6}{z}=2$$
$$\frac{7}{x}-\frac{9}{y}-\frac{11}{z}=9.$$

31.
$$\frac{4}{x}+\frac{3}{y}+\frac{2}{z}=1$$
$$\frac{8}{x}+\frac{1}{y}=\tfrac{5}{3}$$
$$\frac{6}{y}+\frac{1}{z}=-\tfrac{3}{2}.$$

32.
$$\frac{2}{x}+\frac{3}{y}=5$$
$$\frac{2}{z}-\frac{1}{y}=1$$
$$\frac{7}{x}-\frac{9}{z}=3.$$

33. $\begin{aligned} 4xy - 2x + y &= 0 \\ yz - y + z &= 0 \\ 14zx - 3z - 2x &= 0. \end{aligned}$ [*Hint*]

34. $$\frac{9}{10x} - \frac{1}{5y} = \frac{1}{2z} - \frac{3}{10x}$$
$$= \frac{7}{10y} + \frac{3}{4z} = \frac{3}{4}.$$

---

35. $5x + 4y + 3z + 2t = 3$

$\quad 2(3x + 2y) = 5z + 4t = 2$

$\quad\quad\quad\quad 3y + z = -1.$ [*Solved*]

36. $\begin{aligned} x + 2y + 3z + 4t &= 1 \\ 3x + 2y &= 6 \\ 4y + z &= 1 \\ 3z + t &= 2 \end{aligned}$

37. $\begin{aligned} x + 2y - z &= 6 \\ y + 3z - t &= -2 \\ z + 4t - x &= -4 \\ t + 5x - y &= 14. \end{aligned}$

38. Solve for a, b, c. d and e :—

$\quad\quad a + b + c + d + e = 15$

$\quad\quad\quad\quad\quad 2a + b = 4$

$\quad\quad\quad\quad\quad 3b - c = 3$

$\quad\quad\quad\quad\quad 4c + d = 16$

$\quad\quad\quad\quad\quad 3e - 15 = 0.$

39. Solve for p, q, r, s, t and u :—

$\quad 3p - q = 5q - r = q + s = 2t - q = 5$

$\quad\quad\quad 2s - 3t - 4r + 1 = 0.$

40. Solve for l, m, n, p, q :—

$\quad 2l + 3m = 3n + 4p = 6q + p = 6$

$\quad l + m + n = 3m + 4n - 3q = 5.$

SOLUTIONS & HINTS—EXERCISE 31

1. $\begin{aligned} x + y + z &= 6 \\ 2x + y - z &= 1 \\ 3x - 2y + z &= 2 \end{aligned}$ $\left.\begin{array}{l} \dots\dots(i) \\ \dots\dots(ii) \\ \dots\dots(iii) \end{array}\right\}$ Group 1.

[Group 2 can be most easily obtained by eliminating z, for the coefficients of this letter are 1 or -1 and elimination can be performed by simple addition or subtraction of the given equations without multiplying them by any numbers.]

Eq. (*i*) + Eq. (*ii*) gives $3x+2y=7$ ......(*iv*) |
Eq. (*ii*) + Eq. (*iii*) „ $5x-y=3$ ......(*v*) | Group 2

[Group 2 may now be solved by any method of the last chapter. We propose to solve it by eliminating *y*]

Eq. (*v*) × 2 gives $10x-2y=6$

Eq. (*iv*) is $3x+2y=7$.

By addition $13x = 13$

∴ $x = 1.$

Substituting this value of *x* in eq. (*v*) we get :—

$$5-y=3$$

or $$-y=3-5=-2$$

or $$y=2.$$

Substituting the values of *x* and *y* in eq. (*i*) we get :—

$$1+2+z=6$$

∴ $$z=6-2-1=3.$$

∴ $$x=1$$
$$y=2 \quad \text{Ans.}$$
$$z=3$$

Verification. When $x=1$, $y=2$, $z=3$, we have :—

L. H. S. of (*i*) $=1+2+3=6=$ R. H. S. of (*i*)
L. H. S. of (*ii*) $=2+2-3=1=$ R. H. S. of (*ii*)
L. H. S. of (*iii*) $=3-4+3=2=$ R. H. S. of (*iii*)

∴ the solution is correct.

5. $3x-y-2z=1$ ....(*i*)
$4x+3y-3z=14$ ....(*ii*) Group 1
$2x-4y+5z=-7$ ....(*iii*)

[Any one of the unknown quantities can be eliminated easily for the sake of obtaining Group 2. However, we propose to eliminate *y*, partly for a change and partly on observing that if we take the pairs (*i*), (*ii*) and (*i*), (*iii*) we shall have to alter only one equation in each case by multiplying it by an appropriate number.]

Eq. (*i*) × 3 gives $9x-3y+6z=3$

Eq. (*ii*)   is $4x+3y-3z=14$

By addition $13x \quad +3z=17$ ... .(*iv*)

Eq. (*i*) × 4 gives $12x-4y+8z=4$

Eq. (*iii*)   is $2x-4y+5z=-7$

By subtraction $10x \quad +3z=11$ ....(*v*)

Thus, we have $13x+3z=17$ } ....(*iv*)
$10x+3z=11$ } ....(*v*)   Group 2.

Now Eq (*iv*)$-$Eq. (*v*) gives $3x=6$

$$\therefore \qquad\qquad x=2.$$

Substituting this value of x in (*iv*) we have :—

$$26+8z=17$$

or $8z=17-26=-9$

or $z=-3$

Substituting the values of x and z in (*i*) we get :—

$$6-y-6=1$$

or $-y=1$

or $y=-1$

$$x=2,\ y=-1,\ z=-3. \quad \textbf{Ans.}$$

Verification.   When $x=2,\ y=-1,\ z=-3,$

L. H. S. of (*i*) $=6+1-6=1$ $=$R. H. S. of (*i*)

L. H. S. of (*ii*)$=8-3+9=14$ $=$R. H. S. of (*ii*)

L. H. S. of (*iii*)$=4+4-15=-7=$R. H. S. of (*iii*)

   .. The solution is correct.

11   Arrange the third equation properly, that is, write it
as $-3x-4y+5z=0$ or $3x+4y-5z=0$.

15   The third equation does not contain z, therefore it can
be put in Group 2. Another such equation can be
obtained by eliminating z between the first two
equations.

19   We may either put equation (*ii*) in Group 2 and
eliminate z between equations (*i*) and (*iii*) to get
another equation of that form, or put eq. (*iii*) in

Group 2 and eliminate x between equations (i) and (ii) to get another equation of that form.

23. Get rid of fractions and brackets and arrange the equations in the same form.

27. We notice that both sides of eq. (iii) can be divided by 3. Doing this and re-arranging suitably we have :—

$$-\frac{4}{x}+\frac{2}{y}+\frac{15}{z}=2 \qquad \ldots\ldots\ldots(i)$$

$$\frac{13}{x}+\frac{4}{y}-\frac{3}{z}=14 \qquad \ldots\ldots\ldots(ii) \qquad \Big\}\ \text{Group 1}$$

$$\frac{3}{x}-\frac{2}{y}+\frac{3}{z}=3 \qquad \ldots\ldots\ldots(iii)$$

{The letter that can be eliminated most easily is y.]

Eq. $(i) \times 2$ gives $\quad -\dfrac{8}{x}+\dfrac{4}{y}+\dfrac{30}{z}=4$

Eq. (ii) is $\quad \dfrac{13}{x}+\dfrac{4}{y}-\dfrac{3}{z}=14$

By subtraction $\quad -\dfrac{21}{x} \qquad +\dfrac{33}{z}=-10 \ \ldots(iv)$

Also, eq. (i)+eq. (iii) gives

$$-\frac{1}{x} \qquad +\frac{18}{z}=5 \qquad \ldots(v)$$

Group 2

Re-writing Group 2 :—

$$\frac{-21}{x}+\frac{33}{z}+10=0$$

$$\frac{-1}{x}+\frac{18}{z}-5=0$$

By cross multiplication :—

$$\cdot\quad\frac{\dfrac{1}{x}}{(33)(-5)-(18)(10)}=\frac{\dfrac{1}{z}}{(10)(-1)-(-21)(-5)}$$

$$=\frac{1}{(-21)(18)-(-1)(33)}$$

or $\quad \dfrac{\dfrac{1}{x}}{-165-180} = \dfrac{\dfrac{1}{z}}{-10-105} = \dfrac{1}{-378+33}$

or $\quad \dfrac{\dfrac{1}{x}}{-345} = \dfrac{\dfrac{1}{z}}{-115} = \dfrac{1}{-345}$

$\therefore \quad \dfrac{1}{x} = \dfrac{-345}{-345} = 1$, whence $x = 1$

and $\quad \dfrac{1}{z} = \dfrac{-115}{-345} = \tfrac{1}{3}$, whence $z = 3$.

Substituting these values of x and z in eq. (i), we get :—

$$-4 + \dfrac{2}{y} + \dfrac{15}{3} = 2$$

or $\quad \dfrac{2}{y} = 2 + 4 - 5 = 1$

or $\quad\quad y = 2$

$\therefore \quad\quad x = 1, \quad y = 2, \quad z = 3.$ **Ans.**

Test :—

When $x = 1$, $y = 2$, $z = 3$, we have :—

L. H. S. of $\quad(i) = \dfrac{-4}{1} + \dfrac{2}{2} + \dfrac{15}{3} = -4 + 1 + 5 = 2 = $ R.H.S. of $\quad(i)$

L. H. S. of $\quad(ii) = \dfrac{13}{1} + \dfrac{4}{2} - \dfrac{3}{3} = 13 + 2 - 1 = 14 = $ R.H.S. of $\quad(ii)$

L. H. S. of $\quad(iii) = \dfrac{3}{1} - \dfrac{2}{2} + \dfrac{3}{3} = 3 - 1 + 1 = 3 = $ R.H.S. of (iii)

$\therefore$ The solution is correct.

33. Dividing equation (i) by xy we get :—

$$4 - \dfrac{2}{y} + \dfrac{1}{x} = 0 \quad\text{or}\quad \dfrac{1}{x} - \dfrac{2}{y} = -4.$$

Similarly, divide eq. (ii) by yz and eq. (iii) by zx; we get three equations in $\dfrac{1}{x}$, $\dfrac{1}{y}$ and $\dfrac{1}{z}$.

35. $\quad\begin{aligned}5x + 4y + 3z + 2t &= 3 &\quad&\ldots\ldots\ldots(i)\\ 3x + 2y &= 1 &\quad&\ldots\ldots\ldots(ii)\\ 5z + 4t &= 3 &\quad&\ldots\ldots\ldots(iii)\\ 3y + z &= -1 &\quad&\ldots\ldots\ldots(iv)\end{aligned} \right\}$ Group 1

[**Note.** The second equation, *viz.*, $2(3x+2y)=2$ has been simplified by cancelling the common factor 2.]

Preliminary observations :—

There are four alternatives before us (1) to get Group 2 free from x ; in this case we can put equations (*iii*) and (*iv*) in that group, (2) to get Group 2 free from y, in which case we can put eq. (*iii*) in that group, etc. etc.

We choose the fourth alternative, *viz.*, getting Group 2 free from t for reasons which are quite obvious.

Solution :—

Eq. (*i*) $\times$ 2 gives $\quad 10x+8y+6z+4t=6$

Eq. (*iii*) $\qquad$ is $\qquad\qquad\qquad 5z+4t=2.$

By subtraction, $\quad 10x+8y+z=4 \qquad \dots\dots(v)$

Also, $\qquad\qquad\qquad 3x+2y \quad =1 \qquad \dots\dots(ii)$ Group 2

and $\qquad\qquad\qquad\qquad 3y+z=-1 \qquad \dots(iv)$

[We now note that eq. (*ii*) is free from z and the same letter can be easily eliminated between (*v*) and (*iv*), hence we obtain Group 3 free from z.]

Eq. (*v*) $-$ Eq. (*iv*) gives $10x+5y=5$

Also, Eq. (*ii*) $\qquad$ is $\qquad 3x+2y=1$ $\Big\}$ Group 3

Solving Group 3 as usual we have $x=1, \ y=-1.$

Substituting the value of y in Eq. (*iv*) [Group 2], we have .

$$-3+z=-1$$

or $\qquad\qquad . \quad z=-1+3=2.$

Substituting this value of z in Eq. (*iii*) [Group 1], we have :

$$10+4t=2$$

or $\qquad\qquad 4t=2-10=-8$

or $\qquad\qquad t=-2.$

$\therefore \quad x=1, \ y=-1, \ z=2, \ t=-2.$ **Ans.**

Verification of the result is left to the students as an exercise.

62. Some Special Methods and Devices can be used with advantage for the solutions of equations of the next exercise. They will be explained in the typical solutions.

EXERCISE 32

Solve th. 'ollowing equations :—

1.
$$4x-2y-5z=0$$
$$5x-8y+2z=0$$
$$7x-5y-3z=14.$$

2.
$$x-2y+z=0$$
$$6x-8y+3z=0$$
$$2x+3y+5z=33.$$

3.
$$5x-7y-4z=0$$
$$9x-12y-9z=0$$
$$3x-3y+5z=11.$$

4.
$$4x-13y+8z=0$$
$$6x-17y+10z=0$$
$$5x+7y-5z=36.$$

5.
$$6x-7y+2z=0$$
$$5z-3x-4y=0$$
$$7x+8y+9z=49.$$

6.
$$8x+7y+2z=0$$
$$2x-5y-4z=0$$
$$8x+4y-5z-10=0.$$

7.
$$5x-y-2z=0$$
$$3x-7y+6z=0$$
$$7x+5y-8z=8.$$

8.
$$\frac{x}{4}-\frac{y}{20}-\frac{z}{5}=0$$
$$10x+3y-3z=0$$
$$6x-4y-7z=1.$$

9.
$$\frac{3}{x}+\frac{2}{y}-\frac{3}{z}=0$$
$$\frac{5}{x}-\frac{6}{y}+\frac{2}{z}=0$$
$$2x+3y+4z=6. \quad [Hint]$$

10.
$$\frac{4}{x}-\frac{4}{y}+\frac{1}{z}=0$$
$$\frac{5}{x}-\frac{6}{y}+\frac{2}{z}=0$$
$$\frac{x}{6}+\frac{y}{4}+\frac{z}{3}=1.$$

11.
$$\frac{2}{x}+\frac{3}{y}+\frac{4}{z}=3$$
$$\frac{1}{x}-\frac{8}{y}+\frac{2}{x}=0$$
$$\frac{6}{x}-\frac{3}{y}-\frac{8}{z}=0.$$

12.
$$\frac{3}{x}-\frac{4}{y}+\frac{5}{z}=1$$
$$\frac{9}{x}+\frac{8}{y}-\frac{25}{z}=0$$
$$\frac{12}{x}+\frac{12}{y}-\frac{35}{z}=0.$$

13.
$$4x-3y=0$$
$$2x+3z=0$$
$$5x+y+2z=15. \quad [Hint]$$

14.
$$5x-2y=0$$
$$7y-5z=0$$
$$6x+5y-5z=4.$$

15.
$$5x+7y=0$$
$$3y-5z=0$$
$$3x+2y+4z+1=0.$$

16.
$$5x-9z=0$$
$$5y+7z=0$$
$$x+2y+3z-10=0.$$

17. $2x=3y=4z$
$x+2y-3z=5.$ [Hint]

19. $6x=-4y=3z$
$6x+5y+4z=13.$

18. $12x=20y=15z$
$2x-3y-2z+14=0.$

20. $12x=-9y=-8z$
$4x-3y+4z=1.$

———

21. $x+y=10$
$y+z=6$
$z+x=8.$ [Solved]

23. $x+y=2$
$y+z=-10$
$z+x=4.$

25. $2x+3y=\frac{3}{5}$
$3y+4z=\frac{5}{6}$
$2z+x=\frac{2}{3}.$ [Hint]

27. $\dfrac{1}{x}+\dfrac{1}{y}=\frac{1}{2}$

$\dfrac{1}{y}+\dfrac{1}{z}=\frac{1}{4}$

$\dfrac{1}{z}+\dfrac{1}{x}=\frac{1}{6}.$ [Hint]

29. $\dfrac{xy}{x+y}=\frac{2}{3}$

$\dfrac{xz}{x+z}=\frac{3}{4}$

$\dfrac{yz}{y+z}=\frac{6}{5}.$ [Hint]

31. $6(x+y)=xy$
$12(y+z)=-yz$
$4(z+x)=3zx.$ [Hint]

33. $x-y+z=4$
$y+z-x=0$
$x+y-z=6.$ [Hint]

22. $x+y=12$
$y+z=18$
$z+x=14.$

24. $12(x+y)=5$
$24(y+z)=7$
$8(z+x)=3.$

26. $3x+4y=\frac{1}{6}$
$4y+5z=-\frac{7}{12}$
$5z+3x=\frac{1}{4}.$

28. $\dfrac{1}{x}+\dfrac{1}{y}=\frac{7}{12}$

$\dfrac{1}{y}+\dfrac{1}{z}=\frac{9}{20}$

$\dfrac{1}{z}+\dfrac{1}{x}=\frac{3}{15}.$

30. $\dfrac{7xy}{10(x+y)}=1$

$\dfrac{11yz}{80(y+z)}=1$

$\dfrac{2zx}{3(z+x)}=1.$ [Hint]

32. $\frac{6}{5}(y+z)=yz$
$\frac{3}{4}(z+x)=zx$
$\frac{2}{3}(x+y)=xy.$

34. $x+y+z=12$
$x-y+z=4$
$x+y-z=0.$

———

35. $xy=12$
$yz=20$
$zx=15.$ [Solved]

36. $xy=4$
$yz=\frac{1}{2}$
$zx=2.$

37 $xy = \frac{1}{2}$

       $yz = -\frac{8}{3}$

       $zx = -\frac{8}{15}$.

38. $x(y+z) = 33$

       $y(z+x) = 40$

       $z(x+y) = 49$.   [*Hint*]

39. $x(y+z) = 27$

       $y(z+x) = 32$

       $z(x+y) = 35$.

40. $\dfrac{1}{x}\left(\dfrac{1}{y} + \dfrac{1}{z}\right) = 14$

       $\dfrac{1}{y}\left(\dfrac{1}{z} + \dfrac{1}{x}\right) = 18$

       $\dfrac{1}{z}\left(\dfrac{1}{x} + \dfrac{1}{y}\right) = 20$

SOLUTIONS & HINTS—EXERCISE 32

1. $4x - 2y - 5z = 0$ ...(i)

       $5x - 8y + 2z = 0$ ...(ii)

       $7x - 5y - 3z = 14$ ...(iii)

From equations (i) and (ii) we have by cross * multiplication :—

$$\frac{x}{(-2)(2)-(-5)(-8)} = \frac{y}{(-5)(5)-(4)(2)} = \frac{z}{(4)(-8)-(-2)(5)}$$

or $\dfrac{x}{-4-40} = \dfrac{y}{-25-8} = \dfrac{z}{-32+10}$

or $\dfrac{x}{-44} = \dfrac{y}{-33} = \dfrac{z}{-22}$

or $\dfrac{x}{4} = \dfrac{y}{3} = \dfrac{z}{2}$ [Multiplying each fraction by -11]

Let each of these fractions be equal to k.

Then **, $x = 4k$, $y = 3k$, $z = 2k$ ...(iv)

---

*   In these equations, instead of the usual constant terms we have terms containing z. The method is, however, applicable. Only instead of the form $x = y = 1$, we have $x = y = z$. This can be easily proved as in Article 55.

** $\dfrac{x}{4} = k$; Multiplying both sides by 4, we get $x = 4k$

Similarly $y = 3k$, $z = 2k$.

Substituting these values of x, y, z in Eq. (iii) we have :—

$$7(4k)-5(3k)-3(2k)=14$$

or

$$28k-15k-6k=14$$

or

$$7k=14$$

or

$$k=2.$$

Substituting this value of k in results (iv) we have :—

$$x=4\times 2=8,\quad y=3\times 2=6,\quad z=2\times 2=4.$$

$\therefore$ $x=8$, $y=6$, $z=4$.   **Ans.**

9.   Applying the method of cross multiplication to the first two equations we have :—

$$\frac{\frac{1}{x}}{2}=\frac{\frac{1}{y}}{3}=\frac{\frac{1}{z}}{4}=k \text{ (suppose)}$$

$\because$ $\dfrac{1}{x}=2k,\qquad \dfrac{1}{y}=3k,\qquad \dfrac{1}{z}=4k.$

or $x=\dfrac{1}{2k},\quad y=\dfrac{1}{3k},\quad z=\dfrac{1}{4k}$

Substitute these values of x, y, z in the third equation and get the value of k as before.

13.   The first two equations may be written

$$4x-3y+0\ =0$$
$$2x+0\ +3z=0$$

Cross-multiply and proceed as before.

17. $2x=3y=4z=k$ (suppose).

$\therefore$ $x=\dfrac{k}{2},\quad y=\dfrac{k}{3},\quad z=\dfrac{k}{4}$

Substitute these values in the third equation and get the value of k as before.

21. $x+y=10$ ...(i)

$y+z=6$ ...(ii)

$z+x=8$ ...(iii)

Adding these three equations we have :—

$$2x+2y+2z=24,$$

or $x+y+z=12$ ...(iv) [dividing both sides by 2]

Eq. (iv)−Eq. (ii) gives $x=12-6=6$

Eq. (iv)−Eq. (iii) ,, $y=12-8=4$ } Ans.

Eq. (iv)−Eq. (i) ,, $z=12-10=2$

25. If we add the three equations after multiplying the third equation by 2, we get :—

$$4x+6y+8z=\tfrac{3}{2}+\tfrac{5}{6}+\tfrac{4}{3}=\tfrac{22}{6}=\tfrac{11}{3}.$$

Divide both sides by 2 and then proceed as in Q. 21.

27. Proceed exactly as in Q. 21 getting the value of :—

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$ and then subtracting the given

equations from this result in turn.

29. Inverting both sides of the first equation we get

$$\frac{x+y}{xy}=\frac{3}{2}$$

or $$\frac{x}{xy}+\frac{y}{xy}=\frac{3}{2}$$

or $$\frac{1}{y}+\frac{1}{x}=\frac{3}{2}.$$

Get two other similar equations and proceed as before.

30. The first equation may be written as :—

$$\frac{xy}{x+y}=\frac{10}{7}$$ [transfering 7 and 10 to R. H. S.]

Rewrite the other equations similarly and proceed as in the last question.

31. The first equation may be written as :—

$$\frac{x+y}{xy}=\frac{1}{6}$$ [transferring xy to L.H.S. and 6 to R.H.S.]

Rewrite the other equations similarly and proceed as in the last two questions.

33. Adding the three given equations we get :—

$$x+y+z=10$$

From this equation subtract the given equations in turn.

35    $xy=12$ ...(i)

$yz=20$ ...(ii)

$zx=15$ ...(iii)

Multiplying the three equations we get :—

$$x^2y^2z^2=12\times20\times15=3600$$

Taking sq. root of both sides we have :—

$$xyz=60 \qquad ...(iv)$$

Dividing Eq. (iv) by Eq. (ii) we have :—

$$\frac{xyz}{yz}=\frac{60}{20}, \text{ or } x=3$$

Similarly, $y=4$ } **Ans.**

and $z=5$

Note. xyz can also be equal to -60, for the square of -60 is also $+3600$. If we take this value of xyz, we get

$$x=-3, \ y=-4, \ z=-5. \quad \textbf{Ans.}$$

Verify these values by substitution.

38. Add up the equations ; divide the resulting equation by 2 ; from this result subtract the given equations in turn. As a result we get equations similar to those of Q. 35.

CHAPTER XII

PROBLEMS LEADING TO SIMULTANEOUS EQUATIONS

63. In many problems, for the sake of convenience, we denote more than one unknown quantities by the letters x, y, z etc. In such cases we should be able to form as many independent equations from the statement of the question as the number of unknown quantities to be determined. [See Art. 58].

EXERCISE 33

[Verify your solution in each case]

1. The sum of two numbers is 54 and their difference is 12 ; find the numbers. [*Solved*]

2. The sum of two numbers is 100 and their difference is 56 ; find the numbers.

3. One third of the sum of two numbers is 28 and one-half of their difference is 8 ; find the numbers.

4. Half of the sum of two numbers is 60 and three times their difference is 54 ; find the numbers.

5. Find two numbers such that three times the first added to four times the second is equal to 93 and three times the second exceeds four times the first by 1.

6. Find two numbers such that the first with half the second may make 40, and also the second with a third of the first may make 40.

7. Find two numbers such that half the first with a third of the second may make 96, and that a fourth of the first with a fifth of the second may make 54.

8. A number is divided into two parts such that one-third of the smaller exceeds one-fifth of the greater by 6, and twice the smaller exceeds the greater by 75. Find the number and its parts.

♥ 9. Two numbers are in the ratio 4 : 5 ; if half of the first be added to one-third of the second, the result is 33 ; find the numbers. [*Hint*]

10. Two numbers are in the ratio $\frac{2}{3} : \frac{3}{4}$; if one-fourth of the first be subtracted from one-third of the second, the remainder is 6 ; find the numbers.

11. A number is divided into two parts in the ratio 5 : 7. If 9 times the larger part together with 13 times the smaller exceed 11 times the whole by 36, find the number and its parts.

♦ 12. The sum of two numbers is 27 and the difference of their squares is 81 ; find the numbers. [*Hint*]

13. The difference of two numbers is 5 and the difference of their squares is 135 ; find the numbers.

14. Find two numbers whose difference is 34, such that the larger divided by the smaller gives a quotient 3 and a remainder 2 [*Hint*]

15. A number is divided by another; the quotient obtained is 7 and the remainder is 3; if one-fifth of the first be divided by one-fourth of the second, the quotient obtained is 5 but the remainder is the same as before; find the numbers.

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16. The value of a fraction not in its lowest terms, is $\frac{2}{3}$; if 3 be added to the numerator the value becomes $\frac{7}{9}$; find the fraction. [*Hint*]

17. The value of a fraction, not in its lowest terms, is $\frac{4}{5}$; if 8 be subtracted from the numerator, the value becomes $\frac{2}{5}$; find the fraction.

18. The value of a fraction not in its lowest terms is $\frac{5}{6}$. If 12 be subtracted from the numerator and 8 added to the denominator, the value becomes $\frac{3}{8}$; find the fraction.

19. Two numbers are in the ratio 3 : 5. If 5 be added to each the ratio becomes 5 : 8. Find the numbers. [*Hint*]

20. Two numbers are in the ratio 7 : 5. If the first be increased by 4 and the second decreased by 4, the ratio becomes 5 : 3; find the numbers.

21. A fraction becomes equal to $\frac{3}{4}$ when 1 is added to its numerator and equal to $\frac{1}{2}$ when 10 added to its denominator; find the fraction.

22. The denominator of a fraction exceeds the numerator by 11. If the numerator is increased by 2, the fraction becomes equal to $\frac{2}{3}$. Find the fraction.

23. Find the fraction which is doubled when 2 is added to both the numerator and the denominator and trebled when 8 is added to each.

24. The denominator of a fraction exceeds the numerator by 5. If the numerator be increased by 2 and the denominator decreased by 2 the value of the fraction becomes $\frac{5}{6}$. Find the fraction.

25. If $\frac{1}{2}$ be added to the numerator of a certain fraction, the fraction is increased by $\frac{1}{30}$, and if   be subtracted from the denominator its value becomes $\frac{1}{2}$;   it.

26.   A certain number of two digits is five times the sum of its digits, and if 9 be added to it the digits will be interchanged ; find the number. [*Hint*]

27.   A certain number of two digits is four times the sum of its digits, and if 27 be added to it the digits will be reversed ; find the number.

28.   A certain number between 10 and 100 is 8 times the sum of its digits ; and if the sum of the digits be added to the number the result is 81 ; find the number.

29.   The sum of a number of two digits and of the number formed by reversing the digits is 165 and the difference of the digits is 1 ; find the number.

30.   A number of two digits is 12 times the difference of the digits and is less than the number formed by reversing the digits by 36 ; find the number.

31.   A number consists of three digits whose sum is 15. If the left-hand and middle digits be interchanged the number is increased by 90. If the left-hand digit be halved and the other two digits interchanged the number is diminished by 345. Find the number. [*Hint*]

32.   A number consists of three digits. The left-hand digit is greater than the middle one by 2. If 99 be subtracted from the number the extreme digits change their places and if the number be divided by the sum of the digits, the quotient is 44 and remainder 6. Find the number.

————

33.   Five tables and three chairs cost Rs. 84, and four tables and five chairs cost Rs. 88 ; find the cost of each article. [*Hint*]

34.   Three clocks and four watches cost Rs. 220, while five clocks and seven watches cost Rs. 875 ; find the cost of each article.

35.   Six horses and seven cows can be bought for £500, and thirteen cows and eleven horses can be bought for £922. What is the value of each animal ?

36.   8 pens and 7 pencils cost Rs. 3, 5a. and 6 pens and

10 pencils cost Rs. 3, 6a., what will 4 pens and 5 pencils cost ? [*Hint*]

37. The cost of two coats exceeds the cost of 12 shirts by Rs. 4, while the cost of 3 coats and 4 shirts is Rs. 182; find the cost of 4 coats and 5 shirts.

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38. Six years ago a man was six times as old as his son; four years hence thrice his age will be equal to eight times his son's; find their present ages. [*Hint*]

39. Eight years ago a man was four times as old as his son and two years hence their ages will be in the ratio 5 : 2; find their present ages.

40. Four times B's age exceeds A's age by 20 years; 8 years hence A will be twice as old as B; find their present ages.

41. Twice B's age is equal to what A's age was thirteen years ago; nineteen years ago A was eighteen times as old as B was; find their present ages.

42. Ten years ago a father's age was five times the sum of the ages of his two children; and 10 years hence his age will be to the sum of their ages as 7 is to 5; find his present age.

43. " I am thrice as old as you were when I was as old as you are," said Krishna to Hari. " Three years hence I shall be as old as you were three years ago," replied Hari. Find their present ages. [*Hint*]

44. " When I was as old as you are, I was six times as old as you were," said A to B. " In eighteen years' time I shall be only three-fourth of your age " replied B. Find their present ages.

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45. The sum of three numbers is 77. Three times the first added to the third is less than four times the second by 8, and the excess of the third over the second is 18. Find the numbers. [*Hint*]

46. The sum of three numbers is 177; the first is to the

second as 3 is to 4 and the second is to the third as 5 is to 6; find the numbers.

47.   Divide Rs. 750 among A, B and C so that A gets twice as much as B, and B gets Rs. 350 less than the joint share of A and C.

48.   Divide Rs. 210 among three persons A, B and C so that A's share may exceed the joint share of B and C by Rs. 30 and also twice the share of B may exceed the difference between A and C's shares by Rs. 20.

49.   A, B, C and D have Rs. 580 between them; A has twice as much as C and B has three times as much as D; also C and D together have Rs. 100 less than A. Find how much each has.   [*Hint*]

50.   A, B, C and D have Rs. 810 between them; A has three times as much as C, and B five times as much as D; also A and B together have Rs. 150 less than eight times what C has. Find how much each has.

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51.   If B were to give Rs. 50 to A they would have equal sums of money; if A were to give Rs. 44 to B, the money of B would be double that of A; find the money which each actually has.   [*Hint*]

52.   A says to B, " Give me 16 rupees and I shall have as much money as you will then have." B replies, " Give me 24 rupees and I shall have twice as much as you will have." How much money has each ?

53.   A and B lay a wager of 40 rupees; if A loses he will have as much as B will then have; if B loses he will have half of what A will then have; find the money of each.

54.   If B gives Rs. 20 to A he is left with $\frac{3}{10}$ of what A then has; but if A gives Rs. 24 to B he is left with $\frac{2}{5}$ of what B then has. Find how much each has.

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Miscellaneous Problems :—

55.   A man buys 4 horses and 9 cows for Rs. 670. He sells the horses at a profit of 10 p. c. and the cows at a profit

of 20 p. c. ; his whole gain is Rs. 94 ;   what price did he pay for a horse ? [*Hint*]

56.   Two pounds of tea and eight pounds of sugar cost Rs. 6 ; but if tea rose 50 p. c. and sugar 25 p. c., they would cost Rs. 8.   Find the prices of tea and sugar per pound.

57.   8 lbs. of tea and 10 lbs. of coffee together cost £1, 16s. 8d, ; but if tea were to rise 10 p. c. and coffee to fall 12½ p. c., the same quantities would cost £1, 16s. 7d.   Find the prices of tea and coffee per lb.

58.   A party of travellers coming to a hotel find that there are two rooms too few for each to have one. If they sleep two in a room, 2 rooms are left empty. How many rooms will be left empty if they sleep four in a room ?   [*Hint*]

59.   A and B can together do a piece of work in 8 days. A does twice as much work as B in the same time. How long will it take them to do the work separately. [*Hint*]

60.   2 men and 8 boys can do a piece of work in 8 days, and 4 men and 2 boys can do it in 2¼ days.   How long will 8 men and 3 boys take to do it ? [*Hint*]

61.   A and B working together can earn Rs. 11 in two days ; B and C Rs. 51 in 8 days and C and A Rs. 29 in 4 days ; find what each man can earn alone per day.

62.   A purse contains a number of crowns and sovereigns ; the number of sovereigns is one more than ⅙ of the number of crowns, and the total value of sovereigns is the same as the value of the crowns.   Find the number of each.

63.   A certain sum of money is divided equally among a certain number of men.   Had there been 8 men more each would have got Re. 1 less, and had there been 2 fewer, each would have received Re. 1 more. Find the number of men and the sum of money. [*Hint*]

64.   If the length and breadth of a rectangular courtyard were increased by 5 yards each, the area would be increased by 250 sq. yds. If the length were decreased by 3 yards and breadth increased by 2 yards, the area would be diminished by 16 sq.   Find the length and breadth of the courtyard.

65. Two persons, $2\frac{3}{4}$ miles apart, setting out at the same time are together in $5\frac{1}{2}$ hours if they walk in the same direction, and in 80 minutes if they walk in opposite directions. Find their rates of walking. [*Hint*]

66. A takes $1\frac{1}{2}$ hours longer than B to walk 15 miles; but if he doubles his pace he takes one hour less time than B; find their rates of walking.

67. A traveller walks a certain distance at a certain rate, had he gone one mile an hour faster, he would have walked it in three-fourths of the time; had he gone one mile an hour slower he would have been 2 hours longer on the road. Find the distance and his rate of walking. [*Hint*]

68. A boat goes up-stream 30 miles and down-stream 44 miles in 10 hours. It also goes up-stream 40 miles and down-stream 55 miles in 13 hours. Find the rate of the stream and of the boat in still water. [*Hint*]

69. A person lent two sums of money at simple interest; one to A for 3 years at 4 p. c. and the other to B for 4 years at 5 p. c. The total amount paid by A was greater than that paid by B by Rs. 204, and the sum borrowed by A was nine times the interest paid by B. Find the two principal sums.

70. If I lend a sum of money at 12 p. c. simple interest, the interest for a certain time exceeds the loan by Rs. 150; but if I lend it at 8 p. c. for half the time, the loan exceeds the interest by Rs. 450. How much do I lend?

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71. In a triangle ABC, angle A is five times the difference of angles B and C and also one-fifth of their sum; find the angles, supposing angle B to be greater than angle C. [*Hint*]

72. In a quadrilateral ABCD, angle A is half the sum of angles B and C; angle B is less than the sum of angles C and D by 100 degrees; and angle D is to B as 4 is to 3; find the angles.

73. ABCD is a cyclic quadrilateral; angle A is to B as 3 : 4 and angle C is to D as 21 : 16. Find the angles.

74. An exterior angle of a regular polygon is to an exterior angle of another as 2 : 5 but the sum of the interior

angles of the former is three times the sum of the interior angles of the latter. Find the number of sides of each.

[*Hint*]

SOLUTIONS & HINTS—EXERCISE 33

1. Let the numbers be x and y, x being the greater.
The sum of the numbers is given equal to 54

$$\therefore \qquad x+y=54 \qquad \text{...}(i)$$

Similarly $\quad x-y=12 \qquad \text{...}(ii)$

Eq. (i)+Eq. (ii) gives $2x=66$, $\quad \therefore \quad x=33$ $\Big\}$ **Ans.**
Eq. (i)−Eq. (ii) $\quad,, \quad 2y=42$, $\quad \therefore \quad y=21$

Verification. Sum of 33 and 21 is 54, and difference is 12, therefore the solution is correct.

9. Let the numbers be x and y.

According to the first condition $\quad \dfrac{x}{y}=\tfrac{4}{5}$ or $5x=4y$, etc.

12. If the numbers are x and y, we easily have :—

$$x+y=27 \qquad \text{...}(i)$$

and $\quad x^2-y^2=81 \qquad \text{...}(ii)$

Dividing Eq. (ii) by Eq. (i) we have :—

$$\frac{x^2-y^2}{x+y}=\tfrac{81}{27} \quad \text{or} \quad x-y=3 \qquad \text{...}(iii)$$

Solve equations (i) and (iii).

14. Remember that Dividend=(Divisor × Quotient)

+Remainder.

16. Let the numerator be x and denominator y.
Then, clearly :—

$$\frac{x}{y}=\tfrac{2}{3}, \text{ whence } \qquad 3x=2y \qquad \text{...}(i)$$

and $\quad \dfrac{x+3}{y}=\tfrac{7}{9}$, whence $9x+27=7y \qquad \text{...}(ii)$

Solve these equations.

19. If the numbers be x and y, then clearly :—

$$\frac{x}{y} = \frac{3}{5} \qquad \text{...}(i)$$

and $\qquad \dfrac{x+5}{y+5} = \dfrac{5}{8} \qquad \text{...}(ii)$

Simplify the equations and solve.

26. Let the left-hand digit be x and right-hand digit y.

$\therefore$ The sum of the digits $= x+y$

and, the value of the number $= 10x+y$ [See Art. 50, Solved Example No. 12]

If the digits are interchanged, the value of the number becomes $10y+x$.

The first condition of the question states that the number is five times the sum of the digits, that is, $(10x+y)$ is five times $(x+y)$

i.e., $\quad 10x+y=5(x+y) \qquad \text{...}(i)$

The second condition states that if 9 be added to the number, the digits change their places, that is, if 9 be added to $10x+y$, the result is $10y+x$, i.e.,

$$10x+y+9=10y+x \qquad \text{...}(ii)$$

Simplify equations (i) and (ii) and solve.

31. If the digits, from left to right, be x, y and z, we have :—

$$x+y+z=15 \qquad \text{...}(i)$$

The value of the number $= 100x+10y+z$.

In the second case the digits are y, x and z and the number becomes $100y+10x+z$

In the third case the digits are $\dfrac{x}{2}$, z, y and the number

becomes $\dfrac{100x}{2}+10z+y$ or $50x+10z+y$.

The conditions can be translated into equations very easily.

33.   Let each table cost Rs. x and each chair Rs. y.

Then, cost of 5 tables$=$Rs. $5x$

and   ,,   3 chairs$=$Rs. $3y$

$\therefore$ The first condition of the question gives $5x+3y=84$

Similarly $4x+5y=88$.   Solve the two equations.

36.   First find the cost of one pen and one pencil separately.

38.   Let the present age of the man be x years and that of his son y years.

Six years ago their ages were $(x-6)$ years and $(y-6)$ years respectively.

The first condition states that $(x-6)$ is six times $(y-6)$

i.e., $x-6=6(y-6)$ ...(i)

Four years hence their ages will be $(x+4)$ years and $(y+4)$ respectively ; we are given that three times $(x+4)$ is equal to 8 times $(y+4)$.

$\therefore$ $3(x+4)=8(y+4)$ ...(ii)

Solve equations (i) and (ii).

43.   Let Krishna's present age be x years

and Hari's   ,,   ,, y ,,

" When I (Krishna) was as old as you *are* " means when Krishna was y years old.   This was clearly $(x-y)$ years ago for now Krishna is x years old.

Also, $x-y$ years ago   Hari was $y-(x-y)$ that is $2y-x$ years old.

Hence the first condition gives the equation
$$x=3(2y-x)$$

The other equation $[y+3=x-3]$ can be easily obtained.

45.   Let the numbers be x, y and z.

Then, $x+y+z=77$ ...(i)

The next condition states that :—

three times x added to z is less than four times y by 3

$\therefore$ $3x+z$ is less than $4y$ by 3

$\therefore \quad 8x+z=4y-3 \qquad\qquad …(ii)$

The third equation is easy to obtain.

49. Let the four amounts be x, y, z and t rupees respectively.

Then, $\quad x+y+z+t=580$

Also, $\qquad\qquad\qquad x=2z$

$$y=3t$$

$$z+t=x-100$$

Solve these equations.

51. Suppose A has x rupees and B has y rupees

When B gives Rs. 50 to A, he is left with $(y-50)$ rupees, and A's money becomes $(x+50)$ rupees.

$\therefore \quad x+50=y-50 \qquad\qquad …(i)$

Similarly, the second equation is :—

$$y+44=2(x-44) \qquad …(ii)$$

Solve (i) and (ii).

55. Suppose he paid Rs. x for a horse and Rs. y for a cow.

It is easy to get the first equation, $4x+9y=670 \qquad …(i)$

Also, gain on Rs. 100 in case of horses $=$ Rs. 10

$\therefore \quad$ „ $\quad$ Rs. $4x$ $\quad$ „ $\qquad =$ Rs. $\frac{10}{100}\times 4x=$ Rs. $\frac{2x}{5}$

Similarly „ Rs. $9y$ in case of cows $=$ Rs. $\frac{20}{100}\times 9y=$ Rs. $\frac{9y}{5}$

Hence the second equation is $\frac{2x}{5}+\frac{9y}{5}=94 \qquad …(ii)$

58. Let the number of travellers be x and the number of rooms y.

The first equation is, clearly, $x=y+2 \qquad …(i)$

If they sleep 2 in a room, the no. of rooms required is

$\frac{x}{2}$. this number is less than y by 2, etc., etc.

59.   Suppose A and B can do the work in x and y days respectively.

$\therefore$ A's one day's work $= \dfrac{1}{x}$

and B's   „   „ $= \dfrac{1}{y}$ $\therefore$ $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{3}$ ...(

etc., etc.

60.   Suppose one man can do the work in x days, and or boy in y days.

$\bullet \therefore$ one man's one day's work $= \dfrac{1}{x}$

and one boy's   „   „ $= \dfrac{1}{y}$

The first equation is $\dfrac{2}{x} + \dfrac{3}{y} = \dfrac{1}{3}$

etc., etc.

63.   Let the no. of men be x and suppose each gets Rs. y.

$\therefore$ the sum distributed $=$ Rs. xy.

Had there been 3 men more [$i.e.$, $(x+3)$ men], each woul l have got Re. 1 less [$i.e.$, Rs. $(y-1)$], so that the sum dis - tributed would have been Rs. $(x+3)(y-1)$

$\therefore$ $(x+3)(y-1) = xy$

or $xy + 3y - x - 3 = xy$

or $3y - x - 3 = 0$ ...(i)

etc., etc.

65.   Let the rates of walking be x m p.h. and y m.p.h, x being greater than y.

In $1\frac{1}{2}$ hours, the faster man walks $\dfrac{11x}{2}$ miles, ana tl e other $\dfrac{11y}{2}$ miles ; the former should be greater than the latt r by $1\frac{1}{4}$ miles.

$\therefore$ $\dfrac{11x}{2} - \dfrac{11y}{2} = 1\frac{1}{4}$ or $x - y = \frac{1}{2}$ ...(t)

Similarly $\frac{1}{2}x + \frac{1}{2}y = 1\frac{1}{4}$ or $x + y = 2\frac{1}{2}$ ...(ii)

etc.

67. Suppose the rate of walking is x miles per hour and time taken is y hours, so that distance $= xy$ miles.

If the rate had been $(x+1)$ m.p.h., time taken would have been $\frac{3y}{4}$ hours. This gives $(x+1) \times \frac{3y}{4} = xy$

or $\quad (x+1) \times 3y = 4xy$

or $\qquad 3(x+1) = 4x \qquad$ [Dividing by y] $\qquad$ ...(i)
$\quad$ etc.

Important Note. In the last step we have divided both sides of the equation by y. *We can do this only if y is not equal to zero*; for division by zero is not permissible. But here we know that y, the time taken, is not zero.

68. Let the rate of the boat in still water be x m.p.h. and the rate of the stream y m.p.h.

$\therefore$ Speed of the boat up-stream $= (x-y)$ m.p.h.
and $\qquad$,, $\qquad$ down-stream $= (x+y)$,,

It will be seen that the equations to be solved are :—

$$\frac{30}{x-y} + \frac{44}{x+y} = 10 \qquad ...(i)$$

and $\qquad \dfrac{40}{x-y} + \dfrac{55}{x+y} = 13 \qquad ..(ii)$

To solve these equations, put $(x-y) = a$ and $(x+y) = b$.

71. Let angle $A = x$ degrees, and angle $B = y$ degrees and angle $C = z$ degrees.

$\therefore \quad x+y+z = 180 \qquad$ [Angles of a triangle are together equal to two right angles].

$\qquad$ etc., etc.

74. Let the no. of sides be x and y respectively.

$\therefore$ An exterior angle of the former $= \dfrac{360}{x}$ degrees

and $\qquad$,, $\qquad$,, $\qquad$ latter $= \dfrac{360}{y}$,,

$\therefore \quad \dfrac{360}{x} / \dfrac{360}{y} = \frac{2}{3}$ or $\quad \dfrac{y}{x} = \frac{2}{3} \qquad$ ...(i)

Again, sum of interior angles of the former $= 2x-4$ rt. angles
and $\qquad$,, $\qquad$,, $\qquad$ latter $= 2y-4$,,

$\therefore \quad 2x-4 = 3(2y-4) \qquad$ ...(ii)

Solve (i) and (ii).

TEST PAPERS–SET 2.

(CHAPTERS I to XII)
TEST PAPER 1. (Ex. 34)

1. If $a=-2$, $b=-3$ and $c=4$, find the value of :—

$$\frac{(ab+bc+ca)-(a^2+b^2+c^2)}{3abc-(a^3+b^3+c^3)}.$$

2. Multiply :—
   a^3-8+6a^2+8a by a^2+4-2a.

3. Solve the equation :—
   $\frac{1}{8}(x+1)-\frac{1}{3}(x+4)=16-\frac{1}{4}(x+3)$.

4. Solve for x and y :—
   $x-\frac{7}{6}y=7$, $\frac{7}{3}x-2y=25$.

5. Solve for x, y and z :—
   $8x+y+z=15$, $3x-5y+7z=75$, $9x-11z+10=0$.

6. A person bought 83 mangoes for ten rupees.; some h bought at the rate of 9 per rupee and the rest at 1 for two rupees ; how many of each sort did he buy ?

TEST PAPER 2—(Ex. 35)

1. Subtract $11x-[6x-2\{3x-4(y-x)\}-(9x+8y)]$ from 0.

2. Divide $1-32a^5-128a^7$ by $4a^2+1-2a$ by the metho' óf detached coefficients.

3. Solve the equation :—

$$x=4-\frac{5x-12}{3}-\frac{3(x-3)}{5}.$$

4. Solve for a and b :—
   $57a+52b=181$
   $76a-89b=458$.

5. Solve for a, b and c :—

$$\frac{a}{2}+\frac{b}{3}+\frac{c}{4}=\frac{a}{3}+\frac{b}{4}+\frac{c}{5}=\frac{a}{4}+\frac{b}{5}+\frac{c}{6}=1$$

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6. 24 pounds of bread and 3 pounds of cheese cost 5 shillings which is also the sum paid for 16 pounds of bread and 4 pounds of cheese. Hence find the cost of 30 pounds of bread and 6 pounds of cheese.

TEST PAPER 3—(Ex. 36)

1. If $a=-1$, $b=-3$, $c=1$, find the value of :—

$$\frac{3\sqrt{(a^2+b^3+c)(a-b-3c)}}{\sqrt[3]{ab^3c^2}}.$$

2. Multiply $3-3x+\frac{3}{2}x^2-\frac{1}{2}x^3$ by $2+2x+x^2+\frac{x^3}{3}$.

3. Solve the equation $2+\dfrac{3x-1}{15}-\dfrac{x+4}{3}+\dfrac{x-4}{5}=0$ and verify the result by substitution.

4. If the fractions $\dfrac{x+3}{5}$, $\dfrac{8-y}{4}$ and $\dfrac{3(x+y)}{8}$ are equal to each other, find the numerical value of each. [*Hint*]

5. Solve for a, b and c :—

$$\frac{a}{2}-1=\frac{b}{6}+1=\frac{c}{7}+2 \text{ and}$$

$$\frac{a}{2}+\frac{b}{3}+\frac{c}{2}=18.$$

6. Two numbers consist of two digits each ; seven times the sum of the numbers is eleven times their difference. Find the numbers, given that they consist of the same digits reversed.

TEST PAPER 4—(Ex. 37)

1. What should be added to :—

$8\{3x-(4y-5z)\}+6\{4x-(5y-2z)\}+6\{5x-3(y-z)\}$ so that the result may be zero ?

2. Divide $a^3+8b^3+c^3-6abc$ by $a^2+4b^2+c^2-ac-2ab-2bc$.

3. Solve the equation :—

$$\frac{x}{2}-\frac{x-13}{18}-\frac{6x+1}{10}-\frac{1}{3}\left(6-\frac{3x}{2}\right)=0.$$

4. Solve for x and y :—

$$\frac{x}{3} + \frac{y}{4} = 2xy, \quad 1 + \frac{1}{2x} - \frac{1}{y} = 0.$$

5. Solve for x, y and z :—

$$5x - 6y - 3z = 0$$
$$7y - 2z - 4x = 0$$
$$2z - 3x - 4y + 15 = 0.$$

6. A grocer buys 60 lbs. of sugar and 112 lbs. of tea for £4, 6s. 8d. ; selling the sugar at a loss of 10 p. c. and the tea at a gain of 30 p. c., he clears 10 shillings on his outlay ; how much per lb. did he pay for each ?

TEST PAPER 5—(Ex. 38)

1. Subtract the excess of $2[a + c - 3\{ -2(b-1) \}]$ over

$$4\left[-\frac{a}{2} - (3 - \tfrac{2}{3}b) \right] \text{ from } a - (b - c) \text{ and evaluate}$$

the result for $a = 6$, $b = 7$, $c = 8$.

2. Multiply :—

$$x^3 + 8y^3 + 6x^2y + 12xy^2 \text{ by } x^3 - y^3 - 3xy(x - y).$$

3. If $\dfrac{a+2}{2} + \dfrac{a+3}{3} + \dfrac{a+4}{4}$ is equal to $\tfrac{2}{5}$, find the value of

$$\frac{a^2+4}{4} + \frac{a^2+9}{9} + \frac{a^2+16}{16}. \quad [Hint]$$

4. Solve for x and y :—

$$2(y+1) - (x+1) = 4(x+1)(y+1), \quad \frac{4}{y+1} - \frac{3}{x+1} = 9$$

5. Solve for x, y and z :—

$$\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = -4$$
$$\frac{1}{x} - \frac{1}{2y} + \frac{1}{3z} = 1$$
$$\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = 20.$$

6. A man walks from his village to a railway station distant 7 miles, in 2 hours and 20 minutes, and returns in 2 hours and 85 minutes, his rates of walking up-hill, down-hill and on the level being 2, 4 and 3 miles per hour respectively; find the length of level road between the village and the station. [*Hint*]

TEST PAPER 6—(Ex. 39)

1. Simplify by opening brackets and collecting like terms

$$\frac{a}{4}\left(3-\frac{8}{a}\right)-\frac{7}{8}\left(7-\frac{3a}{4}\right)-15\left(\frac{1}{3}-\frac{a}{64}\right).$$

Evaluate the result for $a=8$.

2. What should be added to $x-37x^4+3x^6+7x^2+85x^3$ so that the result may be divisible by $x(x-1)(x+4)-2$? [Answer to contain lowest possible powers of x] [*Hint*]

3. If $3(3x-4)-\dfrac{4(7x-9)}{5}-1\frac{2}{5}\left(6+\dfrac{x-1}{3}\right)$ is equal to

zero, find the value of $13(x-7)$.

4. For what values of p and q will the following relations hold good :—

     $85p+41q=847$ and $41p+85q=539$.

5. Solve for a, b and c :—

     $5ab=6(a+b)$

     $7bc=12(b+c)$
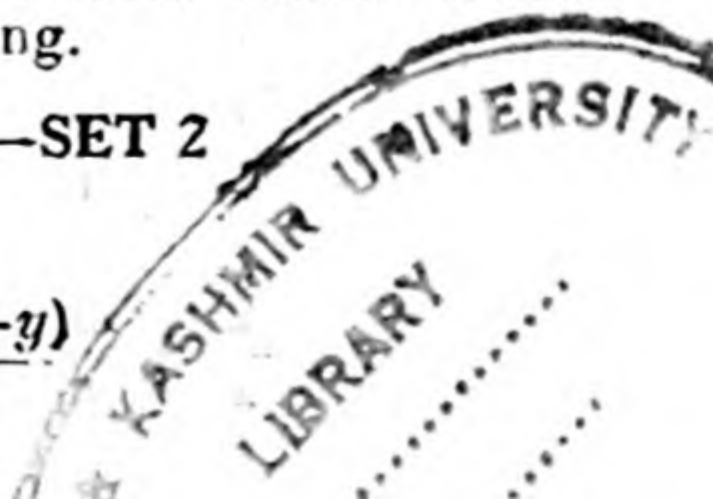
     $3ca=4(c+a)$.

6. A man walks a certain distance. Had he walked half a mile an hour faster than he did he would have taken 15 minutes less, but had he walked half a mile an hour slower he would have taken 20 minutes more. Find the distance and his rate of walking.

HINTS FOR TEST PAPERS—SET 2

Paper 3, Q. 4.

We are given that $\dfrac{x+3}{5}=\dfrac{8-y}{4}=\dfrac{3(x+y)}{8}$

Solving these equations for x and y, we get $x = 12$ and $y = -4$.

With these values of x and y each fraction will be found equal to 8.

Paper 5, Q. 3.

We are given that $\dfrac{a+2}{2} + \dfrac{a+3}{3} + \dfrac{a+4}{4} = \dfrac{5}{6}$

Solve this equation and substitute the value of a obtained from it in the second expression.

Paper 5, Q. 6.

Let level road be x miles, up-hill distance (while going from the village to the railway station) y miles, and down-hill distance z miles.

It is easy to obtain the following equations :—

$$x + y + z = 7$$

$$\frac{x}{3} + \frac{y}{2} + \frac{z}{4} = 3$$

$$\frac{x}{3} + \frac{y}{4} + \frac{z}{2} = 3\tfrac{1}{2}$$

Paper 6, Q. 2.

Remove the brackets in the divisor, $x(x-1)(x+4) - 2$.

Arrange the dividend and the divisor in *descending* powers of x, and divide ; the remainder with sign changed is the reqd. result.

————

CHAPTER XIII
FORMULAE

(General Results in Multiplication)

64. There are certain general results in multiplication which ought to be carefully learnt and committed to memory : by their help a number of products can be written down at once without actually going through the usual process   The other and more important use of these results is the reverse process ; ' *Factorisation* ' which will be dealt with in the next

Definition. Any general result expressed in symbols is called a Formula. Thus, all the general results referred to above are *formulas* which we propose to discuss in this chapter, but, as already remarked, we shall use them here only for *multiplying* and not for *factorising*.

65. It will be seen by actual multiplication that

$(a+b)\times(a+b)=a^2+2ab+b^2$ and $(a-b)\times(a-b)=a^2-2ab+b^2$.

Hence we have :—

Formula 1. $(a+b)^2=a^2+2ab+b^2$

Formula 2. $(a-b)^2=a^2-2ab+b^2$.

The most important thing to be realised by the students is that *a and b may be replaced by any expressions whatever, simple or Compound.* Hence it is far better to remember the above formulas in the following forms :—

$(\text{I Exp.}+\text{II Exp.})^2=(\text{I Exp.})^2+(\text{II Exp.})^2+2\times(\text{I Exp.})\times(\text{II Exp.})$

$(\text{I Exp.}-\text{II Exp.})^2=(\text{I Exp.})^2+(\text{II Exp.})^2-2\times(\text{I Exp.})\times(\text{II Exp.})$

With this device the students will readily understand the following examples :—

Ex. 1. $(2x+3y)^2=(2x)^2+(3y)^2+2\times2x\times3y$
$\qquad\qquad =4x^2+9y^2+12xy.$

Ex 2. $(3a-4b)^2=(3a)^2+(4b)^2-2\times3a\times4b$
$\qquad\qquad =9a^2+16b^2-24ab.$

Ex. 3. $(309)^2\quad=(300+9)^2=(300)^2+(9)^2+2\times300\times9$
$\qquad\qquad =90000+81+5400=95481.$

Ex. 4. $(498)^2\quad=(500-2)^2=(500)^2+(2)^2-2\times500\times2$
$\qquad\qquad =250000+4-2000=248004.$

Ex. 5. $(a+b+c)^2=\{(a)+(b+c)\}^2$
$\qquad\qquad =(a)^2+(b+c)^2+2.a.(b+c)$
$\qquad\qquad =a^2+(b^2+c^2+2bc)+(2ab+2ac)$
$\qquad\qquad =a^2+b^2+c^2+2ab+2ac+2bc.$

[The result may be written : $a^2+b^2+c^2+2a(b+c)+2bc$].

Ex. 6 $(a+b+c+d)^2=\{(a+b)+(c+d)\}^2$
$\qquad =(a+b)^2+(c+d)^2+2(a+b)(c+d)$
$\qquad =(a^2+b^2+2ab)+(c^2+d^2+2cd)+(2ac+2ad+2bc+2bd)$
$\qquad =a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$

[The result may be written :—
$$a^2+b^2+c^2+d^2+2a(b+c+d)+2b(c+d)+2cd]$$

66.   The following formulas can be easily obtained from the two formulas of the last article :—

Formula 3. $(a+b)^2+(a-b)^2=2(a^2+b^2)$

Formula 4. $(a+b)^2-(a-b)^2=4ab.$

Formula 5. $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$

Formula 6. $(a+b+c+d+e+\ldots\ldots)^2$
$$=a^2+b^2+c^2+d^2+e^2+\ldots\ldots$$
$$+2a(b+c+d+e+\ldots\ldots)$$
$$+2b(c+d+e+\ldots\ldots)$$
$$+2c(d+e+\ldots\ldots)$$
$$+\ldots\ldots\ldots$$

Formula 3 is the result of adding the two formulas

$$(a+b)^2=a^2+b^2+2ab$$

and $(a-b)^2=a^2+b^2-2ab$

as they stand ; formula 4 is got from them by subtraction ; formula 5 is the result of solved example 5 ; formula 6 is the generalisation of the results of solved examples 5 and 6 given within square brackets at the end of each solution.

67.   The following *Corollaries* obtained from the formulas of the last two articles by simple transpositions and minor changes may also be noted.   Students should see for them-selves how they are obtained.

Cor. 1. $a^2+b^2=(a+b)^2-2ab.$ [From Formula 1]

Cor. 2. $a^2+b^2=(a-b)^2+2ab.$ [„   „   2]

Cor. 3. $(a+b)^2=(a-b)^2+4ab.$ [„   „   4]

Cor. 4. $(a-b)^2=(a+b)^2-4ab.$ [„   „   4]

Cor. 5. $a^2+b^2+c^2=(a+b+c)^2-2(ab+ac+bc).$
                                              [From Formula 5]

Cor. 6. $2(ab+bc+ca)=(a+b+c)^2-(a^2+b^2+c^2).$
                                              [From Formula 5]

The students should realise that they can do without the

above corollaries and they should use them instead of the direct formulas only if they find them more convenient.

68. We draw the attention of the students to an important fact : **Formula 2 is, in fact. only a Corollary of Formula 1,** because :—

$$(a-b)^2 = \{ (a)+(-b) \}^2$$
$$= (a)^2 + (-b)^2 + 2(a)(-b)$$
$$= a^2 + b^2 - 2ab.$$

That is to say, if in Formula 1 we have $-b$ instead of b we get Formula 2.

The idea of changing the signs of one or more letters in a given formula to obtain new formulas therefrom is very important and will be **referred** to in several other cases as well.

Note. The extensive use and importance of the formulas and their corollaries discussed in the foregoing articles (No. 65 to 67) justify the inclusion of a large number of examples in the next exercise. This will also lay a good foundation to enable the students to handle the other formulas with greater ease and self-confidence.

EXERCISE 40

Find the square of :—

1. $3x+4y$. [*Solved*]   2. $4a+5b$.
3. $2x+5y$.   4. $3x+7y$.
5. $5p+6q$.   6. $7l+1$.

———

7. $8a-3b$. [*Solved*]   8. $5x-7y$.
9. $4n-9t$.   10. $8k-9l$.
11 $s-13t$.   12. $1-20l$.

———

13 $a+\dfrac{1}{a}$. [*Solved*]   14. $x+\dfrac{1}{x}$

15. $b-\dfrac{1}{b}$.   16. $2c-\dfrac{1}{2c}$.

17. $k + \dfrac{3}{k}$.

18. $2t - \dfrac{3}{t}$

___

19. $3ab - 4c^2$. [Solved]

20. $4xy - 5z^2$.

21. $k^2 + 6l^2$.

22. $3ab - 4cd$.

23. $7 - 8m^2$.

24. $1 + 9xyz$.

___

25. $5a^3b^4 - 8ab^5$. [Solved]

26. $4x^4y^2 - 5x^3y^7$.

27. $8x^2y^3 + 3y^4z^2$.

28. $1 - 12k^{12}$.

29. $10a^{10} - 11b^{11}$.

30. $a^{12}b^4 - 4b^6c$.

___

31. $a - b - c$. [Solved]

32. $x - y + z$.

33. $-p + q + r$.

34. $-l - m + n$.

35. $-g + h - k$.

36. $-x - y - z$.

___

37. $2x - 3y + 4z^2$. [Solved]

38. $3a - 4b + 5c^2$.

39. $1 - 3k - 4k^2$.

40. $x^2 - xy + y^2$.

41. $p^2 - 2pq - 3q^2$.

42. $\frac{2}{3}a^2 - a + \frac{3}{2}$.

___

43. $a - 2b + 3c - 4d + 5$. [Solved]

44. $2x - y + 3z - t$.

45. $3x^3 - 2x^2 + x - 1$. [Arrange your result in descending powers of x]

46. $4a^4 + 3a^3 - 2a^2 + a - 1$. [Arrange your result in descending powers of a]

___

Without performing long multiplication, find the square of :—

47. 997. [Solved]

48. 995.

49. 796.

50. 703.

51. $400\frac{1}{2}$.

52. $599\frac{3}{4}$.

___

If $2ab$ be added to or subtracted from $a^2 + b^2$, it becomes a perfect square. Find similar quantities which should be

added to or subtracted from the following expressions to make them perfect squares :—

53. $16x^2+9y^2$. [Solved]   54. $4x^2+25y^2$.
55. $81+4b^4$.   56. $1+9c^{10}$.
57. a^8+b^8.   58. $100x^{16}+121y^2z^2$.

---

59. $x^2+\dfrac{1}{x^2}$.   60. $a^4+\dfrac{1}{a^4}$.

61. $\tfrac{1}{4}x^2y^2+\dfrac{1}{x^2y^2}$.   62. $\dfrac{x^4}{y^4}+\dfrac{y^4}{x^4}$.

If b^2 be added to a^2+2ab, it becomes a perfect square. Find similar quantities which should be added to the following expressions to make them perfect squares :—

63. $4x^4+12x^2y^2$. [Solved]   64. $9a^4+12a^2$.
65. $100-60x^2$.   66. $121-44a^2b^2$.
67. x^8+6x^4.   68. $a^6-8a^3b^2$.

---

69. x^2+2. [Solved]   70. x^4+2.
71. a^4b^4-4.   72. a^8-10.

Find *as shortly as possible* the value of :—

73. $9x^2+42x+49$ when $x=-9$. [Solved]
74. $9a^2-30a+25$ when $a=15$.
75. $25a^2+40ab+16b^2$ when $a=-16$, $b=21$.
76. $9m^2-30mn+25n^2$ when $m=12$, $n=8$.
77. $a^2b^2-10abc+25c^2$ when $a=-3$, $b=-7$, $c=6$.
78. $x^2y^2+22xyz+121z^2$ when $x=7$, $y=9$, $z=-6$.

---

79. $a^2+b^2+c^2-2ab+2ac-2bc$ when $a=20$, $b=19$, $c=9$.
                                                [Hint].
80. $a^2+b^2+c^2+2ab-2ac-2bc$, when $a=15$, $b=16$, $c=31$.
81. $x^2+y^2+z^2-2xy-2xz+2yz$, when $x=25$, $y=16$, $z=8$.
82. $x^2+4y^2+9z^2+4xy+6xz+12yz$, when $x=28$, $y=5$, $z=4$.
83. $4a^2+9b^2+16c^2-12ab+16ac-24bc$, when $a=12$, $b=8$
       and $c=15$.

84. $a^2 + 16b^2 + 25 - 8ab - 10a + 40b$, when $a = 21$, $b = 5$.

Simplify the following by removing brackets and collectin; like terms :—

85. $(a + 3b)^2 - (3a + b)^2$. [Solved]
86. $(2x + 3y)^2 + (2x - 5y)^2$.   87. $(3a - 4b)^2 + (4a + 5b)^2$
88. $(a + b - c)^2 - (a - b + c)^2$.

89. $(a + 2b)^2 - 2(a + 2b)(a - 2b) + (a - 2b)^2$. [Solved]
90. $(2x - 3y)^2 + 2(2x - 3y)(4x + 3y) + (4x + 3y)^2$.
91. $(2a - 3b + 4c)^2 + 2(2a - 3b + 4c)(2a + 3b - 4c)$
$$+ (2a + 3b - 4c)^2$$
92. $4(a - b)^2 - 12(a - b)(a + b) + 9(a + b)^2$. [Hint]
93. $25(x - y)^2 + 10(x - y)(3x + 5y) + (3x + 5y)^2$.
94. $36(l + m)^2 - 12(l + m)(5l + 6m) + (5l + 6m)^2$.

95. If $a + b = 5$ and $ab = 7$, find the value of $a^2 + b^2$.
[Solved]

96. If $a + b = -6$ and $ab = 8$, find the value of $a^2 + b^2$.
97. If $a - b = 1$ and $ab = 7$, find the value of $a^2 + b^2$.
98. If $x - y = -8$ and $xy = -3$, find the value of $x^2 + y^2$.
99. If $a^2 + b^2 = 10$ and $ab = 3$, find the value of $a + b$.
[Hint]

100. If $a^2 + b^2 = 25$ and $ab = -12$, find the value of $a + b$.
101. If $a^2 + b^2 = 36$ and $ab = 10$, find the value of $a - b$.
102. If $x^2 + y^2 = 49$ and $xy = 6\frac{1}{2}$, find the value of $x - y$.

103. Find the value of $a + b$, when $a - b = 5$ and $ab = 2\frac{3}{4}$.
[Solved]

104. Find the value of $a + b$, when $a - b = 6$ and $ab = 3\frac{1}{4}$.
105. Find the value of $a + b$, when $a - b = 7$ and $ab = -6$.
106. Find the value of $a - b$, when $a + b = 8$ and $ab = 3\frac{3}{4}$.
[Hint]

107. Find the value of $x - y$, when $x + y = -8$ and $xy = -4$.

108. If $a + b = 3\sqrt{2}$ and $a - b = \sqrt{6}$, evaluate $a^2 + b^2$ and ab
[Solve]

109. If $a+b=\sqrt{11}$ and $a-b=\sqrt{7}$,   evaluate a^2+b^2 and ab.

110. If $x+y=3\sqrt{2}$ and $x-y=2\sqrt{2}$,   evaluate $xy(x^2+y^2)$.

111. If $l+m=\sqrt{7}$ and $l-m=-\sqrt{6}$, evaluate $\dfrac{l^2+m^2}{lm}$

112. If $a+b=2\sqrt{3}$ and $a-b=\sqrt{5}$,   evaluate a^3b+ab

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113. If $a+b+c=11$ and $ab+bc+ca=45$,   ' evaluate $a^2+b^2+c^2$. [Solved]

114. If $a+b+c=-13$ and $ab+bc+ca=60$,   evaluate $a^2+b^2+c^2$.

115. If $x+y+z=10$ and $xy+yz+zx=-30$, evaluate $x^2+y^2+z^2$.

116. If $x+y+z=15$ and $x^2+y^2+z^2=85$,   evaluate $xy+yz+zx$. [Hint]

117. If $x+y+z=-20$ and $x^2+y^2+z^2=16$,   evaluate $xy+yz+zx$.

118. If $a^2+b^2+c^2=24$ and $ab+bc+ca=6$,   evaluate $a+b+c$. [Hint]

119. If $a^2+b^2+c^2=50$ and $ab+bc+ca=-\frac{1}{2}$, evaluate $a+b+c$.

120. If $x^2+y^2+z^2=40$ and $xy-yz-zx=12$,   evaluate $x+y-z$.

121. If $x^2+y^2+z^2=75$ and $x-y-z=11$,   evaluate $xy-yz+zx$. [Hint]

122. If $-a-b+c=12$, and $-ab+bc+ca=22$, evaluate $a^2+b^2+c^2$.

------

123. If $a+\dfrac{1}{a}=3$,   find the value of $a^2+\dfrac{1}{a^2}$.   [Solved]

124. If $a+\dfrac{1}{a}=5$,   find the value of $a^2+\dfrac{1}{a^2}$

125. If $m+\dfrac{1}{m}=-7$, find the value of $m^2+\dfrac{1}{m^2}$.

126. If $x-\dfrac{1}{x}=6$,   find the value of $x^2+\dfrac{1}{x^2}$ 'Hint]

127. If $x - \dfrac{1}{x} = \sqrt{8}$, find the value of $x^2 + \dfrac{1}{x^2}$.

128. If $x + \dfrac{1}{x} = \sqrt{5}$, find the value of $x^4 + \dfrac{1}{a^4}$. [Solved]

129. If $x + \dfrac{1}{x} = \sqrt{7}$, find the value of $x^4 + \dfrac{1}{x^4}$.

130. If $a - \dfrac{1}{a} = -3$, find the value of $a^4 + \dfrac{1}{a^4}$.

———

131. If $a^2 + \dfrac{1}{a^2} = 7$, evaluate $a + \dfrac{1}{a}$. [Solved]

132. If $a^2 + \dfrac{1}{a^2} = 14$, evaluate $a + \dfrac{1}{a}$.

133. If $x^2 + \dfrac{1}{x^2} = 23$, evaluate $x + \dfrac{1}{x}$.

134. If $x^2 + \dfrac{1}{x^2} = 38$, evaluate $x - \dfrac{1}{x}$. [Hint]

135. If $a^2 + \dfrac{1}{a^2} = 51$, evaluate $a - \dfrac{1}{a}$.

136. If $a^4 + \dfrac{1}{a^4} = 47$, evaluate $a + \dfrac{1}{a}$. [Hint]

137. If $m^4 + \dfrac{1}{m^4} = 194$, evaluate $m + \dfrac{1}{m}$.

138. If $x^4 + \dfrac{1}{x^4} = 84$, evaluate $x - \dfrac{1}{x}$. [Hint]

139. If $p^4 + \dfrac{1}{p^4} = 119$, evaluate $p - \dfrac{1}{p}$.

140. If $k^4 + \dfrac{1}{k^4} = 2$, prove that $k - \dfrac{1}{k} = 0$.

———

Show that the following expressions can never* be negative :—

141. $9a^2 - 12a + 6.$ [*Solved*]   142. $25a^2 - 10a + 6.$
143. $b^2 - 2bc + c^2 + 3.$ 144. $a^2 - 4ab + 4b^2 + c^2.$
145. $a^2 + 6ab + 9b^2 + c^2 + d^2.$
146. $a^2 - 8ab + 16b^2 + c^2 - 6cd + 9d^2.$

---

147.   Prove that $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2} \{ (a-b)^2 + (b-c) + (c-a)^2 \}$; hence show that $a^2 + b^2 + c^2 - ab - bc -ca$ can never be negative.   [*Hint*]

148.   Find the value of $a^2 + b^2 + c^2 - ab - bc - ca$, when $a = 576, \; b = 577, \; c = 579.$ [*Hint*]

149.   If $x = a - b, \; y = b - c, \; z = c - a$, show that $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$ is equal to zero.   [*Hint*]

150.   If $x = a + b - 2c, \; y = b + c - 2a, \; z = c + a - 2b$, show that $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$ is equal to zero.

---

151.   If $x = 3a + 2b$ and $y = 2a - 3b$, find the value of $4x^2 - 12xy + 9y^2$ in terms of a and b.   [*Hint*]

152.   If $a = 3m - n$ and $b = 2m + 3n$, find the value of $9a^2 + 6ab + b^2$ in terms of m and n.

153.   If $x = a + b, \; y = a + 2b, \; z = a + 3b$, find the value of $x^2 + y^2 + 4z^2 + 2xy - 4xz - 4yz$ in terms of a and b.

---

154.   Show that $\left(a + \dfrac{1}{a} \right)^2 - 4 \left(a - \dfrac{1}{a} \right)$ is a perfect square.   [*Solved*]

155.   Show that $\left(a + \dfrac{1}{a} \right)^2 - 6 \left(a - \dfrac{1}{a} \right) + 5$ is a perfect square.

156.   Show that $\left(a - \dfrac{1}{a} \right)^2 + 2 \left(a + \dfrac{1}{a} \right) + 5$ is a perfect square.

---

* It is of course understood that the letters involved stand for *real* quantities. This has not been mentioned, for the students haven't yet come across *imaginary quantities* and therefore they cannot be implied.

FORMULAE

Express the following expressions as perfect squares :—

157. $x^2 + \dfrac{1}{x^2} + 4\left(x + \dfrac{1}{x}\right) + 6.$

158. $x^2 + \dfrac{1}{x^2} - 1 + 2\left(x - \dfrac{1}{x}\right).$

159. $\left(a^2 + \dfrac{1}{a^2}\right)^2 - 4\left(a + \dfrac{1}{a}\right)^2 + 12.$ [*Hint*]

160. $\left(x^2 + \dfrac{1}{x^2}\right)^2 - 6\left(x - \dfrac{1}{x}\right)^2 - 3.$

SOLUTIONS & HINTS—EXERCISE 40

1. $(3x+4y)^2 = (3x)^2 + (4y)^2 + 2(3x)(4y)$
$\qquad = 9x^2 + 16y^2 + 24xy.$ **Ans.**

7. $(8a-3b)^2 = (8a)^2 + (3b)^2 - 2(8a)(3b)$
$\qquad = 64a^2 + 9b^2 - 48ab.$ **Ans.**

13. $\left(a + \dfrac{1}{a}\right)^2 = (a)^2 + \left(\dfrac{1}{a}\right)^2 + 2(a)\left(\dfrac{1}{a}\right)$

$\qquad\qquad = a^2 + \dfrac{1}{a^2} + 2.$ **Ans.**

19. $(3ab-4c^2)^2 = (3ab)^2 + (4c^2)^2 - 2(3ab)(4c^2)$
$\qquad = 9a^2b^2 + 16c^4 - 24abc^2.$ **Ans.**

25. $(5a^3b^4 - 8ab^5)^2 = (5a^3b^4)^2 + (8ab^5)^2 - 2(5a^3b^4)(8ab^5)$
$\qquad = 25a^6b^8 + 64a^2b^{10} - 80a^4b^9.$ **Ans.**

31. $(a-b-c)^2 = \{(a-b)-c\}^2$
$\qquad = (a-b)^2 + (c)^2 - 2(a-b)(c)$
$\qquad = a^2 - 2ab + b^2 + c^2 - 2ac + 2bc$
$\qquad = a^2 + b^2 + c^2 - 2ab - 2ac + 2bc.$ **Ans.**

$\qquad\qquad$ *Or thus* :—

$(a-b-c)^2 = \{a+(-b)+(-c)\}^2$
$\qquad = a^2 + (-b)^2 + (-c)^2 + 2(a)(-b) + 2(a)(-c)$
$\qquad\qquad + 2(-b)(-c)$ [using **Formula 5**]
$\qquad = a^2 + b^2 + c^2 - 2ab - 2ac + 2bc.$ **Ans.**

37. $(2x-3y+4z^2)^2$

$\quad =(2x)^2+(-3y)^2+(4z^2)^2+2(2x)(-3y)+2(2x)(4z^2)$

$\quad\quad\quad +2(-3y)(4z^2)$　　　　　　　　　　[using Formula 5]

$\quad =4x^2+9y^2+16z^4-12xy+16xz^2-24yz^2.$　**Ans.**

43　$(a-2b+3c-4d+5)^2$

$\quad =(a)^2+(-2b)^2+(3c)^2+(-4d)^2+(5)^2$

$\quad\quad\quad +2a(-2b+3c-4d+5)-4b(3c-4d+5)$

$\quad\quad\quad\quad +6c(-4d+5)-8d(5)$　　　　[using Formula 6]

$\quad =a^2+4b^2+9c^2+16d^2+25-4ab+6ac-8ad+10a$

$\quad\quad\quad -12bc+16bd-20b-24cd+30c-40d.$　**Ans.**

47. $(997)^2=(1000-3)^2$

$\quad\quad\quad =(1000)^2+(3)^2-2\times 1000\times 3$

$\quad\quad\quad =1000000+9-6000$

$\quad\quad\quad =994009.$　**Ans.**

53. If we put the given expression in the form $(a)^2+(b)^2$, the quantity $2.a.b$ can at once be obtained.

Given exp.$=16x^2+9y^4=(4x)^2+(3y^2)^2$

$\therefore$　Reqd. quantity　$=2(4x)(3y^2)=24xy^2$

63. If we put the given expression in the form $(a)^2+2.a.b$, the quantity $(b)^2$ can at once be obtained.

Given exp.$=4x^4+12x^2y^2=(2x^2)^2+2(2x^2)(3y^2)$

$\therefore$　Reqd. quantity$=(3y^2)^2=9y^4.$

69. Given exp. $=x^2+2=(x)^2+2(x)\left(\dfrac{1}{x}\right)$

$\therefore$　Reqd. quantity $=\left(\dfrac{1}{x}\right)^2=\dfrac{1}{x^2}.$

73. Given exp.$=9x^2+42x+49=(3x)^2+2(3x)(7)+(7)^2$

$\quad\quad\quad =(3x+7)^2$

$\quad\quad\quad =(-27+7)^2$　　　$[\because\ x=-9]$

$\quad\quad\quad =(-20)^2=400.$　**Ans.**

79　Given exp.$=(a-b+c)^2.$

85. $(a+3b)^2-(3a+b)^2$

$\quad =(a^2+6ab+9b^2)-(9a^2+6ab+b^2)$

$$= a^2 + 6ab + 9b^2 - 9a^2 - 6ab - b^2$$
$$= -8a^2 + 8b^2. \quad \textbf{Ans.}$$

89. $(a+2b)^2 - 2(a+2b)(a-2b) + (a-2b)^2$
$$= x^2 - 2xy + y^2 \qquad \text{[where } x = a+2b \text{ and } y = a-2b\text{]}$$
$$= (x-y)^2$$
$$= \{ (a+2b) - (a-2b) \}^2 \qquad \text{[putting back the values of } x \text{ and } y\text{]}$$
$$= (4b)^2 = 16b^2. \quad \textbf{Ans.}$$

92. Given exp. $= 4x^2 - 12xy + 9y^2$ [where $x = a-b$, $y = a+b$]
$$= (2x - 3y)^2, \text{ etc. etc.}$$

95. $a^2 + b^2 = (a+b)^2 - 2ab$ \qquad [Cor. 1. Art. 67]
$$= (5)^2 - 2 \times 7 \qquad [\because \quad a+b=5 \text{ and } ab=7]$$
$$= 25 - 14 = 11. \quad \textbf{Ans.}$$

99. $(a+b)^2 = a^2 + b^2 + 2ab$ \qquad [Formula 1]
$$= 10 + 2 \times 3 \qquad [\because \quad a^2 + b^2 = 10, \ ab = 3]$$
$$= 10 + 6 = 16$$
$$\therefore \quad a + b = \sqrt{16} = 4 \quad \text{or} \quad -4. \quad \textbf{Ans.}$$

103. $(a+b)^2 = (a-b)^2 + 4ab$ \qquad [Cor. 3, Art. 67]
$$= (5)^2 + 4 \times \tfrac{11}{4} \qquad [\because \quad a - b = 5 \text{ and } ab = \tfrac{11}{4}]$$
$$= 25 + 11 = 36$$
$$\therefore \quad a + b = \sqrt{36} = 6 \quad \text{or} \quad -6. \quad \textbf{Ans.}$$

106. Use Cor. 4, Art. 67. \qquad [*viz*, $(a-b)^2 = (a+b)^2 - 4ab$]

108. $2(a^2 + b^2) = (a+b)^2 + (a-b)^2$ \qquad [Formula 3]
$$= (3\sqrt{2})^2 + (\sqrt{6})^2 \ [\because \ a+b = 3\sqrt{2}, \ a-b = \sqrt{6}]$$
$$= 18 + 6 \ [\because \ (3\sqrt{2})^2 = 3\sqrt{2} \times 3\sqrt{2} = 9 \times 2 = 18]$$
$$= 24.$$
$$\therefore \quad a^2 + b^2 = \tfrac{24}{2} = 12. \quad \textbf{Ans.}$$
Also $4ab = (a+b)^2 - (a-b)^2$ [Formula 4] $= 18 - 6 = 12$
$$\therefore \quad ab = \tfrac{12}{4} = 3.$$

113. $a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab + ac + bc)$ \qquad [Cor. 5, Art. 67]
$$= (11)^2 - 2(45) \text{ [Substituting the given values]}$$
$$= 121 - 90 = 31. \quad \textbf{Ans.}$$

$$Or\ thus :—$$

$$a+b+c=11 \qquad\qquad\qquad\qquad\qquad \text{[given]}$$

$$\therefore \quad (a+b+c)^2=(11)^2$$

$$\text{or} \qquad a^2+b^2+c^2+2(ab+ac+bc)=121$$

$$\text{or} \qquad a^2+b^2+c^2+2\times45=121$$

$$\text{or} \qquad a^2+b^2+c^2=121-90 \qquad \text{[Transposing } 2\times45,\ i.e\ 90]$$

$$=31. \quad \textbf{Ans.}$$

116. Use Cor. 6, Art. 67, viz., $2(ab+ac+bc)=(a+b+c)^2$ $-(a^2+b^2+c^2)$ or proceed as in the alternative solution of Q. 113.

118. $(a+b+c)^2=a^2+b^2+c^2+2(ab+ac+bc)$
$$=24+2(6),\ \ etc.$$

121 $\qquad x-y-z=11$

$$\therefore\ (x-y-z)^2=(11)^2,\ etc.$$

[In the last step you will have to divide by -2 instead of 2]

123. $\quad a^2+b^2=(a+b)^2-2ab.$ $\qquad\qquad$ [Cor. 1, Art. 67]

$$\therefore \quad a^2+\frac{1}{a^2}=\left(a+\frac{1}{a}\right)^2-2\times a\times\frac{1}{a}$$

$$\left[\ \text{Writing}\ \frac{1}{a}\ \text{instead of } b\ \right]$$

$$=(3)^2-2=9-2=7. \quad \textbf{Ans.}$$

$$Or\ thus :—$$

$$a+\frac{1}{a}=3 \quad \text{[given]}$$

$$\therefore \quad \left(a+\frac{1}{a}\right)^2=(3)^2$$

$$\therefore \quad a^2+\frac{1}{a^2}+2.a.\frac{1}{a}=9$$

$$\therefore \quad a^2+\frac{1}{a^2}+2=9$$

$$\therefore \quad a^2+\frac{1}{a^2}=9-2=7. \quad \textbf{Ans.}$$

126. Either use Cor. 2, Art. 67, *viz.*, $a^2 + b^2 = (a-b)^2 + 2ab$, or proceed as in the alternative solution of Q. 123.

128. $\quad x + \dfrac{1}{x} = \sqrt{5}$ [given]

$\therefore \qquad \left(x + \dfrac{1}{x} \right)^2 = (\sqrt{5})^2$

or $\qquad x^2 + \dfrac{1}{x^2} + 2 = 5$

or $\qquad x^2 + \dfrac{1}{x^2} = 5 - 2 = 3$

Squaring again, we have :—

$$\left(x^2 + \dfrac{1}{x^2} \right)^2 = (3)^2$$

or $\qquad x^4 + \dfrac{1}{x^4} + 2 = 9$

or $\qquad x^4 + \dfrac{1}{x^4} = 9 - 2 = 7.$ **Ans.**

Or thus :—

$$x^2 + \dfrac{1}{x^2} = \left(x + \dfrac{1}{x} \right)^2 - 2.x. \dfrac{1}{x}$$
$$= (\sqrt{5})^2 - 2 = 5 - 2 = 3$$

Again $\quad x^4 + \dfrac{1}{x^4} = \left(x^2 + \dfrac{1}{x^2} \right)^2 - 2.x^2. \dfrac{1}{x^2}$
$$= (3)^2 - 2 = 9 - 2 = 7.$$ **Ans.**

131. $\left(a + \dfrac{1}{a} \right)^2 = a^2 + \dfrac{1}{a^2} + 2.a. \dfrac{1}{a} = a^2 + \dfrac{1}{a^2} + 2$

$$= 7 + 2 \qquad \left[\because \ a^2 + \dfrac{1}{a^2} = 7 \right]$$
$$= 9$$

$\therefore \qquad a + \dfrac{1}{a} = \sqrt{9} = 3$ or $\ -3.$ **Ans.**

Or thus :—

$a^2 + \dfrac{1}{a^2} = 7$ [given]

Adding 2 to both sides we have :—

$$a^2 + \frac{1}{a^2} + 2 = 7 + 2 = 9$$

or

$$\left(a + \frac{1}{a} \right)^2 = 9$$

or

$$a + \frac{1}{a} = \sqrt{9} = 3 \quad \text{or} \quad -3. \quad \text{Ans.}$$

134. $\left(x - \frac{1}{x} \right)^2 = x^2 + \frac{1}{x^2} - 2 = 38 - 2,$ etc.

Or thus :—

$$x^2 + \frac{1}{x^2} = 38$$

Subtract 2 from both sides etc.

136. $a^4 + \frac{1}{a^4} = 47$

Add 2 to both sides and thus get the value of $a^2 + \frac{1}{a^2}$

Again add 2 to both sides and get the value of $a + \frac{1}{a}$.

138. First get the value of $x^2 + \frac{1}{x^2}$

[by adding 2 to both sides]

Then get the value of $x - \frac{1}{x}$

[by subtracting 2 from both sides]

141. Given exp. $= 9a^2 - 12a + 6$

$$= (9a^2 - 12a + 4) + 2$$

$$= (3a - 2)^2 + 2$$

The first part, viz., $(3a - 2)^2$, is a positive quantity, because even if the expression within brackets is negative, its square will be positive. The second part, viz., $+2$, is also positive. Hence the given exp. is positive.

Remember that *a squared quantity can never be negative and its least value is zero.*

147. Proceed with the right-hand expression: take the squares; collect like terms; remove the outer bracket; the resulting expression will be the same as that on left-hand side.

148. The given exp. $=\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$

[See last question]

Substitute the given values.

149. The given exp. $=(x+y+z)^2$.

Substitute the given values of x, y and z.

151. The given exp. $=(2x-3y)^2$.

Substitute the given values of x and y.

154. $(a+b)^2=(a-b)^2+4ab$.   [Cor. 3, Art. 67]

$$\therefore \left(a+\frac{1}{a}\right)^2=\left(a-\frac{1}{a}\right)^2+4.a.\frac{1}{a}.$$

$$\left[\text{Writing } \frac{1}{a} \text{ instead of } b\right]$$

$$=\left(a-\frac{1}{a}\right)^2+4 \qquad \ldots(i)$$

Now, the given exp.

$$=\left(a+\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right)$$

$$=\left(a-\frac{1}{a}\right)^2+4-4\left(a-\frac{1}{a}\right) \quad \left[\text{using result } (i)\right]$$

$$=x^2+4-4x \qquad\qquad \left[\text{where } x=a-\frac{1}{a}\right]$$

$$=(x-2)^2$$

$$=\left(a-\frac{1}{a}-2\right)^2 \qquad \text{[Re-writing the value of } x\text{]}$$

$$=\text{a perfect square.}$$

159. The given exp. $= \left(a^2 + \dfrac{1}{a^2} \right)^2 - 4 \left(a + \dfrac{1}{a} \right)^2 + 12$

$$= \left(a^2 + \dfrac{1}{a^2} \right)^2 - 4 \left(a^2 + \dfrac{1}{a^2} + 2 \right) + 12$$

$$= \left(a^2 + \dfrac{1}{a^2} \right)^2 - 4 \left(a^2 + \dfrac{1}{a^2} \right) - 8 + 12$$

etc., etc.

65. Formula 7. $(a+b)(a-b) = a^2 - b^2$

[Verify by actual multiplication]

Since a and b can be replaced by any expressions, we have :—

$(I \text{ Exp.} + II \text{ Exp.})(I \text{ Exp.} - II \text{ Exp}) = (I \text{ Exp.})^2 - (II \text{ Exp.})^2$, a form which is much more convenient for a beginner.

In words, the formula may be stated as follows :—

" *The product of the sum and difference of two quantities is equal to the difference of their squares.*"

Example 1. $(x+3)(x-3) = (x)^2 - (3)^2 = x^2 - 9$.

Example 2. $(3a+4)(3a-4) = (3a)^2 - (4)^2 = 9a^2 - 16$.

Example 3. $(3x^2+7y)(3x^2-7y) = (3x^2)^2 - (7y)^2 = 9x^4 - 49y^2$.

Example 4 $\{ (a+b)+(c) \} \{ (a+b)-(c) \}$
$$= (a+b)^2 - (c)^2 = a^2 + b^2 + 2ab - c^2$$

The above examples are only to illustrate the formula. Solutions of different types of questions in the next exercise have been provided as usual.

EXERCISE 41

Write down the product of :—

1   $2a+3$ and $2a-3$ [Solved] 2. $3x+5$ and $3x-5$

3   $4x+7y$ and $4x-7y$.   4. $\dfrac{8}{l} - m$ and $\dfrac{8}{l} + m$.

5   $-1+10p$ and $1+10p$ [Hint]

6   $-15+9r$ and $15+9r$

7. $5a^3b+6c^5$ and $5a^3b-6c^5$. [Solved]
8. $3a^4 \div 4b^3c$ and $3a^4-4b^3c$.
9. $5x^6-7y^4z^3$ and $5x^6+7y^4z^2$.
10. $-8x^8+3y^3z^3$ and $8x^8+3y^3z^3$.
11. $-9k^9+10l^{10}$ and $-9k^9-10l^{10}$.
12. $\dfrac{3p^3}{q^3}-5r^{12}$ and $-\dfrac{3p^3}{q^3}-5r^{12}$.

———

Write down the following products :—

13. $\{(2x+3y)+4z\}\{(2x+3y)-4z\}$. [Solved]
14. $\{(3a-4)+5b\}\{(3a-4)-5b\}$.
15. $\{(2a+5b)-6c\}\{(2a+5b)+6c\}$.
16. $\{4m+(3p-5q)\}\{4m-(3p-5q)\}$.
17. $\{(a+b)-(c+d)\}\{(a+b)+(c+d)\}$.

———

18. $(x-3y+4z)(x+3y-4z)$. [Solved]
19. $(2x+y-5z)(2x-y-5z)$.
20. $(x^2-xy+y^2)(x^2+xy+y^2)$
21. $(a^2+a-1)(a^2-a+1)$.
22. $(x^4+x^2+1)(x^4-x^2+1)$.
23. $(ax^2-by^2+cz^2)(-ax^2+cz^2+by^2)$.
24. $(x+y+z+t)(x-y-z+t)$
25. $(ax-by+cz-1)(ax+by-cz+1)$.

———

Find the continued product of —

26. $(a+b)(a-b)(a^2+b^2)$. [Solved]
27. $(x-y)(x+y)(x^2+y^2)$.
28. $(2a+3b)(2a-3b)(4a^2+9b^2)$
29. $(4x^2+25y^2)(2x+5y)(2x-5y)$.
30. $\left(x+\dfrac{1}{x}\right)\left(x^2+\dfrac{1}{x^2}\right)\left(x^4+\dfrac{1}{x^4}\right)\left(x-\dfrac{1}{x}\right)$ [Hint,

31. $(16a^4 \div b^4)(2a+b)(4a^2+b^2)(2a-b)$.
32. $(l^8+m^8)(l+m)(l^4+m^4)(l-m)(l^2+m^2)$.

---

33. $(1+a+a^2)(1-a^2+a^4)(1-a+a^2)$. [*Hint*]
34. $(a^2+ab+b^2)(a^4-a^2b^2+b^4)(a^2-ab+b^2)$.
35. $(x^2+1)(x^2-1)(x^2+x\sqrt{2}+1)(x^2-x\sqrt{2}+1)$. [*Hint*]
36. $(a-b-c)(b-c-a)(c-a-b)(a+b+c)$. [*Hint*]

Simplify :—

37. $(a-2b+3c)^2-(a+2b-3c)^2$. [*Solved*]
38. $(2x+3y+4z)^2-(2x-3y-4z)^2$.
39. $(x^2+5x+7)^2-(x^2-5x+7)^2$.
40. $(a^2-\sqrt{ab}+b^2)^2-(\sqrt{ab}-a^2+b^2)^2$.
41. $(2x+y-3z+p)^2-(2x-y+3z-p)^2$.
42. $(3a^2-6ab+8b^2)-4(a^2-3ab+4b^2)^2$.

Use the formula $(a+b)(a-b)=a^2-b^2$ to find the value of :—

43. 3004×2996. [*Hint*]   44. 915×885.
45. $700\cdot6 \times 699\cdot4$.   46. $30\cdot16 \times 29\cdot84$.
47. $3148 \times 3148 - 2852 \times 2852$. [*Hint*]
48. $4152 \times 4152 - 2848 \times 2848$.
49. $(3\cdot289)^2-(1\cdot711)^2$.   50. $(47\cdot13)^2-(12\cdot87)^2$.

SOLUTIONS & HINTS—EXERCISE 41

1. $(2a+3)(2a-3)=(2a)^2-(3)^2=4a^2-9$.
5. Reqd. product$=(10p-1)(10p+1)$.
7. $(5a^3b+6c^5)(5a^3b-6c^5)$
   $=(5a^3b)^2-(6c^5)^2=25a^6b^2-36c^{10}$.
13. $\{(2x+3y)+4z\}\{(2x+3y)-4z\}$
    $=(2x+3y)^2-(4z)^2=4x^2+12xy+9y^2-16z^2$.

18. [We must put the two expressions in the form of the last group (Q 13 to 17).

For this purpose *we combine those terms which have the same sign in the two expressions · they form the first part , the remaining terms form the second part.*]

$$(x-3y+4z)(x+3y-4z)$$

$$= \{ (x)-(3y-4z) \} \{ (x)+(3y-4z) \}$$

[Note that only x has the same sign in the two expressions]

$$=(x)^2-(3y-4z)^2$$

$$=x^2-(9y^2+16z^2-24yz)$$

$$=x^2-9y^2-16z^2+24yz.$$

26. $(a+b)(a-b)(a^2+b^2)$

$=(a^2-b^2)(a^2+b^2)$ [Multiplying the first two factors]

$=(a^2)^2-(b^2)^2=a^4-b^4.$ **Ans.**

30. First multiply the first and the fourth factors.

33. First find the product of the first and the third factors ; it will be found to be $1+a^2+a^4$.

35. Multiply the first two factors and the last two factors separately (the latter product will .be found to be (x^4+1) ; then multiply the two products.

36. The product of the first two factors will be found to be $c^2-a^2-b^2+2ab$ and of the last two $c^2-a^2-b^2-2ab$. The signs of the first three terms in each are the same, hence the expressions should be written as,

$$\{ (c^2-a^2-b^2)+(2ab) \} \text{ and } \{ (c^2-a^2-b^2)-(2ab) \} .$$

37. $(a-2b+3c)^2-(a+2b-3c)^2$

$=x^2-y^2$ [where $x=a-2b+3c, y=a+2b-3c$]

$=(x+y)(x-y)$

$=(2a)(-4b+6c)$ [Replacing x and y by their values]

$=-8ab+12ac.$

43. $3004 \times 2996=(3000+4)(3000-4).$

47. Given exp.$=(3148)^2-(2852)^2$

$$=(3148+2852)(3148-2852).$$

70. Formula 8. $(x+a)(x+b)=x^2+(a+b)x+ab.$

[Verify by actual multiplication]

Also, changing signs of (*i*) *a*, (*ii*) *b*, (*iii*) *a* and *b*, on both sides, we have the following Corollaries :—

Cor. 1. $(x-a)(x+b)=x^2+(-a+b)x-ab.$

Cor. 2. $(x+a)(x-b)=x^2+(a-b)x-ab.$

Cor. 3. $(x-a)(x-b)=x^2+(-a-b)x+ab.$

or $x^2-(a+b)x+ab.$

But instead of referring to these Corollaries it is better to remember that :—

The coefficient of *x*

= The *algebraic sum* of the second terms

The term free from *x*

= The *algebraic product* of the second terms.

EXERCISE 42

Write down the product of :—

1. $x+2$ and $x+3$. [*Solved*]   2. $x+5$ and $x+7$.
3. $a+6$ and $a+4$.   4. $m+9$ and $m+8$.

_____

5. $x-9$ and $x+5$. [*Solved*]   6. $x-8$ and $x+7$.
7. $a+8$ and $a-10$.   . 8. $p+5$ and $p-1$.
9. $b-6$ and $b-8$.   10. $t-1$ and $t-11$.

_____

Distribute the following products :—

11. $(x-8)(7-x)$. [*Solved*]   12. $(x-5)(6-x)$.
13. $(9-x)(-x-10)$.   14. $(-x-10)(-11-x)$.

_____

15. $(2x-5)(2x-7)$. [*Solved*] 16. $(3x-8)(3x+9)$.

17. $(4a-1)(7-4a)$.   18. $\left(1-\dfrac{5}{k}\right)\left(6-\dfrac{5}{k}\right).$

_____

19. $(3x+4y-5)(3x+4y+7)$. [*Solved*]
20. $(2a-5b+1)(2a-5b-10)$.
21. $(x^2-2x+3)(x^2-5x+3)$. [*Hint*]
22. $(3a^2-a-4)(3a^2+2a-4)$.
23. $(5a-4b-6)(5a+4b+1)$. [*Hint*]
24. $(4x^2-2x+1)(4x^2+x+3)$.

SOLUTIONS & HINTS—EXERCISE 42

1. $(x+2)(x+3)=x^2+(2+3)x+2\times3=x^2+5x+6$. **Ans.**

5. $(x-9)(x+5)=x^2+(-9+5)x+(-9)(5)$
$$=x^2-4x-45. \quad \textbf{Ans.}$$

11. $(x-8)(7-x)$
$=-(x-8)(x-7)$ [taking minus sign out of the second bracket]

$=-\{x^2+(-8-7)x+(-8)(-7)\}$
$=-(x^2-15x+56)$
$=-x^2+15x-56$. **Ans.**

15. $(2x-5)(2x-7)$
$=(a-5)(a-7)$ [writing a for $2x$.]
$=a^2+(-5-7)a+(-5)(-7)$
$=a^2-12a+35$
$=(2x)^2-12(2x)+35$ [Putting back the value of a]
$=4x^2-24x+35$.

Or thus directly :—

$(2x-5)(2x-7)=(2x)^2+(-5-7)(2x)+(-5)(-7)$
$$=4x^2-24x+35. \quad \textbf{Ans.}$$

19. $(3x+4y-5)(3x+4y+7)$
$=(a-5)(a+7)$ [where $a=3x+4y$]
$=a^2+(-5+7)a+(-5)(7)$
$=a^2+2a-35$
$=(3x+4y)^2+2(3x+4y)-35$ [Putting back the value of a]
$=9x^2+16y^2+24xy+6x+8y-35$.

21.   Reqd. product $=(x^3-2x+3)(x^2-5x+3)$
$$=\{(x^2+3)-2x\}\{(x^2+3)-5x\}$$
$$=(a-2x)(a-5x) \qquad [\text{where } a=x^2+3]$$
etc.

23.   Reqd. product $=\{x+(-4b-6)\}\{x+(4b+1)\}$
$$[\text{where } x=5a]$$

___

71.   **Formula '9.** $(x+a)(x+b)(x+c)$
$$=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$$

[Verify by actual multiplication]

Corollaries can be obtained as in formula 8 by changing signs. but it is far better to do without them and to remember that :—

(i) The coefficient of x^2 is the *algebraic sum* of the second terms.

(ii) The coefficient of x is the *algebraic sum of the algebraic products* of the second terms taken two by two.

(iii) The term free from $x=$The algebraic product of the second terms.

EXERCISE 43

Write down the values of the following products :—

1. $(x+2)(x+3)(x+4)$.   [Solved]
2. $(x+8)(x+4)(x+1)$.   3. $(x+2)(x+5)(x+6)$.
4. $(x+7)(x+1)(x+4)$.

___

5. $(x-3)(x+4)(x-6)$.   [Solved]
6. $(x+1)(x-5)(x+8)$.   7. $(x+2)(x-5)(x-6)$.
8. $(x+8)(x-2)(x-7)$.   9. $(x-1)(x-2)(x-8)$.
10. $(x-4)(x-5)(x-6)$.   11. $(x+1)(x+6)(x-7)$.
12. $(x-2)(x+8)(x+7)$.   13. $(a+11)(a-12)(a-18)$.
14. $(m-7)(m+18)(m-6)$.   15. $(k+8)(k+12)(k-15)$.

___

16. $(2a+1)(2a+3)(2a+5)$.   [*Hint*]

17. $(3k-1)(3k-2)(3k-4)$.

18. $\left(\dfrac{a}{b}+1\right)\left(\dfrac{a}{b}-4\right)\left(\dfrac{a}{b}-5\right)$

19. $(xy-3)(xy-4)(xy-7)$.

20. $(5a+3b)(5a-4b)(5a-12b)$.

___

Fill in the circular brackets by suitable terms in the follow-ing identities :—

21. $\{x+2\}\{x+(\quad)\}\{x+(\quad)\} \equiv$
$$x^3-4x^2+(\quad)x+10. \quad [Hint]$$

22. $\{x+(\quad)\}\{x+(\quad)\}\{x-3\} \equiv$
$$x^3-6x^2+(\quad)x+30.$$

23. $\{a+(\quad)\}\{a+(\quad)\}\{a+(\quad)\} \equiv a^3-7a+6$.

24. $\{x+a\}\{x+(\quad)\}\{x+(\quad)\} \equiv x^3+x^2-a^2x-a^3$.

___

SOLUTIONS & HINTS—EXERCISE 43

1. $(x+2)(x+3)(x+4)$
$$=x^3+(2+3+4)x^2+(2\times3+2\times4+3\times4)x+2\times3\times4$$
$$=x^3+9x^2+(6+8+12)x+24$$
$$=x^3+9x^2+26x+24. \quad \textbf{Ans.}$$

5. $(x-3)(x+4)(x-6)$
$$=x^3+(-3+4-6)x^2+\{(-3)(4)+(-3)(-6)+(4)(-6)\}x$$
$$+(-3)(4)(-6)$$
$$=x^3-5x^2+(-12+18-24)x+72$$
$$=x^3-5x^2-18x+72.$$

16. Put $2a=x$. At the end replace x by its value $2a$.

21. Let the first two missing terms be a and b.
$$\therefore \quad 2+a+b=-4 \qquad [\text{Coefficient of } x^2]$$
$$\therefore \qquad a+b=-4-2=-6 \qquad \dots(i)$$
Also $2\times a\times b=10$ [The term free from x]
$$\therefore \qquad ab=\dfrac{10}{2}=5 \qquad \dots(ii)$$

From (*i*) and (*ii*) we see that the sum of *a* and $\overset{\triangledown}{b}$ is -6 and product 5. Hence clearly their values are -5 and -1 (or -1 and -5) respectively.

The coefficient of x can now be found.

72.　Formula 10.　$(a+b)^3=a^3+3a^2b+3ab^2+b^3$

　　　　　　　　　or　$a^3+b^3+3ab(a+b)$

Formula 11.　　　$(a-b)^3=a^3-3a^2b+3ab^2-b^3$

　　　　　　　　　or　$a^3-b^3-3ab(a-b)$.

[Verify by actual multiplication]

The following corollaries obtained by simple transpositions are also important :—

Cor. 1.　$a^3+b^3=(a+b)^3-3ab(a+b)$　　[From Formula 10]

Cor. 2.　$a^3-b^3=(a-b)^3+3ab(a-b)$　　[From Formula 11]

Note.　Formula 11 can be obtained from formula 10, and Cor. 2 from Cor. 1, by changing *b* into $-b$.

[See Note on Art. 68]

EXERCISE 44

Find the cube of :—

1.　$2x+3y$.　[*Solved*]　　　　　2.　$5x+4z$.
3.　$3m+7n$.　　　　　　　　　　4.　$ax+by$.
5.　x^2+2y^2.　　　　　　　　　　6.　$\frac{1}{2}x^2+\frac{1}{3}y$.

7.　$3x^2-2y$.　[*Solved*]　　　　8.　a^2-3b.
9.　$-5+2ab$.　　　　　　　　　　10.　$\frac{1}{3}m^2-n$.

11.　$\frac{1}{3}x-\frac{1}{4}a^2$.　　　　　　　　12.　$2x-\dfrac{1}{3x}$.

13.　Prove that $(a+b+c)^3=a^3+b^3+c^3+3ab(a+b)+3bc(b+c)+3ca(c+a)+6abc$.　[*Solved*]

❩ Use the result of the last question (No. 13) to find the walues of :—

14.　$(a-b+c)^3$.　[*Hint*].　　　15.　$(a+b-c)^3$.

16. $-a+b+c)^3$.   17. $(a-b-c)^3$.

18. $(2a-3b+c)^3$.   19. $(3x-2y-1)^3$.

Find the values of the following without using the result of question 13 :—

20. $(a-2b-3c)^3$. [Hint]   21. $(3x+2y-4z)^3$.

22. $(x^2-xy+y^2)^3$.

Find the value of a^3+b^3, when :—

23. $a+b=4$ and $ab=5$. [Solved]

24. $a+b=5$ and $ab=7$.   25. $a+b=-6$ and $ab=8$.

26. $a+b=p$ and $ab=q$.   27. $a+b=2k$ and $ab=3k^2$.

Find the value of a^3-b^3, when :—

28. $a-b=4$, $ab=12$. [Solved] 29. $a-b=1$, $ab=-6$.

30. $a-b=\frac{3}{2}$, $ab=1$.   31. $a-b=p$, $ab=q$.

32. $a-b=2l$, $ab=3l^2$.

33. If $x+\frac{1}{x}=2$, find the value of $x^3+\frac{1}{x^3}$. [Solved]

34. If $x+\frac{1}{x}=3$, find the value of $x^3+\frac{1}{x^3}$.

35. If $a+\frac{1}{a}=-4$, find the value of $a^3+\frac{1}{a^3}$.

36. If $2z+\frac{1}{2z}=5$, find the value of $8z^3+\frac{1}{8z^3}$.

37. If $3x+\frac{1}{4x}=2k$, find the value of $27x^3+\frac{1}{64x^3}$.

38. If $x-\frac{1}{x}=-3\frac{1}{2}$, evaluate $x^3-\frac{1}{x^3}$. [Solved].

39. If $p-\frac{1}{p}=6$, evaluate $p^3-\frac{1}{p^3}$.

40. If $5t-\frac{1}{3t}=\frac{a}{2}$, evaluate $125t^3-\frac{1}{27t^3}$.

41. If $y^2 - \dfrac{3}{y^2} = -\dfrac{k}{3}$, evaluate $y^6 - \dfrac{27}{y^6}$.

---

42. If $x - \dfrac{1}{x} = 2$, evaluate $x^6 + \dfrac{1}{x^6}$. [Hint]

43. If $a - \dfrac{1}{a} = -1$, evaluate $a^6 + \dfrac{1}{a^6}$.

44. If $m^2 + \dfrac{1}{m^2} = 18$, evaluate $m^3 - \dfrac{1}{m^3}$. [Hint]

45. If $x^2 + \dfrac{1}{x^2} = 11$, evaluate $x^3 - \dfrac{1}{x^3}$.

46. If $x^2 + \dfrac{1}{x^2} = 1$, evaluate $x^3 + \dfrac{1}{x^3}$. [Hint]

47. If $a^2 + \dfrac{1}{a^2} = 23$, evaluate $a^3 + \dfrac{1}{a^3}$.

Find the value of ab, when :—
48. $a^3 + b^3 = 28$ and $a + b = 4$. [Solved]
49. $a^3 + b^3 = 9$ and $a + b = 3$.
50. $a^3 + b^3 = -2k^3$ and $a + b = k$.
51. $a^3 - b^3 = 26$ and $a - b = 2$. [Hint]
52. $a^3 - b^3 = 9l^3$ and $a - b = 3l$.

---

53. If $a + b = 4$, evaluate $a^3 + b^3 + 12ab$. [Solved]
54. If $a + b = 5$, evaluate $a^3 + b^3 + 15ab$.
55. If $p - q = 6$, evaluate $p^3 - q^3 - 18pq$.
56. If $m - n + 7 = 0$, show that $m^3 - n^3 + 21mn + 343 = 0$.
   [Hint]
57. If $a + b + c = 0$, show that $a^3 + b^3 + c^3 = 3abc$. [Hint]

Find the value of :—

58. $x^3 + 6x^2 + 12x + 8$ for $x = 28$.   [*Hint*]

59. $x^3 - 9x^2 + 27x - 27$ for $x = 23$.

60. $8x^3 - 36x^2y + 54xy^2 - 27y^3$ for $x = 18$, $y = 12$.

61. $27a^3 - 135a^2 + 225a - 10$ for $a = 15$.   [*Hin*]

62. $64m^3 - 96m^2 + 48m + 1$ for $m = 18$.

Simplify :—

63. $(2a+3b)^3 + 3(2a+3b)^2(2a-3b) + 3(2a+3b)(2a-3b)^2$
$$+ (2a-3b)^3. \quad [Solved]$$

64. $(3a+5b)^3 + 3(3a+5b)^2(2a-5b) + 3(3a+5b)(2a-5b)^2$
$$+ (2a-5b)^3$$

65. $(2p+q)^3 - 3(2p+q)^2(2p-3q) + 3(2p+q)(2p-3q)^2$
$$- (2p-3q)^3$$

66. $(4x-7y)^3 - 3(4x-7y)^2(5x-7y) + 3(4x-7y)(5x-7y)^2$
$$- (5x-7y)^3$$

67. $(3x+y)^3 - (3x-y)^3 - 6y(9x^2 - y^2)$.   [*Hint*]

68. $(p+2q)^3 + (p-2q)^3 + 6p(p^2 - 4q^2)$.

69. $(m+2n)^3 + (2m-n)^3 + 3(m+2n)(2m-n)(3m+n)$. [*Hint*]

70. $(2x-5y)^3 + (x+2y)^3 + 9(2x-5y)(x+2y)(x-y)$.

SOLUTIONS & HINTS—EXERCISE 44

1. $(2x+3y)^3 = (a+b)^3$ [where $a = 2x$, $b = 3y$]
$$= a^3 + 3a^2b + 3ab^2 + b^3$$
$$= (2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3$$
[writing back the values of a and b]
$$= 8x^3 + 3(4x^2)(3y) + 3(2x)(9y^2) + 27y^3$$
$$= 8x^3 + 36x^2y + 54xy^2 + 27y^3.$$

Or thus :—

$(2x+3y)^3 = (2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3 = $ etc.

That is, the steps of substitution and re-substitution may be done mentally.

7. $(3x^2-2y)^3=(3x^2)^3-3(3x^2)^2(2y)+3(3x^2)(2y)^2-(2y)^3$
$$=27x^6-3(9x^4)(2y)+3(3x^2)(4y^2)-8y^3$$
$$=27x^6-54x^4y+36x^2y^2-8y^3.$$

13. $(a+b+c)^3=\{(a+b)+(c)\}^3$
$$=(a+b)^3+3(a+b)^2(c)+3(a+b)(c)^2+(c)^3$$
$$=(a^3+3a^2b+3ab^2+b^3)+3(a^2+2ab+b^2)c+3(a+b)c^2+c^3$$
$$=a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$$
$$=a^3+b^3+c^3+3a^2b+3ab^2+3b^2c+3bc^2+3c^2a+3ca^2+6abc.$$
<div style="text-align:right">[Re-arranging the terms]</div>
$$=a^3+b^3+c^3+3ab(a+b)+3bc(b+c)+3ca(c+a)+6abc.$$

14.   By the last solution we have :—
$$(a+b+c)^3=a^3+b^3+c^3+3ab(a+b)+3bc(b+c)+3ca(c+a)$$
$$+6abc.$$

If we change the sign of b on both sides, the left-hand side becomes $(a-b+c)^3$; on the right-hand side b^3 becomes $(-b)^3$ $=-b^3$, $3ab$ becomes $3a(-b)$ or $-3ab$, etc. etc., and we get the required result.

20. $(a-2b-3c)^3=\{(a-2b)-(3c)\}^3$
$$=(a-2b)^3-3(a-2b)^2(3c)+3(a-2b)(3c)^2-(3c)^3. \text{ etc.}$$

23. $a^3+b^3=(a+b)^3-3ab(a+b)$ [Cor. 1 of Formula 10]
$$=(4)^3-3\times5\times(4) \qquad [\because \quad a+b=4 \text{ and } ab=5]$$
$$=64-60=4.$$

<div style="text-align:center">*Or thus :—*</div>

$$a+b=4 \qquad \text{[given]}$$
$$\therefore \quad (a+b)^3=(4)^3$$
$$\therefore \quad a^3+b^3+3ab(a+b)=64$$
or $a^3+b^3+3\times5\times(4)=64$ $[\because \quad ab=5 \text{ and } a+b=4]$
or $a^3+b^3+60=64$
or $a^3+b^3=64-60=4.$

28. $a^3-b^3=(a-b)^3+3ab(a-b)$ [Cor. 2 of Formula 11.]
$$=(4)^3+3\times12\times(4) \qquad [\because \quad a-b=4, \text{ and } ab=12]$$
$$=64+144=208.$$

Or thus :—

$$a-b=4 \quad \text{[given]}$$

$$\therefore \quad (a-b)^3=(4)^3$$

$$\therefore \quad a^3-b^3-3ab(a-b)=64, \text{ etc.} \quad \text{[See alternative solution of Q. 28]}$$

33. $\quad a^3+b^3=(a+b)^3-3ab(a+b) \qquad \text{[Cor 1, Formula 10]}$

$$\therefore \quad x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)^3-3.x.\frac{1}{x}\left(x+\frac{1}{x}\right)$$

$$\left[\text{writing } x \text{ for } a \text{ and } \frac{1}{x} \text{ for } b\right]$$

$$=\left(x+\frac{1}{x}\right)^3-3\left(x+\frac{1}{x}\right).$$

$$=(2)^3-3(2) \qquad\qquad \left[\because \quad x+\frac{1}{x}=2\right]$$

$$=8-6=2. \quad \textbf{Ans.}$$

Or thus :—

$$x+\frac{1}{x}=2 \quad \text{[given]}$$

$$\therefore \qquad \left(x+\frac{1}{x}\right)^3=(2)^3$$

$$\therefore \qquad x^3+\frac{1}{x^3}+3.x.\frac{1}{x}\left(x+\frac{1}{x}\right)=8.$$

etc.

38. $\quad a^3-b^3=(a-b)^3+3ab(a-b) \qquad \text{[Cor. 2, Formula 11]}$

$$\therefore \qquad x^3-\frac{1}{x^3}=\left(x-\frac{1}{x}\right)^3+3.x.\frac{1}{x}\left(x-\frac{1}{x}\right)$$

$$\left[\text{writing } x \text{ for } a \text{ and } -\frac{1}{x} \text{ for } b\right]$$

$$=\left(x-\frac{1}{x}\right)^3+3\left(x-\frac{1}{x}\right)$$

$$=(-3l)^3+3(-3l) \qquad \left[\because \quad x-\frac{1}{x}=-3l\right]$$

$$=-27l^3-9l. \quad \textbf{Ans.}$$

42   First find the value of $x^2 + \dfrac{1}{x^2}$ [See Ex. 40, Q. 126]

Then apply Cor. 1 of Formula 10 as in Q. 33 of this exercise.

44.   First find the value of $m - \dfrac{1}{m}$ [See Ex. 40, Q. 134]

Then proceed as in Q. 38 of this exercise.

46.   First find the value of $x + \dfrac{1}{x}$ [See Ex. 40, Q. 131]

This will come out to be $\sqrt{3}$ or $-\sqrt{3}$. Also note that $(\sqrt{3})^3 = \sqrt{3} \times \sqrt{3} \times \sqrt{3} = 3\sqrt{3}$ and $(-\sqrt{3})^3) = -3\sqrt{3}$.

48. $a + b = 4$ [given]

$\therefore$ $(a+b)^3 = (4)^3$

or $a^3 + b^3 + 3ab(a+b) = 64$

or $28 + 3ab(4)$ $= 64$ $[\because \ a^3 + b^3 = 28$ and $a+b=4]$

or $28 + 12ab$ $= 64$

or $12ab$ $= 64 - 28 = 36$

$\therefore$ ab $= \frac{36}{12} = 3.$

51. $a - b = 2$ [given]

$\therefore$ $(a-b)^3 = (2)^3$

$\therefore$ $a^3 - b^3 - 3ab(a-b) = 8$

            etc.

53. $a^3 + b^3 + 12ab = a^3 + b^3 + 3ab \times 4 = a^3 + b^3 + 3ab(a+b)$

                                      $[\because \ a + b = 4]$

$= (a+b)^3 = (4)^3 = 64.$ **Ans.**

Or thus :—

          $a + b = 4$

$\therefore$ $(a+b)^3 = (4)^3$

or $a^3 + b^3 + 3ab(a+b) = 64$

or $a^3 + b^3 + 3ab(4)$ $= 64$ $[\because \ a+b=4]$

or $a^3 + b^3 + 12ab$ $= 64.$

56. $m-n+7=0$ [given]

$\therefore \qquad m-n=-7$

$\therefore \quad (m-n)^3=(-7)^3$

$\qquad$ etc.

57. $a+b+c=0$

$\therefore \qquad a+b=-c$

$\therefore \quad (a+b)^3=(-c)^3$, etc.

58. $x^3+6x^2+12x+8$

$=(x)^3+3(x)^2(2)+3(x)(2)^2+(2)^3$

$=(x+2)^3=(28+2)^3 \qquad [\because \quad x=28]$

$=(30)^3=27000.$

61. Given exp. $=27a^3-135a^2+225a-125+115.$

$\qquad =(3a-5)^3+115$

$\qquad\qquad$ etc.

63. $(2a+3b)^3+3(2a+3b)^2(2a-3b)+3(2a+3b)(2a-3b)^2$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +(2a-3b)^3$

$=x^3+3x^2y+3xy^2+y^3 \qquad$ [where $x=2a+3b,\ y=2a-3b$]

$=(x+y)^3$

$=(2a+3b+2a-3b)^3 \quad$ [Re-writing the values of x and y]

$=(4a)^3=64a^3.$

67. $-6y(9x^2-y^2)$ can be written as $-3(3x+y)(3x-y)(2y)$

$\qquad [\because \quad (3x+y)(3x-y)=9x^2-y^2$ and $-3(2y)=-6y]$

Now the given expression will be found equal to

$a^3-b^3-3ab(a-b)$ [where $a=3x+y$ and $b=3x-y$, and

$\qquad\qquad\qquad\qquad$ therefore $a-b=3x+y-3x+y=2y]$

$=(a-b)^3$, etc.

69. Put $m+2n=a$ and $2m-n=b$,

so that $a+b=m+2n+2m-n=3m+n.$

The given exp. $=a^3+b^3+3ab(a+b)=(a+b)^3$, etc.

73. Formula 12. $(a+b)(a^2-ab+b^2)=a^3+b^3.$

Formula 13. $(a-b)(a^2+ab+b^2)=a^3-b^3).$

$\qquad\qquad\qquad\qquad$ [Verify by actual multiplication]

Note. Formula 13 can be obtained from Formula 12 by changing the sign of b. [See Note on Art. 68].

<div align="center">

EXERCISE 45

</div>

Multiply together :—

1. $x+1$ and x^2-x+1. [*Solved*]

2. $x+2$ and x^2-2x+4.   3. $2x+3$ and $4x^2-6x+9$

4. $3a+4b$ and $9a^2-12ab+16b^2$.

5. $5a+2b$ and $25a^2-10ab+4b^2$.

6. $ab+3c^2$ and $a^2b^2-3abc^2+9c^4$.

---

7. $2a-3b$ and $4a^2+6ab+9b^2$. [*Solved*] .

8. $3x-4y$ and $9x^2+12xy+16y^2$.

9. $x^2y^2+xyz^2+z^4$ and $xy-z^2$.

10. $a^4+a^2b^2+b^4$ and a^2-b^2.

11. $3a^2-4bc$ and $9a^4+12a^2bc+16b^2c^2$.

12. $3p^3-2$ and $9p^6+6p^3+4$.

Find the values of the following products :—

13. $\left(x+\dfrac{1}{x} \right)\left(x^2+\dfrac{1}{x^2} -1 \right)$. [*Hint*]

14. $\left(\dfrac{a}{3}-\dfrac{3}{a} \right)\left(\dfrac{a^2}{9}+\dfrac{9}{a^2} +1 \right)$.

15. $\left(\dfrac{1}{2}a+\dfrac{1}{3}b \right)\left(\dfrac{1}{4}a^2+\dfrac{1}{9}b^2-\dfrac{1}{6}ab \right)$.

16. $\left(2x^2-\dfrac{3}{x} \right)\left(4x^4+\dfrac{9}{x^2}+6x \right)$.

Find the values of the following continued products :—

17   $(x-1)(x^2+x+1)(x^6+x^3+1)$. [*Solved*]

18   $(a+b)(a^2-ab+b^2)(a^6-a^3b^3+b^6)$.

19. $(x+1)(x-1)(x^4+x^2+1)$.

20. $(c-2)(c^4+4c^2+16)(c+2)$.

21. $(1+x)(1-x)(1+x+x^2)(1-x+x^2)$. [*Hint*]

22. $(x+2y)(x-2y)(x^2-2xy+4y^2)(x^2+2xy+4y^2)$.

23. $(3m-2n)(3m+2n)(9m^2-6mn+4n^2)(9m^2+6mn+4n^2)$.

24. $\left(a-\dfrac{1}{a}\right)\left(a+\dfrac{1}{a}\right)\left(a^2+\dfrac{1}{a^2}+1\right)$

$$\left(a^2+\dfrac{1}{a^2}-1\right)\left(a^6+\dfrac{1}{a^6}\right).$$

Find the values of the following products :—

25. $(a+b+1)(a^2+2ab+b^2-a-b+1)$. [*Solved*]

26. $(2x+y-1)(4x^2+4xy+y^2+2x+y+1)$.

27. $(x-2y+1)(x^2+2x+1+2xy+2y+4y^2)$.

28. $\left(x+\dfrac{1}{x}+1\right)\left(x^2+\dfrac{1}{x^2}+8-x-\dfrac{1}{x}\right)$. [*Hint*]

SOLUTIONS & HINTS—EXERCISE 45

1. $(x+1)(x^2-x+1)=(x+1)\{(x)^2-(x)(1)+(1)^2\}$
$=(x)^3+(1)^3=x^3+1$. **Ans.**

7. $(2a-3b)(4a^2+6ab+9b^2)$
$=\{(2a)-(3b)\}\{(2a)^2+(2a)(3b)+(3b)^2\}$
$=(2a)^3-(3b)^3=8a^3-27b^3$.

13. In the second bracket -1 may be written as

$$-(x)\times\left(\dfrac{1}{x}\right).$$

17. $(x-1)(x^2+x+1)(x^6+x^3+1)$
$=(x^3-1)(x^6+x^3+1)$ [multiplying the first two factors, as in Q. 1]
$=\{(x^3)-(1)\}\{(x^3)^2+(x^3)(1)+(1)^2\}$
$=(x^3)^3-(1)^3$
$=x^9-1$. **Ans.**

21. Multiply the first and fourth factors ; the product is $1+x^3$; also multiply the second and third factors ;

the product is $1-x^3$ Multiply these two products by the formula $(a+b)(a-b)=a^2-b^2$.

25. $(a+b+1)(a^2+2ab+b^3-a-b+1)$

$= \{ (a+b)+(1) \} \{ (a^2+2ab+b^2)-(a+b)+1 \}$

$= \{ (a+b)+(1) \} \{ (a+b)^2-(a+b)(1)+(1)^2 \}$

$=(a+b)^3+(1)^3$

$=a^3+3a^2b+3ab^2+b^3+1$.

----

74. **Formula 14.** $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$
$$=a^3+b^3+c^3-3abc.$$
[Verify by actual multiplication]

Left-hand side of the above formula may be written as :—
$$(a+b+c) \{ (a^2+b^2+c^2)-(ab+ac+bc) \}$$

and also as :—
$$\tfrac{1}{2}(a+b+c) \{ (a-b)^2+(b-c)^3+(c-a)^2 \}$$
[See Q. 147, of Ex. 40]

EXERCISE 46

Multiply :—

1. $(x+y-z)(x^2+y^2+z^2-xy+xz+yz)$.   [*Solved*]
2. $(p-q+r)(p^2+q^2+r^2+pq-pr+qr)$.
3. $(l-m-n)(l^2+m^2+n^2+lm+ln-mn)$.
4. $(-a+b+c)(a^2+b^2+c^2+ab+ac-bc)$.

----

5. $(2a-3b+4c)(4a^2+9b^2+16c^2+6ab-8ac+12bc)$. [*Solved*]
6. $(a-2b+3c)(a^2+4b^2+9c^2+2ab-3ac+6bc)$.
7. $(4x-5y-1)(16x^2+25y^2+1+20xy+4x-5y)$.
8. $(3p-2q+5)(9p^2+4q^2+25+6pq-15p+10q)$.
9. $(xy+yz-zx)(x^2y^2+y^2z^2+z^2x^2-xy^2z+x^2yz+z^2xy)$.
10. $(a^2-b^2-c^2)(a^4+b^4+c^4+a^2b^2+a^2c^2-b^2c^2)$.

Find the value of $a^3+b^3+c^3-3abc$, when :—

11. $a+b+c=2$ and $ab+bc+ca=1$. [*Solved*]

12. $a+b+c=5$ and $ab+bc+ca=5$

13. $a+b+c=8$ and $ab+bc+ca=22$.

14. $a+b+c=3$ and $a^2+b^2+c^2=5$. [*Hint*]

15. $a+b+c=6$ and $a^2+b^2+c^2=10$.

16. $a^2+b^2+c^2=10$, $ab+bc+ca=3$. [*Hint*]

17. $a^2+b^2+c^2=7$, $ab+bc+ca=21$.

18. $a=475$, $b=476$, $c=478$. [*Solved*]

19. $a=551$, $b=553$, $c=554$,

20. $a=564$, $b=568$, $c=578$.

21. $a=2·36$, $b=-3·79$, $c=1·43$. [*Hint*]

22. $a=3·71$, $b=·64$, $c=-4·35$.

23. If $x=43·7$, $y=31·5$, $z=-75·2$, show that
$x^3+y^3+z^3=3xyz$. [*Hint*]

24. If $a=·143$, $b=·234$, $c=·325$, show that
$a^3-8b^3+c^3+6abc=0$.

Find the value of $x^3+y^3+z^3$ when :—

25. $x+y+z=9$, $xy+yz+zx=25$, $xyz=40$. [*Hint*]

26. $x+y+z=10$, $x^2+y^2+zz=50$, $xyz=100$.

---

27. If $x+y+z=0$, prove that $x^3+y^3+z^3=3xyz$. [*Solved*]

Use the result of last question to prove that :—

28. $(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a)$. [*Solved*]

29. $(a-2b)^3+(2b-3c)^3+(3c-a)^3=3(a-2b)(2b-3c)(3c-a)$.

30. $(x+y-2z)^3+(y+z-2x)^3+(z+x-2y)^3$
$$=3(x+y-2z)(y+z-2x)(z+x-2y).$$

31. If $x+y=1$, show that $x^3+y^3+3xy=1$. [*Hint*]

32. If $2a+3b=4c$, show that $8a^3+27b^3+72abc=64c^3$.

---

33. If $a=237$ $b=327$, $c=-560$, find the value of
$$\frac{a^3+b^3+c^3-3abc}{a^2+b^2+c^2-ab-bc-ca}.$$ [*Solved*]

34. If $a=660, b=-315, c=-351$, find the value of
$$\frac{a^3+b^3+c^3-3abc}{ab+bc+ca-a^2-b^2-c^2}. \quad [Hint]$$

35. If $x=3\cdot16, y=2\cdot43, z=4\cdot41$, find the value of
$$\frac{(3abc)-(a^3+b^3+c^3)}{(ab+bc+ca)-(a^2+b^2+c^2)}.$$

36. If $x=2\cdot13, y=3\cdot14, z=4\cdot15$, find the value of
$$\frac{x^3-8y^3+z^3+6xyz}{x^2+4y^2+z^2+2xy-xz+2yz}.$$

SOLUTIONS & HINTS—EXERCISE 46

1. $(x+y-z)(x^2+y^2+z^2-xy+xz+yz)$
$$=\{(x)+(y)+(-z)\}\{(x)^2+(y)^2+(-z)^2-(x)(y)-(x)(-z)$$
$$-(y)(-z)\}$$
$$=(x)^3+(y)^3+(-z)^3-3(x)(y)(-z)$$
$$=x^3+y^3-z^3+3xyz.$$

5. $(2a-3b+4c)(4a^2+9b^2+16c^2+6ab-8ac+12bc)$
$$=\{(2a)+(-3b)+(4c)\}\{(2a)^2+(-3b)^2+(4c)^2$$
$$-(2a)(-3b)-(2a)(4c)-(-3b)(4c)\}$$
$$=(2a)^3+(-3b)^3+(4c)^3-3(2a)(-3b)(4c)$$
$$=8a^3-27b^3+64c^3+72abc.$$

11. Let us first find the value of $a^2+b^2+c^2$.

Now, $a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)$ [Art. 67, Cor. 5]
$$=(2)^2-2(1) \quad [\because a+b+c=2, ab+bc+ca=1]$$
$$=4-2=2.$$

$\therefore a^3+b^3+c^3-3abc=(a+b+c)\{(a^2+b^2+c^2)-(ab+bc+ca)\}$
$$=(2)\{(2)-(1)\}=2(1)=2. \quad \text{Ans.}$$

14. First find the value of $ab+bc+ca$ by Art. 67, Cor. 6.

16. First find the value of $a+b+c$.

18. $a^3+b^3+c^3-3abc$
$$=\tfrac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$$
$$=\tfrac{1}{2}(475+476+478)\{(475-476)^2+(476-478)^2$$
$$+(478-475)^2\}$$

$$= \tfrac{1}{2} \times 1429 \times \{ (-1)^2+(-2)^2+(3)^2 \}$$
$$= \tfrac{1}{2} \times 1429 \times (1+4+9)$$
$$= \tfrac{1}{2} \times 1429 \times 14 = 10003. \quad \textbf{Ans.}$$

Note. Whenever the values of a, b and c are large, but their differences are small, the above form of the formula should be used.

21 Here any form of the formula will do; for the first factor $= a+b+c = 2\cdot36 - 3\cdot79 + 1\cdot43 = 0$, therefore, whatever the second factor, the product $= 0$.

23. It is required to prove that $x^3+y^3+z^3=3xyz$

or that $x^3+y^3+z^3-3xyz=0$,

therefore the question is of the same form as No. 21 or 22.

25. First find the value of $x^2+y^2+z^2$ by applying Art. 67, Cor. 5.

Then substitute all the known values in the formula :—
$$x^3+y^3+z^3-3xyz=(x+y+z)\{ (x^2+y^2+z^2)-(xy+yz+zx) \}$$
The resulting equation will give the value of $x^3+y^3+z^3$.

27. I Method

To prove that $x^3+y^3+z^3=3xyz$
or that $x^3+y^3+z^3-3xyz=0$
or that $(x+y+z)(x^2+y^2+z^2-xy-yz-zx)=0$
But this is so, because $x+y+z=0$ [given]
and $0 \times$ any finite quantity $=0$.

II Method. See Q. 57, Exercise 44.

Note. The result of this example is important and may be stated in words as follows :—

" *If the sum of three quantities is equal to zero, the sum of their cubes is equal to three times their product.*"

28. Consider the quantities $a-b$, $b-c$ and $c-a$.
Their sum $= a-b+b-c+c-a = 0$

$\therefore$ According to the result of last question :—
$$(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a).$$

31. I Method :— $x + y = 1$

$$\therefore \quad (x + y)^3 = (1)^3, \text{ etc.}$$

II Method :— $x + y = 1$

$$\therefore \quad x + y - 1 = 0$$

$\therefore$ the sum of x, y and -1 is 0

Apply the result of Q. 27.

33. $\dfrac{a^3 + b^3 + c^3 - 3abc}{a^2 + b^2 + c^2 - ab - bc - ca}$

$$= \frac{(a+b+c)(a^2+b^2+c^2-ab-bc-ca)}{(a^2+b^2+c^2-ab-bc-ca)}$$

$$= (a+b+c) = 237 + 327 - 560 = 4. \quad \textbf{Ans.}$$

34. Note that the denominator $= -(a^2 + b^2 + c^2 - ab - bc - ca)$.

CHAPTER XIV

FACTORS

75. The student has already learnt how to get the product of two or more given factors ; we shall now show how the process may in some cases be reversed, that is, given an integral expression, we can find the *elementary* factors of which it is the product.

76. Type 1.

Factor formula :—$ka + kb + kc = k(a+b+c)$, that is :—

If a quantity is common to all the terms of an expression, it is a factor of that expression ; the other factor is found by dividing the expression by that quantity.

For example, take the expression—

$$9x^2yz^2 - 12x^3y^2z^2 - 18x^4z^2.$$

The largest number common to the coefficients 9, 12 and 18 (*i e*, the H. C. F. of 9, 12 and 18) is 3 ; the powers of x in the three terms are x^2, x^3 and x^4 and clearly x^2 (the lowest power of x) is common to them ; y does not occur in the third term, therefore no power of y is common ; the powers of z are z^3, z^2 and z^2 and clearly z^2 is common to all. Thus we

have $3x^2z^2$ as the common factor. Dividing the given expression by $3x^2y^2$ we get $3yz-4xy^2-6x^2$. Hence the required factors are :—

$$3x^2z^2(3yz-4xy^2-6x^2). \quad \textbf{Ans.}$$

Note. Multiply the factors obtained (*i. e.*, remove the bracket in the answer) and see if you get back the original expression. This reverse process is often useful to verify the answer.

EXERCISE 47

Factorise :—

1. $4m+6$.
2. $10x-25$.
3. $am+mc$.
4. $abc-bcd$.
5. ab^3+b^2. [*Hint*]
6. $pa+qa-a$.
7. $10xy-15xz$.
8. $a^2b^3+a^3b^2$.
9. xyz^2-xy^2z.
10. $ap^2q^3-bp^3q^4$.
11. $6m^2-15mn+27m$.
12. $3a^2+9xy+3$.
13. $21m-14mn+28m^2$.
14. $2x^3y^2-4x^2y^2+8xy^3-16y^4$.
15. $12x^2yz+16xy^2z-24xyz^2$.
16. $143x^{10}y^6+121x^5y^8z$.
17. $52a^5b^7c^7-78b^5a^7c^7-91c^5a^7b^7$.
18. $96x^9y^8z^3-112x^8y^5z^2+128x^6y^5z^3$.

---

19. $a(b+c)-b(b+c)$. [*Solved*]
20. $x(y+z)-y(y+z)$.
21. $p(2q+r)+(2q+r)$.
22. $ab(3c-d)-(3c-d)$.
23. $a^2(b-4c)-b(b-4c)$.
24. $7m(5x-1)+2(5x-1)$.
25. $5ab(x^3+y^3)-6mn(x^2+y^2)$.
26. $10a^2(a-b+c)-3b(a-b+c)$.
27. $x(2a+b)-y(2a+b)-z(2a+b)$.
28. $2a(x^2+y^2)-3b(x^2+y^2)+(x^2+y^2)$.

---

29. $a(b-c)^2+b(b-c)$. [*Solved*]
30. $x(y+z)^2-y(y+z)$.
31. $3x(2y-z)^2-(2y-z)$.

32. $a(4x-y)^3-b(4x-y)^2$.   **33.** $8x(y+z)^3+(y+z)^2$.

34. $(a-b)^3+2(a-b)^2-3(a-b)$.

35. $3(x+y)^3+5(x+y)^2+(x+y)$.

---

36. $6x^2(3a-4b)^2-9xy(3a-4b)$.   [*Solved*]

37. $c^3b^2(p-2q)^2+a^2b^3(p-2q)$.

38. $5m^2(a-3b)-5m(a-3b)^2$.

39. $6a^2b^2(2b-5c)-18ab(2b-5c)+9b^2(2b-5c)$.

40. $15(p^2-qr)^3-10(p^2-qr)^2$.

41. $x(x+y)^3-3xy(x+y)$.

42. $(2a+3b)(x-y)+(a-2b)(x-y)+(3a+b)(x-y)$.

43. $(x-y)^3(x-z)^2-(x-z)^3(x-y)^2$.

44. $4a(a-1)^3(b-1)^2+6ab(a-1)^2(b-1)^3$.

SOLUTIONS & HINTS—EXERCISE 47

5. The common factor is b^2 and when the expression divided by b^2 the result is $a+1$ and not a only, mistake which is common among students.

19. $a(b+c)-b(b+c)$

$=ax-bx$ [where $x=b+c$]

$=x(a-b)$

$=(b+c)(a-b)$ [Re-writing the value of x].

                  Or, directly :—

$a(b+c)-b(b+c)=(b+c)(a-b)$

Note. The beginner is advised to follow the first method (that of substitution) in the first instance.

29. $a(b-c)^2+b(b-c)$

$=ax^2+bx$ [where $x=b-c$]

$=x(ax+b)$

$=(b-c)\{a(b-c)+b\}$ [Putting back the value of x]

$=(b-c)(ab-ac+b)$. **Ans.**

Or, **proceed** directly without making any substitution.

36. $6x^2(3a-4b)^2-9xy(3a-4b)$

$\quad = 6x^2z^2 - 9xyz \qquad$ [where $z^* = 3a - 4b$]

$\quad = 3xz(2xz - 3y)$

$\quad = 3x(3a-4b)\{ 2x(3a-4b)-3y \}$

$\qquad\qquad\qquad\qquad$ [Putting back the value of z]

$\quad = 3x(3a-4b)(6ax-8bx-3y).$ **Ans.**

77. Type 2. Expressions of the form $a^2+2ab+a^2$ or $a^2-2ab+b^2$ can be factorised at once by the following formulas :—

$\qquad$ **$a^2+2ab+b^2=(a+b)^2$** $\qquad$ [$i.e.,\quad (a+b)(a+b)$]

$\qquad$ **$a^2-2ab+b^2=(a-b)^2$** $\qquad$ [$i.e.,\quad (a-b)(a-b)$].

EXERCISE 48

Factorise :—

1. $4a^2+12ab+9b^2$. [*Solved*] **2.** $9a^2+24ab+16b^2$

3. $25x^2+4y^2+20xy$. $\qquad$ **4.** $36x^2+25+60x$.

5. $144a^2+24a+1$. $\qquad$ **6.** $9a^2+\dfrac{4}{a^2}+12$.

7. $4x^4+9y^4+12x^2y^2$. $\qquad$ **8.** $x^4+\dfrac{9}{x^4}+6$.

---

9. $25x^2-30xy+9y^2$.   [*Solved*]

10. $36x^2-60xy+25y^2$. $\qquad$ **11.** $9x^4-24x^2+16$.

12. $25a^4-30a^2b^2+9b^4$. $\qquad$ **13.** x^6-8x^3+16.

14. $x^4-10+\dfrac{25}{x^4}$.

---

15. $48a^3-120a^2b+75ab^2$.   [*Solved*]

16. $16a^2b+80ab^2+100b^3$. $\qquad$ **17.** $45a^4b+5ab^3-30a^2b^2$

18. $6x^3y+54xy^3+36x^2y^2$. $\qquad$ **19.** $6x^4-12x^3+6x^2$.

$\quad$ * Remember that an expression must always be replaced by a letter which does not occur in the given expression. For example, here $(3a-4b)$ should not be replaced by x or y or a or b.

20. $20xy^4 + 5xy^2 + 20xy^3$.   **21.** $8a^6 + 4a^4b^2 + 2a^2b^4$.

22. $3m^6 - 12m^4n^2 + 12m^2n^2$.

SOLUTIONS & HINTS—EXERCISE 48

1. $4a^2 + 12ab + 9b^2$

$= (2a)^2 + 2(2a)(3b) + (3b)^2$

$= (2a + 3b)^2$.   **Ans.**

9. $25x^2 - 30xy + 9y^2$

$= (5x)^2 - 2(5x)(3y) + (3y)^2$

$= (5x - 3y)^2$.   **Ans.**

15. $48a^3 - 120a^2b + 75ab^2$

$= 3a(16a^2 - 40ab + 25b^2)$

$= 3a(4a - 5b)^2$.   **Ans.**

78.   Type 3.   Expressions in the form of the difference of two squares can be factorised by the formula :—

$$a^2 - b^2 = (a + b)(a - b).$$

EXERCISE 49

Resolve into factors :--

1. $4a^2 - 9b^2$.   [Solved]   **2.** $9x^2 - 16y^2$.

3. $25c^2 - 36d^2$.   **4.** $100 - 9z^2$.

5. $49 - 64k^2$.   **6.** $121 - \dfrac{1}{c^4}$.

7. $a^2b^2 - 16c^4$.   **8.** $p^4 - \dfrac{9}{q^4}$.

Resolve into elementary factors :—

9. $x^4 - 81y^4$.   [Solved]   **10.** $1 - x^4$.

11. $16a^4 - 1$.   **12.** $c^4 - 256$.

13. $a^4 - 625b^4$ **14.** $a^8 - \dfrac{1}{a^8}$.

---

15. $4a^3 - 25ax^2$. [Solved].   16. $18a^2b^3 - 32$.

17. $pq - q^5p^3$.   18. $18a^3b^3 - 242ab^7$.

19. $324x^{17}a^3 - 484x^6a^3$.   20. $3x^5y - 243xy^5$.

———

Factorise :—

21. $x^4 - (2y - 3z)^2$. [Solved]   22. $4a^4 - (2b - c)^2$

23. $(2x+1)^2 - 9x^4$.   24. $16(2x-1)^2 - 25z^2$.

25. $(a^2 + bc)^2 - a^2(b+c)^2$.   26. $4(xy+1)^2 - 9(x-1)^2$

———

27. $a^2 + 2ab + b^2 - c^2$. [Solved]

28. $4a^2 + 4ab + b^2 - 4c^2$.   29. $25x^2 - 10x + 1 - 36y^2$

30. $x^2 - y^2 + 6y - 9$. [Hint]   31. $64a^2 - 25b^2 - 20bc - 4c^2$

32. $49x^2 - 1 - 14xy + y^2$.

33. $4a^2 - 4ab + b^2 - c^2 - 9d^2 + 6cd$. [Hint]

34. $p^2 + 25q^2 - 9r^2 - s^2 - 10pq + 6rs$.

———

SOLUTIONS & HINTS—EXERCISE 49

1. $4a^2 - 9b^2 = (2a)^2 - (3b)^2 = (2a + 3b)(2a - 3b)$.

9. $x^4 - 81y^4 = (x^2)^2 - (9y^2)^2 = (x^2 + 9y^2)(x^2 - 9y^2)$
   $$= (x^2 + 9y^2)\{ (x)^2 - (3y)^2 \}$$
   $$= (x^2 + 9y^2)(x + 3y)(x - 3y).$$

15. $4a^3 - 25ax^2 = a(4a^2 - 25x^2) = a\{ (2a)^2 - (5x)^2 \}$
    $$= a(2a + 5x)(2a - 5x).$$

21. $x^4 - (2y - 3z)^2 = (x^2)^2 - (2y - 3z)^2$
    $$= \{ x^2 + (2y - 3z) \} \{ x^2 - (2y - 3z) \}$$
    $$= (x^2 + 2y - 3z)(x^2 - 2y + 3z)$$

27. $a^2 + 2ab + b^2 - c^2$
    $$= (a^2 + 2ab + b^2) - c^2$$
    $$= (a + b)^2 - (c)^2$$
    $$= (a + b + c)(a + b - c)$$

30. Given exp. $= x^2 - (y^2 - 6y + 9) = (x)^2 - (y-3)^3$ etc.

33. Given exp. $= (4a^2 - 4ab + b^2) - (c^2 + 9d^2 - 6cd)$ etc.

---

79. **Type 4.** Expressions in the form of the *sum or difference of two cubes* are factorised by the following formulas :—

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

EXERCISE 50

Factorise :—

1. $8x^3 + 27y^3$. [*Solved*]   2. $y^3 + 64z^3$.

3. $125a^3 + b^3$.   4. $8z^3 + 343$.

5. $27b^6 + 125$.   6. $x^9 + \dfrac{64}{y^9}$.

---

7. $64a^3 - 27b^3$. [*Solved*]†   8. $125p^3 - 8q^3$.

9. $729a^6 - 8$.   10. $x^{12} - 1000$.

11. $8z^{15} - 125$.   12. $512 - \dfrac{x^6}{y^6}$.

---

13. $a^6 - b^6$. [*Solved*]   14. $x^6 - \dfrac{1}{x^6}$.

15. $z^6 - 64$.   16. $729a^6 - 1$.

17. $x^9 - 512$.   18. $x^{12} - y^{12}$.

---

19. $27a^7 - ab^6$. [*Hint*]   20. $16p^7 - 54pq^6$.

21. $a^2 + a^2 b^6$.   22. $4a + 2048a^{10}$.

---

23. $(2a - 3b)^3 + 64c^3$. [*Solved*] 24. $(3x + 2y)^3 - 125z^3$.

25. $x^3 - 3x^2y + 3xy^2 - y^3 - 1$. [*Hint*]

26. $27b^3 - a^3 - 3a^2 - 3a - 1$. 27. $8(p + 2q)^3 - 125r^3$.

28. $27a^3 - 343(x^3 - 3x^2 + 3x - 1)$.

SOLUTIONS & HINTS—EXERCISE 50

1. $8x^3 + 27y^3 = (2x)^3 + (3y)^3$

$$= (2x+3y)\{ (2x)^2 - (2x)(3y) + (3y)^2 \}$$
$$= (2x+3y)(4x^2 - 6xy + 9y^2). \quad \text{Ans.}$$

7. $64a^3 - 27b^3 = (4a)^3 - (3b)^3$

$$= (4a-3b)\{ (4a)^2 + (4a)(3b) + (3b)^2 \}$$
$$= (4a-3b)(16a^2 + 12ab + 9b^2).$$

13. $a^6 - b^6 = (a^3)^2 - (b^3)^2 = (a^3 + b^3)(a^3 - b^3)$
$$= (a+b)(a^2 - ab + b^2)(a-b)(a^2 + ab + b^2). \quad \text{Ans}$$

Note. We could proceed thus :—

$$a^6 - b^6 = (a^2)^3 - (b^2)^3 = (a^2 - b^2)(a^4 + a^2 b^2 + b^4) \text{ etc.}$$

but the first method is far more convenient. *Remember that if an expression can be expressed both as a difference of two squares and as a difference of two cubes, we must give preference to the first form, i.e., difference of two squares.*

19. $27a^7 - ab^6 = a(27a^6 - b^6) = a[(3a^2)^3 - (b^3)^3]$ etc.

23. $(2a-3b)^3 + 64c^3 = (x)^3 + (4c)^3$ [where $x = 2a-3b$]

$$= (x+4c)\{ x^2 - x \times 4c + (4c)^2 \} = (x+4c)\{ x^2 - 4cx + 16c^2 \}$$
$$= (2a-3b+4c)\{ (2a-3b)^2 - 4c(2a-3b) + 16c^2 \}$$

[Putting back the value of x]

$$= (2a-3b+4c)(4a^2 + 9b^2 - 12ab - 8ac + 12bc + 16c^2). \quad \text{Ans}$$

25. Given exp. $= (x^3 - 3x^2 y + 3xy^2 - y^3) - 1$
$$= (x-y)^3 - (1)^3 \text{ etc.}$$

80. Type 5. (Perfect cubes)

The formulas are :—

$$a^3 + 3a^2 b + 3ab^2 + b^3 = (a+b)^3 \quad [i.e., (a+b)(a+b)(a+b)]$$
$$a^3 - 3a^2 b + 3ab^2 - b^3 = (a-b)^3 \quad [i.e., (a-b)(a-b)(a-b)]$$

Note. The left-hand sides may be written as

$$a^3 + b^3 + 3ab(a+b) \text{ and } a^3 - b^3 - 3ab(a-b).$$

EXERCISE 51

Factorise :—

1. $x^3 + 6x^2 y + 12xy^2 + 8y^3$. [*Solved*]

2. $27x^3 + 27x^2 y + 9xy^2 + y^3$.

3. $125a^3 + 150a^2 + 60a + 8.$

4. $m^3 + \dfrac{1}{m^3} + 3m + \dfrac{3}{m}.$

---

5. $27a^3 - 54a^2b + 36ab^2 - 8b^3.$ *[Solved]*

6, $64k^3 - 144k^2 + 108k - 27.$

7. $a^3x^3 - 3a^2bx^2y + 3ab^2xy^2 - b^3y^3.$

8. $a^6 - 9a^4b^2 + 27a^2b^4 - 27b^6.$

---

9. $16x^4 + 24x^3y + 12x^2y^2 + 2xy^3.$ *[Hint]*

10. $4a^3b - 36a^2b^2 + 108ab^3 - 108b^4.$

---

11. $a^6 - 3a^4b^2 + 3a^2b^4 - b^6.$ *[Hint]*

12. $x^9 - 3x^6y^3 + 3x^3y^6 - y^9.$

SOLUTIONS & HINTS—EXERCISE 51

1. $x^3 + 6x^2y + 12xy^2 + 8y^3$
$= (x)^3 + 3(x)^2(2y) + 3(x)(2y)^2 + (2y)^3$
$= (x + 2y)^3.$

5. $27a^3 - 54a^2b + 36ab^2 - 8b^3$
$= (3a)^3 - 3(3a)^2(2b) + 3(3a)(2b)^2 - (2b)^3$
$= (3a - 2b)^3.$

9. Given exp. $= 2x(8x^3 + 12x^2y + 6xy^2 + y^3)$

11. First show that the given exp. $= (a^2 - b^2)^3.$
Then factorise it further by replacing $a^2 - b^2$ by its factors
$(a + b)(a - b).$

• 81. **Type 6.** (Quadratic Expressions of the form
$x^2 + px + q$ or $x^2 + pxy + qy^2.$

Method 1.

We know that :—
$$x^2 + (a + b)x + ab = (x + a)(x + b) \qquad \dots(i)$$

Hence if we express x^2+px+q in the form of L. H. S. of (i), that is, if we find two quantities a and b, whose sum is p and product q, we get the two factors comprising the R. H. S. of (i).

Let us illustrate this method by factorising the exp. x^2+5x+6.

Here we require two numbers whose sum is 5 and product 6. By trial, we easily find that these numbers are 2 and 3. Hence we have :—

$$x^2+5x+6=x^2+(2+3)x+2\times3=(x+2)(x+3). \quad \text{Ans.}$$
[By formula 1]

After finding the numbers 2 and 3,

Some authors and teachers prefer to proceed as follows :—
$$x^2+5x+6$$

$=x^2+2x+3x+6$ [Splitting the middle term into two parts with coefficients 2 and 3]

$=x(x+2)+3(x+2)$ [Combining the first two and the last two terms separately]

$=(x+2)(x+3)$. **Ans.** [Taking out the common factor $(x+2)$ from the two parts]

There is at least one advantage of the above procedure : if the two numbers found by trial be wrong, the mistake is automatically checked, for then we don't get a common factor from the two parts, as $(x+2)$ in the above process.

The factors of the exp. $x^2+pxy+qy^2$ are obtained in the same manner by the use of the formula :—

$$x^2+(a+b)xy+aby^2=(x+ay)(x+by) \qquad \text{...(ii)}$$

For example :—
$$x^2+7xy+10y^2$$
$$=x^2+(2+5)xy+2\times5\times y^2$$
$$=(x+2y)(x+5y)$$
    Or thus :—
$$x^2+7xy+10y^2$$
$$=x^2+2xy+5xy+10y^2$$

$$= x(x+2y)+5y(x+2y)$$
$$= (x+2y)(x+5y).$$

Important Note. To find by trial the two quantities *a* and *b* whose sum is *p* and product *q* we should proceed as follows :—

Find different pairs of factors of q and select that pair whose sum is p.

For example, let the exp. be $x^2+16x+48$.

The different pairs of factors of 48 are :—

(*i*) 1×48,   (*ii*) 2×24,   (*iii*) 3×16,   (*iv*) 4×12, (*v*) 6×8.

We require that pair of factors whose sum is 16 ; evidently it is the fourth pair, 4×12.

Thus 4 and 12 are the numbers whose sum is 16 and product 48.

However, there is a **definite method** for finding these numbers, which, though lengthy, is instructive. We give it below, but the student is advised to adopt it only when the trial method is found very tedious.

Taking the same example, suppose the required numbers to be *a* and *b*.

$$\therefore \quad a+b=16 \qquad\qquad ...(1)$$
$$\text{and} \quad ab=48 \qquad\qquad ...(2)$$

Thus we have to solve equations (1) and (2) and *the student will learn inccidentally the method of solving this new type of simultaneous equations* which will be dealt with again in Chapter XXV.

Now, $(a-b)^2=(a+b)^2-4ab$ [Cor. 4, Art. 67.]
$$= (16)^2-4 \times 48$$

[Substituting from equations (1) and (2)]
$$= 256-192=64.$$

$\therefore \qquad a-b=\sqrt{64}$

or $\qquad a-b=8$ ...(3)   [Taking only the positive value]

Adding equations (1) and (3) we get :—
$$2a=16+8=24$$
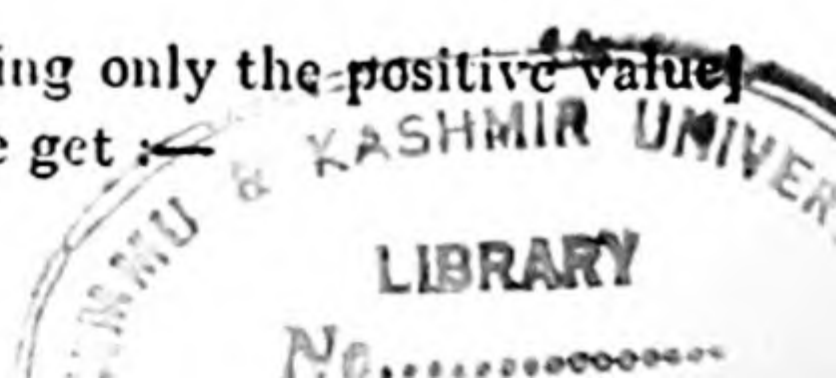
$$\therefore \qquad a=12.$$

Substituting this value of a in (1) we have :—

$$12+b=16$$

$$\therefore \qquad b=16-12=4$$

If we take the negative value of $\sqrt{64}$, the values found are $a=4$, $b=12$.

In any case, the two numbers are 12 and 4.

Method 2.

(By expressing the exp. x^2+px+q or $x^2+pxy+qy^2$ as the difference of two squares)

The method will be clear from the following two examples :—

Ex. 1.
$$\begin{aligned}
x^2+6x+8 &=(x^2+6x+9^*)-1 \\
&=(x+3)^2-(1)^2 \\
&=(x+3+1)(x+3-1) \\
&=(x+4)(x+2).
\end{aligned}$$

Ex. 2. Factorise $x^2+7xy+10y^2$.

Here half the coefficient of x is $\frac{7}{2}y$ and its square is $\frac{49}{4}y^2$ [See Foot Note]. Adding and subtracting this quantity we have :—

$$\begin{aligned}
x^2+7xy+10y^2 &=x^2+7xy+\tfrac{49}{4}y^2-\tfrac{49}{4}y^2+10y^2 \\
&=(x^2+7xy+\tfrac{49}{4}y^2)-\tfrac{9}{4}y^2 \\
&=(x+\tfrac{7}{2}y)^2-(\tfrac{3}{2}y)^2 \\
&=(x+\tfrac{7}{2}y+\tfrac{3}{2}y)(x+\tfrac{7}{2}y-\tfrac{3}{2}y) \\
&=(x+5y)(x+2y). \quad \textbf{Ans.}
\end{aligned}$$

EXERCISE 52

Factorise :—

[Form x^2+px+q]

1. $x^2+7x+12$. [Solved] 2. $x^2+8x+12$.

---

* **How to get this number 9 ?** For this see Exercise 40, Q. 63. But it may be added here that if the coefficient of x^2 is unity, as in the present case, we have only to halve the coefficient of x and square it. Thus we have $(\tfrac{6}{2})^2=(3)^2=9$.

3. $x^2+13x+12.$ 4. $x^2+9x+8.$
5. $x^2+7x+10.$ 6. $x^2+9x+14.$
7. $x^2+12x+20.$ 8. $x^2+9x+20.$

———

[Form x^2-px+q]

9. $x^2-8x+15.$ [Solved]   10. $x^2-16x+15.$
11. $x^2-10x+16.$ 12. $x^2-9x+18.$
13. $x^2-11x+18$ 14. $x^2-19x+18.$
15. $x^2-11x+24.$ 16. $x^2-14x+24.$

———

[Form x^2+px-q]

17. $x^2+5x-24.$ [Solved]   18. $x^2+10x-24.$
19. $x^2+2x-24.$ ·20. $x^2+23x-24.$
21. $x^2+11x-26.$ 22. $a^2+12a-28.$
23. $a^2+3a-28.$ 24. $k^2+27k-28.$

———

[Form x^2-px-q]

25. $m^2-m-30.$ [Solved]   26. $p^2-13p-30.$
27. $q^2-7q-30.$ 28. $l^2-29l-30.$
29. $a^2-4a-32.$ 30. $b^2-14b-32.$
31. $x^2-8x-33.$ 32. $x^2-15x-34.$

———

[Forms $x^2 \pm px \pm q$]

33. $x^2+12x+35.$ 34. $x^2+16x-36.$
35. $x^2-5x-36.$ 36. $k^2-15k+36.$
37. $a^2+20a+36.$ 38. $a^2+9a-36.$
39. $k^2+5k-36.$ 40. $m^2-13m+36.$
41. $a^2+14a+40.$ 42. $a^2-3a-40.$
43. $18a-40+a^2.$ [Hint]   44. $a^2+40-41a.$
45. $k^2-42-19k.$ 46. $k^2-42-k.$

47. $(x+7)^2-(33x+1)$. [*Hint*]

48. $(x+7)(x-7)+2x+1$.

[Forms $-x^2 \pm px \pm q$]

49. $-x^2+8x+48$. [*Solved*] 50. $-x^2-13x+48$.

51. $-x^2+27x-50$. 52. $-x^2-5x+50$.

53. $-a^2+28a-52$. 54. $52-9a-a^2$.

55. $72+14a-a^2$. 56. $108+3m-m^2$.

[Forms $x^2 \pm pxy \pm qy^2$]

57. $x^2-13xy+42y^2$. [*Solved*] 58. $x^2+xy-42y^2$.

59. $a^2-ab-56b^2$. 60. $a^2+15ab+56b^2$.

61. $a^2-15ab+56b^2$. 62. $ab-56b^2+a^2$.

63. $a^2-12ab+32b^2$. 64. $m^2-11mn-60n^2$.

65. $p^2-13pq-48q^2$. 66. $88y^2+x^2+19xy$.

67. $x^2-6xy-135y^2$. 68. $p^2-pq-240q^2$.

69. $x^2-31xy-180y^2$. 70. $a^2+2ab-224b^2$.

[Forms reducible to $x^2 \pm px \pm q$ or $x^2 \pm pxy \pm qy^2$]

71. x^4-5x^2-36. [*Solved*] 72. a^4+11a^2-60.

73. a^6+19a^3+88. 74. p^6-26p^3-27.

75. a^8-22a^4-75. 76. $a^8-79a^4b^4-162b^8$.

77. $m^6-2m^4n^2-195n^4$. 78. $p^{12}-p^6q^4-210q^8$.

79. $2x^3+4x^2-160x$. [*Hint*] 80. $3x^3-33x^2-240x$.

81. $4a^3-80a^2+384a$. 82. $a^4+3a^3-180a^2$.

83. $2k^4+48k^3l+270k^2l^2$. 84. $3x^4-60x^3y+252x^2y^2$.

85. $x^6-123x^4-124x^2$. 86. $8z^6+88z^4-180z^2$.

87. $(a+b)^2-2(a+b)-24$. [*Solved*]

88. $(2a-b)^2-(2a-b)-30$.

89. $(x+3y)^2-21(x+3y)+90.$

90. $(2x-y)^2+15z(2x-y)-54z^2.$

91. $(3-4a)^2+8b(3-4a)-84b^2.$

92. $(x^2+x)^2-14(x^2+x)+24.$ [*Hint*]

93. $(a^2-4a)^2+16(a^2-4a)+48.$

94. $(x^2+5x)^2-8(x^2+5x)-84.$

95 $(x^2-5xy)^2-9(x^2-5xy)(xy-4y^2)+18(xy-4y^2)^2.$ [*Hint*]

96. $(a^2-3ab)^2-5(a^2-3ab)b^2-50b^4.$

SOLUTIONS & HINTS—EXERCISE 52

1   [$12=1\times12$ or 2×6 or 3×4; we take the last pair, because $3+4=7$]

$$x^2+7x+12=x^2+(3+4)x+3\times4$$
$$=(x+3)(x+4).$$

Or thus :—

$$x^2+7x+12=x^2+3x+4x+12$$
$$=x(x+3)+4(x+3)$$
$$=(x+3)(x+4).$$

Or thus :—

[The number which should be added to x^2+7x to make it a perfect square is $(\frac{7}{2})^2$ or $\frac{49}{4}$.]

$x^2+7x+12=x^2+7x+\frac{49}{4}-\frac{49}{4}+12$ [adding and subtracting $\frac{49}{4}$]

$$=(x^2+7x+\frac{49}{4})-\frac{1}{4}$$
$$=(x+\frac{7}{2})^2-(\frac{1}{2})^2$$
$$=(x+\frac{7}{2}+\frac{1}{2})(x+\frac{7}{2}-\frac{1}{2})$$
$$=(x+4)(x+3).$$

Note. We have done the above example by different methods in order to illustrate them in one place.

In the other examples we shall employ only one of these methods.

9. $x^2-8x+15.$

[Only *two negative numbers* can give us *negative sum* and

positive product. Hence the possible factors of 15 are -1×-15, -3×-5; clearly, we need the second pair].

Given exp.$= x^2 - 3x - 5x + 15$
$$= x(x-3) - 5(x-3)$$
$$= (x-3)(x-5).$$

17. $x^2 + 5x - 24.$

[If larger number is positive and smaller negative, **only** then can we have their sum positive and product negative. Hence possible factors of -24 are :—

$$(-1)(24), \ (-2)(12), \ (-3)(8), \ (-4)(6).$$

Evidently, we need the third pair.]

Given exp.$= x^2 - 3x + 8x - 24$
$$= x(x-3) + 8(x-3)$$
$$= (x-3)(x+8).$$

25. $m^2 - m - 30.$

[If the sum of two numbers is negative and the product also negative, the larger number must be negative and smaller positive. Hence the possible factors of -30 are :—

$$(1)(-30), \ (2)(-15), \ (3)(-10), \ (5)(-6).$$

Clearly, we require the last pair]

Given exp.$= m^2 + (5-6)m + (5)(-6) = (m+5)(m-6).$

43. Arrange suitably.

47. Remove brackets and arrange suitably.

49. Given exp.$= -x^2 + 8x + 48$ $\qquad \left\{ \begin{array}{l} -48 = (1)(-48), \\ (2)(-24), \ (3)(-16), \\ (4)(-12), \ (6)(-8). \end{array} \right\}$
$$= -(x^2 - 8x - 48)$$
$$= -\{ x^2 + (4-12)x + 4(-12) \}$$
$$= -\{ (x+4)(x-12) \}$$
$$= (x+4)(12-x).$$

Note that if there is a negative sign before an expression consisting of a number of factors, it will change the signs of only one factor, when removed.

In the above example we have removed the negative sign after changing the signs of the factor $x-12$; the other factor $(x+4)$ does not change.

57. $x^2-13xy+42y^2$ $\left\{\begin{array}{l} 42=(-1)(-42),\ (-2)(-21),\\ \quad\quad (-3)(-14),\ (-6)(-7),\\ \text{we require the last pair.} \end{array}\right\}$

$=x^2-6xy-7xy+42y^2$

$=x(x-6y)-7y(x-6y)$

$=(x-6y)(x-7y).$

71.   Given exp. $=x^4-5x^2-36$

$=(x^2)^2-5x^2-36$

$=a^2-5a-36$ [where $a=x^2$]

$=(a-9)(a+4)$ [By methods explained before

$=(x^2-9)(x^2+4)$ [Putting back the value of a]

$=(x+3)(x-3)(x^2+4).$ **Ans.**

[Factorising x^2-9].

79.   Given exp. $=2x(x^2+2x-80),$ etc.

87. $(a+b)^2-2(a+b)-24$

$=x^2-2x-24$ [where $x=a+b$]

$=x^2+(4-6)x+(4)(-6)$

$=(x+4)(x-6)$

$=(a+b+4)(a+b-6)$ [Putting back the value of x]

92   As before, we get the two factors

$(x^2+x-2)(x^2+x-12)$

Each of these factors can be further split into two factors by the methods of this chapter.

95   Write a for x^2-5xy and b for $xy-4y^2$.

---

82. Type 7   [Quadratic Expressions of the form

$$Ax^2+Bx+C \text{ or } Ax^2+Bxy+Cy^2]$$

There are several methods of procedure, but to explain all of them would be rather puzzling the student. One of them is the *trial method*, which we do not recommend. Out of the remaining we give the two best methods.

Method 1.

General Discussion.

Consider the follo.. ing relation :—

$$(ax+b)(cx+d)=acx^2+(ad+bc)x+bd \qquad ...(i)$$

It will be noted that in the right-hand expression :—

(i) the second coefficient, viz., $ad+bc$, is the sum of two quantities ad and bc, and

(ii) the product of the first and the third coefficients is equal to the product of the same quantities, ad and bc, (each being equal to $abcd$).

Now. the given expression Ax^2+Bx+C is of the same form as R. H. S. of (i) and from the above two facts it is clear that B can be split into two parts corresponding to ad and bc, for :—

the sum of these parts is B .

and the product is AC   (=product of the first and the third coefficients)

Having done this, we can proceed as follows :—

$$acx^2+(ad+bc)x+bd$$
$$=acx^2+adx+bcx+bd$$
$$=ax(cx+d)+b(cx+d)$$
$$=(cx+d)(ax+b).$$

Now take a particular.

Example :—Factorise $18x^2+13x+2$.

Solution :—I Coeff. $\times$ III Coeff.$=18\times2=36$.

Find two numbers whose product is 36 and sum 18 (II Coeff.) These numbers are clearly 9 and 4. Hence we have :—

Given exp.$=18x^2+13x+2$
$$=18x^2+9x+4x+2$$
$$=9x(2x+1)+2(2x+1)$$
$$=(2x+1)(9x+2). \textbf{ Ans.}$$

---

* This general discussion may be omitted on first reading, or may be even omitted altogether. The method can be easily picked up from the solutions of particular cases given here and in the next exercise.

Method 2.

General Case.

Given exp. $= Ax^2 + Bx + C$

$$= \frac{1}{A}\left(A^2x^2 + ABx + AC\right) \quad \text{[Multiplying and dividing by A]}$$

$$= \frac{1}{A}\left\{(Ax)^2 + B.(Ax) + AC\right\}$$

$$= \frac{1}{A}\left\{y^2 + By + AC\right\} \quad \text{[writing } y \text{ for } Ax]$$

The expression within brackets is of Type 6 and therefore can be easily factorised. Then y can be replaced by its value Ax. It will be seen that the two factors are jointly divisible by A. so that $\frac{1}{A}$ disappears from outside.

Example Factorise $18x^2 - 5x - 2$.

Solution.

Given exp. $= 18x^2 - 5x - 2$

$$= \tfrac{1}{18}(18 \times 18x^2 - 18 \times 5x - 18 \times 2)$$

$$\text{[Multiplying and dividing by 18]}$$

$$= \tfrac{1}{18}\left\{(18x)^2 - 5(18x) - 36\right\}$$

$$= \tfrac{1}{18}\left\{a^2 - 5a - 36\right\} \quad \text{[where } a = 18x]$$

$$= \tfrac{1}{18}(a - 9)(a + 4)$$

$$= \tfrac{1}{18}(18x - 9)(18x + 4)$$

$$= (2x - 1)(9x + 2). \quad \textbf{Ans.}$$

$$\text{[Dividing 1st. factor by 9 and 2nd by 2]}$$

Note. The form $Ax^2 + Bxy + Cy^2$ is dealt with exactly in the same way as the form $Ax^2 + Bx + C$. For examples see next exercise.

EXERCISE 53

Resolve into factors :—

1. $6x^2 + 13x + 6$. [*Solved*]   2. $2x^2 + 7x + 3$.
3. $3x^2 + 13x + 4$.   4. $2x^2 + 13x + 20$.
5. $6x^2 + 17x + 5$.   6. $2(x^2 + 7) + 11x$.

7. $3x^3 - 14x + 8$. [Solved]   8. $6x^2 - 19x + 10$.

9. $3x^2 - 17x + 20$.   10. $6(x^3 + 1) - 13x$.

11. $2(6x^2 + 1) - 11x$.   12. $1 + 15x^2 - 8x$.

————

13. $8x^2 - 18x - 5$. [Solved] 14. $2x^2 - 3x - 20$.

15. $6x^2 + 19x - 7$.   16. $8x^2 - 14x - 15$.

17. $5x^3 + 33x - 14$.   18. $6x^2 + 11x - 10$.

19. $-12m^2 + 13m + 4$.   [Solved]

20. $-12k^2 - 7k + 12$.   21. $2 + 7a - 30a^2$.

22. $8 - 17a - 28a^2$.

————

23. $15a^2 + 14ab - 8b^2$.   [Solved]

24. $20a^2 - 7ab - 3b^2$.   25. $10a^2 + 7ab - 12b^2$.

26. $12x^2 - 4xy - 5y^2$.   27. $21x^2 - 25xy - 4y^2$.

28. $20x^2 - 23xy + 6y^2$.   29. $12x^2 - 7xy - 10y^2$.

30. $25x^2 - 15xy + 2y^2$.   31. $2x^2 - 49xy - 25y^2$.

32. $8a^2 - 65ab + 8b^2$.

————

33. $10x^3 + 28x^2 - 6x$. [Hint] 34. $9a^3 - 51a^2 + 60a$.

35. $6a^4 - a^3 - 12a^2$.   36. $12a^4 + 2a^3 - 24a^2$.

37. $24x^3y - 102x^2y^2 + 90xy^3$. 38. $10a^5 - 15a^4b - 45a^3b^2$.

————

39. $9x^4 - 82x^2y^2 + 9y^4$.   [Solved]

40. $18x^4 + 25x^2y^2 - 3y^4$.   41. $3m^4n^4 - 10m^2n^2p^2 + 8p^4$.

42. $\dfrac{x^4}{4} + 2x^2 - 5$.   43. $7x^6 - x^3 - 8$.

44. $8x^6 - 65x^3 + 8$.   45. $8x^6 + 215x^3 - 27$.

46. $4a^8 - 17a^4b^4 + 4b^8$.

————

47. $(5x^2 + 8x)^2 - 2(5x^2 + 8x) - 15$.   [Solved]

48. $(3x^2 + 14x)^2 - 8(3x^2 + 14x) + 15$.

49. $(6x^2-x)^2-17(6x^2-x)+60.$
50. $x^2(x+1)^2-14(x+1)x+24.$
51. $(a^2+a-6)(a^2+a-20)-15.$ [*Hint*]
52. $(a^2+8a+7)(a^2+8a+15)-9.$
53. $(a^2-4a)(a^2-4a-1)-20.$
54. $2(a^2+b^2)^2+5(a^2+b^2)ab+2a^2b^2.$
55. $5(x^2-6x)^2-27(x^2-6x)(x^2-x)+36(x^2-x)^2.$

SOLUTIONS & HINTS—EXERCISE 53

1.   Given exp. $=6x^2+13x+6.$

We require two numbers whose sum$=13$ and product
$=6\times6=36.$

By trial, we find these numbers to be 9 and 4.

Hence we have :—

Given exp. $=6x^2+9x+4x+6$
$$=3x(2x+3)+2(2x+3)$$
$$=(2x+3)(3x+2).\ \textbf{Ans.}$$

<div align="center">Or thus :—</div>

Given exp. $=\tfrac{1}{6}\{\,36x^2+78x+36\,\}$ [multiplying and divid-
ing by 6]
$$=\tfrac{1}{6}\{\,(6x)^2+13(6x)+36\,\}$$
$$=\tfrac{1}{6}\{\,y^2+13y+36\,\} \qquad [\text{where } y=6x]$$
$$=\tfrac{1}{6}(y+9)(y+4)$$
$$=\tfrac{1}{6}(6x+9)(6x+4)\ [\text{Replacing } y \text{ by its value } 6x]$$
$$=(2x+3)(3x+2)\ [\text{Dividing 1st factor by 3 and}$$
second factor by 2]

7. $3x^2-14x+8$
$$=3x^2-2x-12x+8 \qquad\qquad \left\{\begin{array}{l}3\times8=24\\24=(-2)\times(-12)\end{array}\right\}$$
$$=x(3x-2)-4(3x-2)$$
$$=(3x-2)(x-4).$$

13. $8x^2 - 18x - 5$

$= 8x^2 - 20x + 2x - 5 \qquad \left\{ \begin{array}{l} 8 \times (-5) = -40 \\ -40 = (-20) \times 2 \end{array} \right\}$

$= 4x(2x - 5) + 1(2x - 5)$

$= (2x - 5)(4x + 1).$

19. Given exp. $= -12m^2 + 13m + 4 \qquad \left\{ \begin{array}{l} 12 \times -4 = -48 \\ -48 = (-16) \times 3 \end{array} \right\}$

$= -(12m^2 - 13m - 4)$

$= -\{ 12m^2 - 16m + 3m - 4 \}$

$= -\{ 4m(3m - 4) + 1(3m - 4) \}$

$= -(3m - 4)(4m + 1)$

$= (4 - 3m)(4m + 1).$

23. Given exp. $= 15a^2 + 14ab - 8b^2 \qquad \left\{ \begin{array}{l} 15 \times (-8) = -120 \\ -120 = 20 \times (-6) \end{array} \right\}$

$= 15a^2 + 20ab - 6ab - 8b^2$

$= 5a(3a + 4b) - 2b(3a + 4b)$

$= (3a + 4b)(5a - 2b).$

33. Given exp. $= 10x^3 + 28x^2 - 6x \qquad \left\{ \begin{array}{l} 5 \times (-3) = -15 \\ -15 = 15 \times (-1) \end{array} \right\}$

$= 2x(5x^2 + 14x - 3)$

$= 2x\{ 5x^2 + 15x - x - 3 \}$

$= 2x\{ 5x(x + 3) - 1(x + 3) \}$

$= 2x(x + 3)(5x - 1).$

39. Given exp. $= 9x^4 - 82x^2y^2 + 9y^4$

$= 9a^2 - 82ab + 9b^2 \qquad$ [where $a = x^2$ and $b = y^2$]

$= 9a^2 - 81ab - ab + 9b^2 \qquad \left\{ \begin{array}{l} 9 \times 9 = 81 \\ 81 = (-81)(-1) \end{array} \right\}$

$= 9a(a - 9b) - b(a - 9b)$

$= (a - 9b)(9a - b)$

$= (x^2 - 9y^2)(9x^2 - y^2) \qquad$ [Putting back the
values of a and b

$= (x + 3y)(x - 3y)(3x + y)(3x - y).$

47. Given exp. $= (5x^2 + 8x)^2 - 2(5x^2 + 8x) - 15$

$= a^2 - 2a - 15 \qquad$ [where $a = 5x^2 + 8x$]

$= (a - 5)(a + 3)$

$= (5x^2 + 8x - 5)(5x^2 + 8x + 3)$

[The first factor cannot be further factorised, because we cannot find two numbers whose sum may be 8 and product -25].

Now, $5x^2+8x+3=5x^2+5x+3x+3$
$$=5x(x+1)+3(x+1)$$
$$=(x+1)(5x+3).$$

$\therefore$ Given exp.$=(5x^2+8x-5)(x+1)(5x+3).$

51. Given exp.$=(a^2+a-6)(a^2+a-20)-15$
$$=(x-6)(x-20)-15 \qquad \text{[where } x=a^2+a\text{]}$$
$$=x^2-26x+120-15$$
$$=x^2-26x+105.$$

Factorise this exp.

83. Type 8. [Expressions Reducible to the form a^2-b^2]
Such expressions can be conveniently divided into three groups :—

Group 1. Expressions which can be divided into two parts which are perfect squares and have minus sign between them. For example :—

(i) $a^2+2ab+b^2-4c^2=(a^2+2ab+b^2)-(4c^2)=(a+b)^2-(2c)^2$
$$=(a+b+2c)(a+b-2c).$$

(ii) $4x^2-4x+1-y^2+2yz-z^2=(4x^2-4x+1)-(y^2-2yz+z^2)$
$$=(2x-1)^2-(y-z)^2$$
$$=(2x-1+y-z)(2x-1-y+z).$$

etc. etc.

Such expressions have already been dealt with.

[See Exercise 49. Q. 27 to 34]

Group 2. Expressions which reduce to Group 1 after the addition and subtraction of a *suitable new term.* For example :—

(i) $x^4+64=x^4+64+16x^2-16x^2$ [adding and subtracting $16x^2$. How to get this term ? See Ex. 40. Q 53 to 62]
$$=(x^4+64+16x^2)-(16x^2)$$
$$=(x^2+8)^2-(4x)^2$$
$$=(x^2+8+4x)(x^2+8-4x).$$

(ii) $a^2-2ab-c^2+2bc=a^2-2ab+b^2-b^2-c^2+2bc$

$$\text{[adding and subtracting } b^2]$$

$$=(a^2-2ab+b^2)-(b^2+c^2-2bc)$$
$$=(a-b)^2-(b-c)^2$$
$$=(a-b+b-c)(a-b-b+c)$$
$$=(a-c)(a-2b+c).$$

etc. etc.

Group 3. Expressions which reduce to Group 1 after a term is split into two suitable parts. For example :—

(i) $x^4-13x^2+4=x^4-4x^2+4-9x^2$ [Splitting $-13x^2$ into $-4x^2-9x^2$]

$$=(x^4-4x^2+4)-(9x^2)$$
$$=(x^2-2)^2-(3x)^2$$
$$=(x^2-2+3x)(x^2-2-3x).$$

(ii) $x^4+2x^2+9=x^4+6x^2+9-4x^2$ [Splitting $2x^2$ into $+6x^2-4x^2$]

$$=(x^2+3)^2-(2x)^2$$
$$=(x^2+3+2x)(x^2+3-2x).$$

(iii) $4x^2+4xy-8y^2+6y-1$
$$=4x^2+4xy+y^2-9y^2+6y-1$$ [Splitting $-8y^2$ into y^2-9y^2]

$$=(4x^2+4xy+y^2)-(9y^2-6y+1)$$
$$=(2x+y)^2-(3y-1)^2$$
$$=(2x+y+3y-1)(2x+y-3y+1)$$
$$=(2x+4y-1)(2x-2y+1)$$

etc. etc.

EXERCISE 54

Resolve into factors :—

1. $4a^2-4ab+b^2-9c^2$ [Solved]

2. $x^2+6xy+9y^2-16z^2.$ 3. $16x^2-25y^2+1-8x.$

4. $x^4-4x^2+4y^4+4x^2y^2.$ 5. $6b-9b^2-1+25a^2.$ [Hint]

6. $24x+9y^2-16x^2-9.$

7. $x^2 - 10x + 25 - y^2 + 12yz - 36z^2$. [Hint]
8. $4a^4 - b^2 + 9 - 12a^2 - 4c^2 + 4bc$.
9. $x^2 + y^2 - z^2 + 1 + 2xy - 2x - 2y$. [Hint]
10. $4a^2 - 4 + b^2 + c^2 - 4ab - 4ac + 2bc$.
11. $2ab + 2a - 2b + 4c^2 - a^2 - b^2 - 1$. [Hint]
12. $4xz + 2yz - 4xy - 4x^2 - y^2 - z^2 + 16$.

—————

13. $a^4 + 4$. [Solved]
14. $a^4 + 64$.
15. $a^4 + 324$.
16. $4x^4 + 81y^2$.
17. $64a^4b^4 + 81$.
18. $4a^8 + 625b^4$.
19. $4x^5 + 16xy^4$. [Hint]
20. $64a^6 + a^2$.
21. $72x^5 + 18xy^4$.
22. $243y^6 + 12z^4y^2$.

—————

23. $9x^4 - 16x^2 + 4$. [Solved]
24. $4x^4 - 29x^2 + 25$.
25. $9x^4 - 33x^2 + 16$.
26. $16a^4 - 24a^2b^2 + b$
27. $100p^4 - 141p^2q^2 + 9q^4$.
28. $36a^4 - 165a^2b^2 + 49b$.
29. $a^5b - 3a^3b^3 + ab^5$. [Hint]
30. $2p^7 - 22p^5q^2 + 2p^3q^4$.

—————

31. $x^4 + x^2 + 1$. [Solved]
32. $x^4 + x^2y^2 + y^4$.
33. $4a^4 + 8a^2 + 9$.
34. $16a^4 + 7a^2 + 1$.
35. $x^4 + 2x^2y^2 + 9y^4$.
36. $25x^4 + x^2y^2 + y^4$.
37. $4a^4 + 4a^2b^2 + 25b^4$.
38. $25a^4 + 26a^2b^2 + 9b^4$

—————

39. $a^4 - 19a^2 + 9$. [Hint]
40. $9a^4 - a^2 + 16$.
41. $25a^4 - 6x^2y^2 + 9y^4$.
42. $16x^4 - 9x^2y^2 + 25y^4$.

—————

43. $36a^4 - 13a^2 + 1$. [Hint]
44. $9x^4 - 10x^2 + 1$
45. $4x^4 - 37x^2y^2 + 9y^4$
46. $25a^4 - 29a^2b^2 + 4b^4$

—————

47. $x^8 + x^4 + 1$. [Hint]
48. $a^8 + a^4b^4 + b^8$.
49. $a^8 - 47a^4 + 1$.
50. $a^8 + 14a^4b^4 + 81b^8$

—————

51. $x^2-4xy-z^2+4yz$. [*Solved*]
52. $9x^2-6x-16y^2+8y$.   53. $a(a-2b)-c(c+2b)$. [*Hint*]
54. $2a(2a-5b)-3c(3c-5b)$.
55. $x^4+4y^4+2x^2+4y^2+1$. [*Hint*]
56. $4x^4+y^2+1-4x^2y-2y$.

––––––

57. $a^2+4ab+3b^2-c^2+2bc$. [*Hint*]
58. $4a^2-12ab+5b^2-9+12b$.
59. $x^2-9-24y^2-30y-2xy$. [*Hint*]
60. $4x^2-25-7y^2-12xy+40y$.

––––––

SOLUTIONS & HINTS—EXERCISE 54

1. Given Exp.$=4a^2-4ab+b^2-9c^2$
$$=(4a^2-4ab+b^2)-(9c^2)$$
$$=(2a-b)^2-(3c)^2$$
$$=(2a-b+3c)(2a-b-3c).$$

5. Given Exp.$=25a^2-(9b^2+1-6b)$, etc.

7. Given Exp.$=(x^2-10x+25)-(y^2-12yz+36z^2)$.

9. Given Exp.$=(x^2+y^2+1+2xy-2x-2y)-(z^2)$.

11. Given Exp.$=(4c^2)-(a^2+b^2+1-2ab-2a+2b)$.

13. Given Exp.$=a^4+4$
$$=a^4+4+4a^2-4a^2 \qquad \left\{\begin{array}{l}\sqrt{a^4}=a^2 \\ \sqrt{4}=2 \\ 2\times a^2\times 2=4a^2\end{array}\right\}$$
$$=(a^2+4+4a^2)-(4a^2)$$
$$=(a^2+2)^2-(2a)^2.$$
$$=(a^2+2+2a)(a^2+2-2a)$$
$$=(a^2+2a+2)(a^2-2a+2).$$

19. Given Exp.$=4x^3+16xy^4=4x(x^4+4y^4)$, etc.

23. Given Exp.$=9x^4-16x^2+4$
$$=9x^4-12x^2+4-4x^2 \qquad \left\{\begin{array}{l}\sqrt{9x^4}=3x^2 \\ \sqrt{4}=2 \\ 2\times 3x^2\times 2=12x^2\end{array}\right\}$$
$$=(9x^4-12x^2+4)-(4x^2)$$
$$=(3x^2-2)^2-(2x)^2$$
$$=(3x^2-2+2x)(3x^2-2-2x)$$
$$=(3x^2+2x-2)(3x^2-2x-2).$$

29. ab is common to all the terms.

31. Given Exp. $= x^4 + 2x^2 + 1 - x^2$ $\left\{\begin{array}{l} \sqrt{x^4} = x^2 \\ \sqrt{1} = 1 \\ 2 \times x^2 \times 1 = 2x^2 \end{array}\right\}$
$$= (x^2 + 1)^2 - (x)^2$$
$$= (x^2 + 1 + x)(x^2 + 1 - x)$$
$$= (x^2 + x + 1)(x^2 - x + 1).$$

39. Given Exp. $= a^4 - 19a^2 + 9$ $\left\{\begin{array}{l} \sqrt{a^4} = a^2 \\ \sqrt{9} = 3 \\ 2 \times a^2 \times 3 = 6a^2 \end{array}\right\}$
$$= a^4 + 6a^2 + 9 - 25a^2$$
$$\text{etc.}$$

43. $-13a^2$ may be split either into $-12a^2 - a^2$ or into $+12a^2 - 25a^2$.

Also note that each of the two factors found can be further resolved into two factors each.

47. One of the two factors found will be the exp. of Q. 31, which can be further resolved into two factors.

51. Given Exp. $= x^2 - 4xy - z^2 + 4yz$
$$= x^2 - 4xy + 4y^2 - 4y^2 - z^2 + 4yz$$
$$\qquad\qquad \text{[Adding and subtracting } 4y^2\text{]}$$
$$= (x^2 - 4xy + 4y^2) - (4y^2 + z^2 - 4yz)$$
$$= (x - 2y)^2 - (2y - z)^2$$
$$= (x - 2y + 2y - z)(x - 2y - 2y + z)$$
$$= (x - z)(x - 4y + z).$$

53. Open the brackets and proceed as in Q. 51.

55. Adding and subtracting $4x^2y^2$, the expression becomes :
$$(x^4 + 4y^4 + 2x^2 + 4y^2 + 1 + 4x^2y^2) - (4x^2y^2)$$

Both parts are perfect squares.

57. Split $3b^2$ into two parts $4b^2 - b^2$; then the exp. becomes $(a^2 + 4ab + 4b^2) - (b^2 + c^2 - 2bc).$

59. Split $-24y^2$ into two parts.

Type 9. Factorisation by suitable Grouping of Terms.

The general principle under-lying the method can be out-lined as follows :—

Divide the given expression into two or more parts such that when each part is factorised by the fore-going methods, there is

factor common to all the parts. Then we get two factors of the expression by Type 1.

The difficulty lies in dividing the expression into parts : for this the only guiding principle is experience and common sense. A thorough study of the solutions and hints to the next exercise will enable the student to handle fairly difficult examples independently.

EXERCISE 55

Factorise :—

1. $ab+ac+bd+cd$. [*Solved*] 2. $ax+bx+ay+by$.
3. $ax-4a+bx-4b$. 4. $ab-ac+b^2-bc$.
5. x^3-x^2+3x-3. 6. $x^3-3ax-x^2a+3a^3$.
7. $6ab+6-9b-4a$. [*Hint*] 8. $ac^2-by+bcy-ac$.
9. $a^2-2bc+2ab-xc$. 10. $2a^3+3-a^2-6a$.
11. $pq+1+p+q$. 12. $5a^3-2-2a^2+5a$.
13. $2a^3-3-2a+3a^2$. 14. $4x^3+45-9x-20x^2$.
15. $2a^4-1+a^3-2a$. 16. $x^4+y^4+x^3y+xy^3$.
17. $lm(n^2+1)+n(l^2+m^2)$. [*Hint*]
18. $ab(x^2+y^2)-xy(a^2+b^2)$. 19. $bx(a^2+y^2)-ay(b^2+x^2)$.
20. $(pr+qs)^2+(qr-ps)^2$. 21. $(ax+by)^2+(ay-bx)^2$.
22. $(ax-by)^2-(ay-bx)^2$.

___

23. $(a+2b)^2+3a+6b$. [*Hint*]
24. $(2a-3b)^2+8a-12b$. 25. $(3a+1)(3a-1)-6ab+2b$.
26. $(a+3)(a-5)+3ab+9b$.
27. $4bx-2b-(2x-1)(4x^2+2x+1)$.
28. $2ax+3a-(2x+3)(3x-2)$.
29. $(2x-y)^3-10x+5y$.

___

[The next seven questions (Nos. 30 to 86) can be easily reduced to the forms of the previous seven questions (Nos. 23 to 29) respectively. For example, the expression of Q. 30 can be written as $(9x^2+6xy+y^2)+(12x+4y)$ or $(3x+y)^2+$

$4(3x+y)$, which is exactly like Q. 23. Similarly Q. 31 is like Q. 24, Q. 32 like Q. 25, etc.]

30. $9x^2+6xy+y^2+12x+4y$.

31. $16x^2-40xy+25y^2+12x-15y$.

32. $9x^2-16y^2-15x+20y$.   33. $x^2+2x-24-xy+4y$.

34. $2a-5b-8a^3+125b^3$.   35. $6a-2y-6a^2-13ay+5y^2$.

36. $a^3-6a^2b+12ab^2-8b^3-3a+6b$.

――――

[The next seven questions are again of the same type as Q. 23 to 29 or 30 to 36. Only the terms are not in the same order.]

37. $4x^2+10x+9y^2+15y+12xy$.

38. $a^4+a^2b-2a^2-b+1$.   39. $a^4-a^2c+b^2c-b^4$.

40. $m^2+11m+2mn-2n-12$.

41. $2x-x^3+2y-y^3$.

42. $6x-2x^2+5xy-9y+12y^2$.

43. $x^3+4x+y^3+4y+3x^2y+3xy^2$.

――――

44. x^3+2x^2+2x+1.   45. $a^3-2a^2b+2ab^2-b^3$.

46. $ay^2-y^2-ay+y+a-1$. [Hint].

47. $2ap+2aq-cp-bq-bp-cq$.

48. $x^5+x^4+x^3+x^2+x+1$.

――――

SOLUTIONS & HINTS—EXERCISE 55

1. Given Exp. $=ab+ac+bd+cd$
$$=(ab+ac)+(bd+cd)$$
$$=a(b+c)+d(b+c)$$
$$=(b+c)(a+d).$$

7. Combine the first and the third terms together and the other two together. Thus :—$(6ab-9b)-(4a-6)$.

17. Open the brackets and arrange suitably.

23. *Don't open the bracket.* The two terms outside the bracket are equal to $3a(a+2b)$. It will be seen that $(a+2b)$ is common to the two parts.

46. Divide the expression into three parts, grouping together first and second terms, third and fourth, fifth and sixth.

---

85. Type 10. [Expressions of the forms (i) $a^3+b^3+c^3$ $-3abc$, and (ii) $a^3+b^3+c^3$ (where $a+b+c=0$)]

Factors of (i) are $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$

[Formula 14, Art. 74]

Factors of (ii) are $3abc$.   [Result of Q. 27, Ex. 46]

Examples :—

(i) $8a^3+b^3-64c^3+24abc$

$=(2a)^3+(b)^3+(-4c)^3-3(2a)(b)(-4c)$

$=\{(2a)+(b)+(-4c)\}\{(2a)^2+(b)^2+(-4c)^2-(2a)(b)$

$\qquad\qquad\qquad\qquad\qquad -(2a)(-4c)-(b)(-4c)\}$

$=(2a+b-4c)(4a^2+b^2+16c^2-2ab+8ac+4bc)$.

(ii) $(x-y)^3+(y-z)^3+(z-x)^3$

$=a^3+b^3+c^3$ [where $a=x-y$, $b=y-z$, $c=z-x$]

$=a^3+b^3+c^3-3abc+3abc$ [Adding and subtracting $3abc$]

$=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)+3abc$

$=(0)(a^2+b^2+c^2-ab-ac-bc)+3abc$

$\qquad\qquad\qquad$ [$\because a+b+c=x-y+y-z+z-x=0$]

$=0+3abc=3abc$

$=3(x-y)(y-z)(z-x)$ [Rewriting the values of a, b, c]

EXERCISE 56

Factorise :—

1. $x^3-y^3-z^3-3xyz$. [*Solved*]

2. $a^3-b^3-1-3ab$.

3. $a^3-3ab+b^3+1$.

4. $a^3+8b^3+6ab-1$.

5. $a^3-27b^3+64c^3+36abc$.

6. $125x^3-8y^3-30xy-1$

7. $27x^3+64y^3-125z^3+180xyz$.

8. $1-216a^3-27b^3-54ab$.

9. $x^6-512y^6+240x^2y^2+1000$.

10. $x^3 - \dfrac{1}{x^3} + y^3 + 3y.$ [Hint]

11. $a^3 - b^3 + 6b + \dfrac{8}{a^3}$

————

12. $a^3 - \dfrac{1}{a^3} + 14.$ [Hint]   13. $x^3 - \dfrac{1}{x^3} \div 4.$

14. $a^2 - \dfrac{1}{a^3} - 36.$ 15. $a^3 - \dfrac{1}{a^3} - 14.$

16. $8x^3 + \dfrac{1}{x^3} - 4.$

————

17. $(x-2y)^3 + (2y-3z)^3 + (3z-x)^3.$ [See solved Example 2, Art. 85]

18. $(2a-3b)^3 + (a+b)^3 - (3a-2b)^3.$ [Hint]

19. $(a+2b-3c)^3 + (b+2c-3a)^3 + (c+2a-3b)^3.$

20. $(a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3.$

21. $x^3(y-z)^3 + y^3(z-x)^3 + z^3(x-y)^3.$ [Hint]

————

22. $(x+y)^3 + (y+z)^3 + (z+x)^3 - 3(x+y)(y+z)(z+x).$ [Solved]

23. $(a+1)^3 + (a+2)^3 + (2a+3)^3 - 3(a+1)(a+2)(2a+3).$

24. $(a-b)^3 - (b-c)^3 + (c-a)^3 + 3(a-b)(b-c)(c-a).$ [Hint]

————

SOLUTIONS & HINTS—EXERCISE 56

1. $x^3 - y^3 - z^3 - 3xyz$

   $= (x)^3 + (-y)^3 + (-z)^3 - 3(x)(-y)(-z)$

   $= \{ (x)+(-y)+(-z) \} \{ (x)^2+(-y)^2+(-z)^2-(x)(-y)$
   $\qquad\qquad\qquad\qquad\qquad -(x)(-z)-(-y)(-z) \}$

   $= (x-y-z)(x^2+y^2+z^2+xy+xz-yz).$

10 Given Exp. $= (x)^3 + \left(-\dfrac{1}{x} \right)^3 + (y)^3 - 3(x)\left(-\dfrac{1}{x} \right)(y)$

   etc.

12. Given Exp. $=a^3+\dfrac{1}{a^3}+8-6$ $[\because \quad 8+6=14]$

$$=(a)^3+\left(-\dfrac{1}{a}\right)^3+(2)^3-3(a)\left(-\dfrac{1}{a}\right)(2), \text{ etc.}$$

18. Write $-(3a-2b)^3$ as $+(2b-3a)^3$.

21. $x^3(y-z)^3$ may be written as $\{\,x(y-z)\,\}^3$.
Similarly modify the other two parts of the expression.

22. Given Exp. $=(x+y)^3+(y+z)^3+(z+x)^3-3(x+y)(y+z) \times$
$$(z+x)$$

$$=a^3+b^3+c^3-3abc\,[\text{where } a=x+y,\ b=y+z,$$
$$c=z+x]$$

$$=\tfrac{1}{2}(a+b+c)\{\,(a-b)^2+(b-c)^2+(c-a)^2\,\}$$
$$[\text{Art. 74}]$$

Now, $a+b+c=x+y+y+z+z+x=2(x+y+z)$
$(a-b)^2=(x+y-y-z)^2=(x-z)^2=x^2+z^2-2xz.$

etc., etc.

Substituting these values we get :—
Given Exp. $=\tfrac{1}{2}\times 2(x+y+z)\times 2(x^2+y^2+z^2-xy-xz-yz)$
$$=2(x+y+z)(x^2+y^2+z^2-xy-xz-yz).$$

24. Write $-(b-c)^3$ as $+(c-b)^3$
and $3(a-b)(b-c)(c-a)$ as $-3(a-b)(c-b)(c-a)$
Then, given Exp. $=x^3+y^3+z^3-3xyz$
$$[\text{where } x=a-b,\ y=c-b,\ z=c-a]$$
$$=\tfrac{1}{2}(x+y+z)\{\,(x-y)^2+(y-z)^2+(z-x)^2\,\}$$
etc.

---

86. Type 11. Expressions of the forms
  (i) $(x+a)(x+b)(x+c)(x+d)+k$, and
  (ii) $(x+a)(x+b)(x+c)(x+d)+kx^2$

The general method is quite evident : in each case we have to multiply the four binomials, add the single term to the product and then factorise. But the expression finally obtained must be of the fourth degree and therefore difficult

to factorise   However, we shall take only those expressions which can be reduced to quadratic (i.e., second degree) expressions by a simple device.

In case (i) multiply the binomials in pairs, choosing the pairs in such a way that the second degree and first degree terms in the two products are the same.   These two terms then can be replaced by a single letter.

For example, $(x+1)(x+2)(x+3)(x+4)$ should be arranged as $\{(x+1)(x+4)\}\{(x+2)(x+3)\}$,   which is equal to $(x^2+5x+4)(x^2+5x+6)$. and replacing x^2+5x by a, we get $(a+4)(a+6)$.   Now the product will be a second degree expression in a.

[A hint for choosing the pairs of brackets in the above example is : $1+4=2+3$.   But this won't apply when the coefficients of x in the four binomials are not the same]

Case (ii) differs from case (i) only in this that here the two products of the binomials should have the second degree and the constant terms the same.

More details will be clear from the solutions and hints to the next exercise.

EXERCISE 57

Factorise :—

1. $(x+1)(x+2)(x+3)(x+4)-3$.   [Solved]

2. $(x+1)(x+2)(x+4)(x+5)+2$.

3. $(x+1)(x+3)(x+5)(x+7)-9$.

4. $x(x+1)(x+5)(x+6)+6$.   [Hint]

5. $x(x+2)(x+3)(x+5)+9$.

6. $x(x+3)(x+4)(x+7)+20$.

————

7. $(x-1)(x-2)(x+3)(x+4)+4$.   [Hint]

8   $(x+1)(x+3)(x-4)(x-6)+13$.

9. $(x+2)(x+3)(x-5)(x-6)+12$.

10. $(a-2)(a+1)(a+2)(a+5)+36$.

11 $(x+2)(x-3)(x+4)(x-5)+40$.

12 $(a+1)(a+4)(a+7)(a+10)-40$.

13. $(a+3)(a+4)(a-8)(a-9)-64$.
14. $x(x-4)(x-6)(x-10)+80$.
15. $x(x+3)(x-3)(x-6)+56$.

---

16. $(2x+1)(2x+3)(2x+5)(2x+7)-65$. [*Hint*]
17. $2x(2x+1)(2x-3)(2x-4)-12$.
18. $(2x+1)(x-1)(x-4)(2x-5)-52$. [*Hint*]
19. $x(2x+1)(x-2)(2x-3)-63$.
20. $x(3x+2)(x-2)(3x-4)-21$.

---

21. $x(3x-1)(3x+1)(3x+2)-1$. [*Hint*]
22. $x(x^2-1)(x+2)-3$. [*Hint*]
23. $(2a+3)(4a^2-1)(2a+5)-9$.
24. $4(a^2-1)(2a+1)(2a+5)-45$.

---

25. $(a-3)(a-2)(a-6)(a-4)-12a^2$. [*Solved*]
26. $(a+2)(a+4)(a-4)(a-8)+9a^2$.
27. $(x+2)(x-3)(x+4)(x-6)-84x^2$.
28. $(a-1)(3a-1)(3a+2)(a+2)+8a^2$.

---

29. Prove that the product of any four consecutive numbers increased by unity is a perfect square.
[*Solved*]

30. Prove that the product of any four consecutive odd numbers increased by 16 is a perfect square.

31. Prove that the product of any four consecutive even numbers increased by 16 is a perfect square.

32. There are four numbers : the second exceeds the first by 2, the third exceeds the second by 4 and the fourth exceeds the third by 2. Show that their product increased by 36 is a perfect square.

SOLUTIONS & HINTS—EXERCISE 57

1. Given Exp. $=(x+1)(x+2)(x+3)(x+4)-3$

$= \{ (x+1)(x+4) \} \{ (x+2)(x+3) \} -3$

[See Art. 86]

$=(x^2+5x+4)(x^2+5x+6)-3$

$=(a+4)(a+6)-3$ [where $a=x^2+5x$]

$=a^2+10a+24-3$

$=a^2+10a+21$

$=(a+3)(a+7)$

$=(x^2+5x+3)(x^2+5x+7)$

[Restoring the value of a]

Note. The student should study carefully why $(x+1)$ has been paired with $(x+4)$ and $(x+2)$ with $(x+3)$. This will be clear from the third line of process : in each product we get x^2+5x, which is .put equal to a in the next step. Study Art. 86.

4. Given Exp. $= \{ x(x+6) \} \{ (x+1)(x+5) \} +6$

$=(x^2+6x)(x^2+6x+5)+6$

$=a(a+5)+6$ [where $a=x^2+6x$]

      etc.

7. Combine $(x-1)$ with $(x+3)$, and $(x-2)$ with $(x+4)$.

16. Put $2x=y$ and the expression becomes much simpler.

18. Combine $(2x+1)$ with $(x-4)$ and $(x-1)$ with $(2x-5)$

[Why ?]

21. Multiplying and dividing the given exp. by 3 we get

$\frac{1}{3}[3x(3x-1)(3x+1)(3x+2)-3]$.

Now put $3x=a$ and proceed as before.

22. Write (a^2-1) as $(x+1)(x-1)$.

25. Given Exp. $=(a-8)(a-2)(a-6)(a-4)-12a^2$

$= \{ (a-8)(a-4) \} \{ (a-2)(a-6) \} -12a^2$

[Why ?—See Art. 86]

$=(a^2-7a+12)(a^2-8a+12)-12a^2$

$=(a^2+12-7a)(a^2+12-8a)-12a^2$

$$=(x-7a)(x-8a)-12a^2 \quad [\text{where } x=a^2+12]$$
$$=x^2-15ax+56a^2-12a^2$$
$$=x^2-15ax+44a^2$$
$$=(x-11a)(x-4a)$$
$$=(a^2+12-11a)(a^2+12-4a)$$
$$\qquad\qquad [\text{Restoring the value of } x]$$
$$=(a^2-11a+12)(a^2-4a+12).$$

29. Let the consecutive numbers be x, $x+1$, $x+2$, $x+3$.
Hence we have to show that $x(x+1)(x+2)(x+3)+1$ is a perfect square.

Now, $x(x+1)(x+2)(x+3)+1$
$$= \{ (x)(x+3) \} \{ (x+1)(x+2) \} +1$$
$$=(x^2+3x)(x^2+3x+2)+1$$
$$=a(a+2)+1 \qquad [\text{where } a=x^2+3x]$$
$$=a^2+2a+1$$
$$=(a+1)^2$$
$$=(x^2+3x+1)^2 \qquad [\text{Restoring the value of } a]$$

which is a perfect square.

CHAPTER XV

REMAINDER THEOREM & ITS APPLICATION ;

Factors of Expressions in Cyclic Order and Symmetric Expressions

87. Remainder Theorem.

Let any expression in x be denoted by E, and suppose when it is divided by $x-a$, the quotient is Q and the remainder R. Then we know that :—

$$E \equiv Q(x-a)+R \qquad ...(i)$$

Relation (i) is an identity and therefore true for all values of x.

If we put $x-a=0$ (*i.e*, $x=a$), R. H. S. of (i) becomes equal to $Q(a-a)+R=0+R=R$.

Therefore, **L. H. S.** of (i) should also become equal to **R** for the same value of x, that is, for $x=a$.

This shows that the value of E for $x=a$ is equal to **R** (the remainder).

This is the **Remainder Theorem** ; which may be briefly stated as follows :—

When an expression in x is divided by x−a, the remainder is equal to the value of the expression for x=a (found by putting the divisor, x−a equal to zero).

This shows that when an expression in x is divided by $x-a$, *the remainder may be found without performing the actual division*, for we have only to put $x=a$ in the expression and the value so obtained is the required remainder.

Example. Find, without actual division, the remainder when $3x^3-2x^2+5x-20$ is divided by $x-2$.

Solution. If $x-2=0$, we get $x=2$.

Substituting this value of x in the given exp., we get

$$\text{Remainder}=3(2)^3-2(2)^2+5\times2-20$$
$$=3\times8-2\times4+10-20$$
$$=24-8+10-20=6. \quad \text{Ans.}$$

Let us verify our result by actual division.

$$x-2)\overline{3x^3-2x^2+5x-20}(3x^2+4x+13$$
$$\underline{3x^3-6x^2}$$
$$4x^2+5x$$
$$\underline{4x^2-8x}$$
$$13x-20$$
$$\underline{13x-26}$$
$$6$$

We find that the answer is correct.

83. Application of the Remainder Theorem.

The main application of the remainder theorem has been explained in the last article, *viz.*, finding without actual division the remainder when an expression in x is divided by a binomial of the form $x-a$. This, however, leads to another important use of this theorem, *viz.* :—

To find whether an expression in x is divisible by a binomial

of the form $x-a$, [*that is whether* $(x-a)$ *is a factor of the expression*].

For, if it is so divisible, the remainder must be zero and this can be tested by the remainder theorem without actual division.

Example. Is $x+2$ a factor of x^3+3x^2+6x+8 ?

Solution. If $x+2=0$, $x=-2$.

Substituting this value of x in the given exp., we have :—
$$\text{Remainder}=(-2)^3+3(-2)^2+6(-2)+8$$
$$=-8+3\times4-12+8=-8+12-12+8=0.$$

$\therefore$ the expression is exactly divisible by $x+2$, that is, $x+2$ is a factor of the expression.

EXERCISE 58

Find, without actual division, the remainder when :—

1. x^3-3x^2-4x+8 is divided by $x-1$. [*Solved*]
2. x^3+x^2-3x-4 is divided by $x-2$.
3. $2x^3-x^2+3x+1$ is divided by $x-3$.
4. x^4-3x^3-5x-2 is divided by $x-4$.
5. $3x^4+2x^2-5x-1$ is divided by $x+1$. [*Hint*]
6. x^6+x^5+x+1 is divided by $x+2$.

———————

7. Is $x-1$ a factor of $5x^5-4x^4+x^3-3x^2+5x-4$? [*Hint*]
8. Is $x-2$ a factor of $x^3-7x^2+14x-8$?
9. Is $x+2$ a factor of $2x^3-x^2-13x-6$?
10. Is $x+3$ a factor of x^3-5x^2-6x+6 ?

———————

11. Is $2x-1$ a factor of $2x^3-7x^2+11x-4$? [*Solved*]
12. Is $2x+1$ a factor of $2x^3-x^2-13x-6$?
13. Is $2x-3$ a factor of $2x^3-5x^2+5x-3$?
14. Is $3x+2$ a factor of $3x^3+5x^2+5x+4$?

———————

15. Show that $x+1$ is a factor of x^5+1, but not a factor of x^6+1. [*Solved*]

16. Show that $x+1$ is a factor of x^8-1, but not a factor of x^9-1.

17. Show that $x-1$ is a factor of $x^{10}-1$ and also of $x^{11}-1$.

18. Show that $x-1$ is neither a factor of $x^{12}+1$ nor of $x^{13}+1$.

19. Show that $x+1$ is a factor of x^n+1 only when n is add, and a factor of x^n-1 only when n is even.

20. Show that $x-1$ is always a factor of x^n-1 but never a factor of x^n+1.

---

21. Prove without actual division that $2x^4-6x^3+3x^2+3x-2$ is divisible by x^2-3x+2. [Hint]

22. Prove without actual division that $x^4+2x^3-2x^2+2x-3$ is divisible by x^2+2x-3.

---

23. For what value of k is $x^3+kx+2k-2$ divisible by $x+1$? [Solved]

24. For what value of a is $2x^3+ax^2-18x+6a$ divisible by $x-3$?

25. For what value of a is $2x^3+ax^2+11x+a+3$ divisible by $2x-1$?

26. For what values of a and b is the expression x^3+ax^2+bx+6 divisible by x^2+3x+2? [Solved]

27. For what values of m and n is the expression $x^3+2mx^2+nx-80$ divisible by x^2-x-6?

28. For what values of k and l is the expression $2x^3+2kx^2+4lx+3l$ divisible by $2x^2-x-1$?

SOLUTIONS & HINTS—EXERCISE 58

1. If $x-1=0$, we have $x=1$.

Substituting this value of x in the given exp., x^3-3x^2-4x+8, we have :—

Remainder $=(1)^3-3(1)^2-4(1)+8=1-3-4+8=2$.   Ans.

Note. It will be noticed that the value of an expression in x for $x=1$ is found by merely adding the coefficients.

5. If $x+1=0, x=-1$. Put $x=-1$ in the given exp.

7. Find the remainder when the given exp. is divided by $x-1$. If the remainder is 0, $x-1$ is a factor.

11. If $2x-1=0$, $2x=1$ or $x=\frac{1}{2}$.

Substituting this value of x in the given exp., we get

Remainder $= 2(\frac{1}{2})^3 - 7(\frac{1}{2})^2 + 11 \times \frac{1}{2} - 4$

$= 2 \times \frac{1}{8} - 7 \times \frac{1}{4} + \frac{11}{2} - 4$

$= \frac{1}{4} - \frac{7}{4} + \frac{11}{2} - 4 = 0.$

$\therefore$ $2x-1$ is a factor of the given expression.

15. If $x+1=0$, $x=-1$

Substituting this value of x in $x^5 + 1$, we get remainder $= (-1)^5 + 1 = -1 + 1 = 0$

$\qquad$ $[\because (-1)^5 = -1 \times -1 \times -1 \times -1 \times -1 = -1]$

$\therefore$ $x+1$ is a factor of $x^5 + 1$.

Substituting in $x^6 + 1$, we get remainder $= (-1)^6 + 1 = 1 + 1$ $= 2$ (not 0) $\qquad [\because$ Product of even number of negative terms is positive]

$\therefore$ $x+1$ is not a factor of $x^6 + 1$.

21. $x^2 - 3x + 2 = (x-1)(x-2)$

If we show that the given exp. is divisible by $x-1$ as well as by $x-2$, then, since the two factors are different from each other, it will follow that the exp. is divisible by their product, i.e., by $x^2 - 3x + 2$.

23. The remainder when $x^3 + kx + 2k - 2$ is divided by $x+1$ is equal to $(-1)^3 + k(-1) + 2k - 2$

$= -1 - k + 2k - 2$

$= k - 3$

We require that this remainder should be 0.

$\therefore$ we have $k - 3 = 0$

or $k = 3$, which is the value required.

26. $x^2 + 3x + 2 = (x+1)(x+2)$

$\therefore$ The exp. must be divisible by $x+1$ as well as by $x+2$.

Remainder corresponding to $x+1$ is $(-1)^3+a(-1)^2+b(-1)$
$+6=-1+a-b+6=a-b+5$

Remainder corresponding to $x+2$ is $(-2)^3+a(-2)^2+b(-2)$
$+6=-8+4a-2b+6=4a-2b-2$.

Both these remainders must be 0 ; hence we have :—

$$a-b+5=0 \qquad\qquad\qquad\qquad ..(i)$$
$$4a-2b-2=0. \text{ or } 2a-b-1=0 \qquad\qquad ...(ii)$$

We have to solve these equations to get the values of a and b.

By subtraction $-a+6=0$, or $a=6$

Substituting this value of a in (i) we have $6-b+5=0$ or $b=11$

$$\therefore \qquad a=6, \quad b=11. \quad \textbf{Ans}$$

———

89   Factorisation by the use of Remainder Theorem.

The method is in reality a *trial method*, for we go on testing the divisibility of the given expression by different binomial factors of the type $x-a$ or $bx+c$ by employing the Remainder Theorem. However, several useless tests can be avoided by bearing in mind that :—

If the expression is arranged according to descending powers of x and there is no factor common to all its terms, divisibility by $bx+c$ should be tested only if :—

(i) *b is a factor of the first coefficient,*

(ii) *c is a factor of the last coefficient, and*

(iii) *b and c have no common factor*

Take, for example, the exp. $3x^3-8x^2-5x+6$

[The exp is arranged according to descending powers of x and no factor is common to all its terms]

The different factors of 3 are 1 and 3

The different factors of 6 are 1, 2, 3 and 6.

Hence we may test the divisibility of the exp. only by the following binomials :—

$x+1, x-1, \quad x+2, x-2, \quad x+3, x-3, \quad x+6, x-6, \quad 3x+1,$
$3x-1, \quad 3x+2, 3x-2.$

Note that we have not included the binomials $3x+3$, $8x-3$, $3x+6$, $3x-6$, because there is a common factor to the terms of each of these binomials.

It will be seen that the exp. is divisible by $x+1$, $x-3$ and $3x-2$.

In practice, however, as soon as a factor is found, the expression is divided by it and then the test is applied to the quotient obtained. But when the quotient obtained is of second degree it is factorised by the methods of the last chapter.

EXERCISE 59

Factorise :—

1. $x^3-9x^2+18x-10$. [Solved] 2. $a^3-8a^2+25a-18$.
3. $3x^3+8x^2-8x-3$. 4. x^3-x-6.
5. x^3+x^2+x-14. 6. $2x^3-3x^2-4$.
7. $x^3-5x+12$. 8. $x^3-5x-12$.
9. $8x^3+4x-3$. 10. $10x^3+9x^2-1$.

---

11. $x^3+6x^2+11x+6$. [Solved] 12. $x^3+10x^2+29x+20$.
13. $x^3-6x^2-13x+42$. 14. x^3+8x^2+x-42.
15. $x^3+9x^2+6x-16$. 16. $x^3+4x^2-11x-30$.
17. $4x^3+13x^2-9$. 18. $2x^3+5x^2-4x-3$.

---

19. $12a^3+4a^2b-3ab^2-b^3$. [Solved]
20. $x^3-31xy^2-30y^3$. 21. $x^3-19xy^2-80y^3$.
22. $x^3-8x^2y+4y^3$. 23. $9a^3-4ab^2-b^3$.
24. $4a^3+4a^2b+ab^2-2b^3$.

---

25. $x^4-4x^3+2x^2+x+6$. 26. $a^4-7a^3+5a^2+34a-24$.
27. $x^4-2x^3-13x^2+14x+24$. 28. $2a^4-15a^3+14a^2+51a+20$.
29. $2a^4-7a^3+4a^2+7a-6$. 30. $4a^4+7a^3-6a^2-7a+2$

SOLUTIONS & HINTS—EXERCISE 59

1   Given Exp. $= x^3 - 9x^2 + 18x - 10$.

The possible factors are $x-1$, $x+1$, $x-2$, $x+2$, $x+5$, $x-5$, $x+10$, $x-10$.   [See Art. 89]

Let us test $x-1$.   If $x-1=0$, $x=1$.

Put $x=1$, then the Given Exp. $= 1 - 9 + 18 - 10 = 0$.

$\therefore$ $x-1$ is a factor.

Dividing the given exp. by $x-1$ we get $x^2 - 8x + 10$ as quotient.

$$x-1)\overline{x^3 - 9x^2 + 18x - 10}(x^2 - 8x + 10$$
$$\underline{x^3 - x^2}$$
$$-8x^2 + 18x$$
$$\underline{-8x^2 + 8x}$$
$$10x - 10$$
$$\underline{10x - 10}$$
$$\times$$

$\therefore$ Given Exp. $= (x-1)(x^2 - 8x + 10)$.   **Ans.**

Note 1. $x^2 - 8x + 10$ cannot be further factorised.

Note 2.   The sum of the coefficients in the given exp. is equal to zero : this is a sufficient test to show that $x-1$ is a factor.

11.   Given Exp. $= x^3 + 6x^2 + 11x + 6$.

The possible factors are $x+1$, $x+2$, $x+3$, $x+6$.

[We have omitted $x-1$, $x-2$ etc., because all the coefficients in the given exp. are positive and therefore it cannot have a factor of this type].

If we put $x = -1$, the exp. becomes $= -1 + 6 - 11 + 6 = 0$

[Or, we may say, the sum of the coefficients of odd powers is equal to the sum of the coefficients of even powers, (remembering that the constant term is always taken as an even power of x)].

$\therefore$ $x+1$ is a factor

$\therefore$ Given Exp.=

$(x+1)(x^2+5x+6)$

$=(x+1)(x+2)(x+3)$. **Ans.**

$$x+1)\overline{x^3+6x^2+11x+6}(x^2+5x+6$$
$$\underline{x^3+\ \ x^2}$$
$$\ \ 5x^2+11x$$
$$\ \ \underline{5x^2+\ \ 5x}$$
$$\ \ \ \ 6x+6$$
$$\ \ \ \ \underline{6x+6}$$
$$\ \ \ \ \ \times$$

19. Given Exp.$=12a^3+4a^2b-3ab^2-b^3$.

Possible factors are $a-b$, $a+b$, $2a-b$, $2a+b$, $3a-b$, $3a+b$, $4a-b$, $4a+b$, $6a-b$, $6a+b$, $12a-b$, $12a+b$.

If $a=b$, Given Exp.$=12b^3+4b^3-3b^3-b^3$ is not $=0$

If $a=-b$, Given Exp.$=-12b^3+4b^3+8b^3-b^3$ is not $=0$

If $2a=b$, $\left((i.e.,\ a=\dfrac{b}{2}\right)$ Given Exp.$=$

$$\frac{12b^3}{8}+\frac{4b^3}{4}-\frac{3b^3}{2}-b^3 = \frac{3b^3}{2}+b^3-\frac{3b^3}{2}-b^3=0$$

$\therefore$ $2a-b$ is a factor.

Dividing the given exp. by $2a-b$, we get the quotient $6a^2+5ab+b^2$ which is $=6a^2+3ab+2ab+b^2$

$\ \ \ \ =3a(2a+b)+b(2a+b)$

$\ \ \ \ =(2a+b)(3a+b)$

$\therefore$ Given exp.$=(2a-b)(2a+b)(3a+b)$

$$2a-b)\overline{12a^3+4a^2b-8ab^2-b^3}(6a^2+5ab+b^2$$
$$\underline{12a^3-6a^2b}$$
$$\ \ 10a^2b-8ab^2$$
$$\ \ \underline{10a^2b-5ab^2}$$
$$\ \ \ \ 2ab^2-b^3$$
$$\ \ \ \ \underline{2ab^2-b^3}$$
$$\ \ \ \ \ \times$$

90. Symmetrical Expressions.

An expression is said to be *symmetrical with respect to a pair of letters* if it remains unaltered in value when these letters are interchanged. For example $x+2y+2z$ is a symmetrical expression with respect to y and z; for by interchanging y and z it becomes $x+2z+2y$, which is equivalent to $x+2y+2z$. But $x+2y+2z$ is not symmetrical with respect to x and y, for when these letters are interchanged we get $y+2x+2z$ which is not equal to the original expression.

91. An expression is said to be *symmetrical* if it remains unaltered in value when *any pair* of letters in it are interchanged.

Then $a+b+c$, $ab+bc+ca$, $a^2+b^2+c^2$ are all symmetrical expressions.

It is easy to deduce from the above that *in a symmetrical expression all terms of the same type have the same coefficient.*

Definition. Terms which can be obtained from one another by successively interchanging the letters involved are said to belong to the *same type*. Thus, if a, b, c are the letters involved a^2b and b^2a are terms of the same type, for each can be obtained from the other by interchanging a and b. Similarly b^2c, c^2b. c^2a, a^2c are the terms belonging to the same type as a^2b and b^2a.*

92. Cyclic Order.

An expression is said to be written in *cyclic order* when each term in it can be derived from the preceding term by changing the first letter into the second, the second into the third, and so on till the last into the first. Thus with three letters a, b and c, the following are in cyclic order :—

(i) $ab+bc+ca$

(ii) $a^2(b-c)+b^2(c-a)+c^2(a-b)$

(iii) $(a+b-c)(b-c)+(b+c-a)(c-a)+(c+a-b)(a-b)$

Let us test the third expression In the first term of this expression change a into b, b into c and c into a. We get $(b+c-a)(c-a)$, which is the second term. By the same changes, the second term will give rise to the third term and the third to the first again.

---

* Note that b^2c can be obtained from b^2a by interchanging a and c. One may object that there is no letter c in b^2a and therefore we have only to *change a into c* and not *interchange a* and c. Answer to this objection is that *our statement is not wrong* ; if c had been present in the term we would have changed it into a The superfluity of the statement can be tolerated for the sake of its generality

The order in which the letters are changed is conveniently indicated by the arrow-heads in the diagram given here.
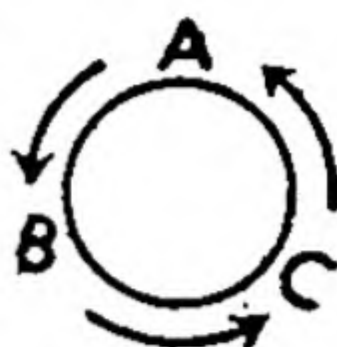
93. Sigma Notation.

An expression consisting of the sum of a number of terms of *the same type* may be written in an abbreviated form by placing the Greek letter Σ, called **Sigma**, before any one term of the type. Thus, with respect to three letters a, b, c,

Σa stands for $a+b+c$,

Σab „ $ab+bc+ca$,

$\Sigma a^3(b-c)$ „ $a^3(b-c)+b^3(c-a)+c^3(a-b)$,

      etc., etc.

Note. With four letters a, b, c, d, Σab does not mean $ab+bc+cd+da$. for though in cyclic order, this expression *does not contain all terms of the type ab.*

In this case $\Sigma ab = ab+ac+ad+bc+bd+cd$.

Similarly, Σa^2b (with regard to a, b, c) is not equal to $a^2b+b^2c+c^2a$, but $a^2b+a^2c+b^2c+b^2a+c^2a+c^2b$.

EXERCISE 60

Arrange the following expressions in *cyclic order* :—

1. $a+c+b$.
2. $b^2+a^2+c^2$.
3. $ab+ac+bc$.
4. $a^3b+ac^3+b^3c$.
5. $(a-b)(a-c)(b-c)$. [*Hint*]
6. $(a+b)(c+b)(a+c)$.
7. $x^2(y+z)+z^2(y+x)+y^2(x+z)$.
8. $x^2(y^3-z^3)+z^2(x^3-y^3)+y^2(z^3-x^3)$.

Write completely the *cyclic expressions* in a, b, c (taken in this order) whose first terms are —

9. a^2b. [*Hint*]
10. ab^3.
11. ab^2c.
12. $a(b^2-c^3)$.

13. $a^3(b-c)$.

14. $\dfrac{ab}{a-b}$.

Write out in full the following expressions which are *symmetric* with regard to a, b, c :—

15. Σa^3.

16. Σbc.

17. $\Sigma(a^2-bc)$.

18. $\Sigma c^2(b-c)$.

19. Σa^2b^3 [*Hint*].

20. $\Sigma a(b-c)(c-a)$.

21. $\Sigma\dfrac{a^2}{(a-b)(a-c)}$.

22. $\Sigma\dfrac{a^3}{bc(b-c)}$.

SOLUTIONS & HINTS—EXERCISE 60

5. Since the reqd. order is $a \to b \to c \to a$, we cannot have $(a-c)$. But $(a-c)$ cannot be changed into $(c-a)$ unless we place a negative sign outside ; for the value of the expression is not to be changed.

9. We require a *cyclic* expression and not a *symmetric* one. Therefore we need not necessarily take all the terms of the type a^2b. All we have to take care of is the order of the letters a, b, c.

19. Unlike question 9, we must take all the terms of the type a^2b^3 and should not mind the cyclic order.

---

94. **Factors of expressions in cyclic order** are often obtained by arranging them according to the powers of the letters involved turn by turn at different stages of the process, each stage giving us a new factor.

Example. Factorise $x^2(y-z)+y^2(z-x)+z^2(x-y)$.

Solution. First arrange according to the powers of x; then :—

Given Exp. $= x^2(y-z)-x(y^2-z^2)+(y^2z-z^2y)$

$\qquad = x^2(y-z)-x(y+z)(y-z)+yz(y-z)$

$\qquad = (y-z)\{x^2-x(y+z)+yz\}$.

Now arrange the exp. within { } according to powers of y; then :—

Given Exp. $=(y-z)\{ y(z-x)-(xz-x^2) \}$
$\qquad =(y-z)\{ y(z-x)-x(z-x) \}$
$\qquad =(y-z)(z-x)(y-x)$
$\qquad =-(x-y)(y-z)(z-x).$ [Putting the result in cyclic order]

95. Use of Remainder Theorem for the example of the last article is very interesting and instructive.

Given Exp. $=x^2(y-z)+y^2(z-x)+z^2(x-y)$

If $x=y$, the exp. $=y^2(y-z)+y^2(z-y)+z^2(0)=0$

$\therefore$ $x-y$ is a factor of the exp.   [Remainder Theorem]

Similarly $y-z$ and $z-x$ are also factors.

$\therefore$ Given Exp. $=(x-y)(y-z)(z-x) \times$ some other factors

But the given exp. is of third degree and $(x-y)(y-z)(z-x)$ is also of the third degree, hence the other factor can only be a constant (free from x, y, z).   Let this factor be k.

$\therefore$ $x^2(y-z)+y^2(z-x)+z^2(x-y)\equiv k(x-y)(y-z)(z-x).$

Since this is an identity ; it must be true for all values of x, y and z.   Put $x=0, y=1, z=-1$, then we have :—

$\qquad 0(1+1)+1(-1-0)+1(0-1)=k(0-1)(1+1)(-1-0)$
or $0-1-1=k\times(-1)(2)(-1)$
or $\qquad -2=2k$
or $\qquad k=-1$

$\therefore$ $x^2(y-z)+y^2(z-x)+z^2(x-y)\equiv -(x-y)(y-z)(z-x).$

96. Some Symmetric Expressions can also be factorised by the methods of Articles 94 and 95, but several miscellaneous methods and artifices have also to be used, some of which will be illustrated in the solutions and hints to the next exercise.

<center>**EXERCISE 61**</center>

Factorise :—

1. $a^2(b-c)+b^2(c-a)+c^2(a-b).$ [Hint]
2. $ab(a-b)+bc(b-c)+ca(c-a).$

3. $a(b^2-c^2)+b(c^2 \quad a^3 +c(a^2-b^3)$.

4. $a^3(b-c)+b^3(c-a)+c^3(a-b)$.   [*Hint*]

5. $a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)$

6. $(a-b)^3+(b-c)^3+(c-a)^3$.   [*Hint*]

7. $\Sigma a^4(b-c)$.   [*Hint*]   8. $\Sigma a^3(b^2-c^2)$.

9. $\Sigma yz(y^3-z^3)$.   10. $\Sigma x^2(y^3-z^3)$.

---

11. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.   [*Hint*]

12. $x(y^2+z^2)+y(z^2+x^2)+z(x^2+y^2)+2xyz$.

13. $(a+b+c)(ab+bc+ca)-abc$.   [*Hint*]

14. $a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc$.

15. $ab(a-b)-bc(b+c)+ca(c+a)$.

16. $(b-c)(b+c)^2+(c-a)(c+a)^2+(a-b)(a+b)^2$.   [*Hint*]

---

17.   (i) $ab(a+b)+bc(b+c)+ca(c+a)+3abc$.   [*Solved*]

      (ii) $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$.   [*Hint*]

      (iii) $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3abc$.

18. $(a+b)(b+c)(c+a)+abc$.

19. $x^2(y+z)+y^2(z+x)+z^2(x+y)+x^3+y^3+z^3$.   [*Hint*]

20. $a(b-c)^2+b(c-a)^2+c(a-b)^2+9abc$.

21. $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$.

22. $\Sigma(b+c)(b-c)^3$.   [*Hint*]

SOLUTIONS & HINTS—EXERCISE 61

1.   See Solved example, Art. 94.

4.   Given Exp. $=a^3(b-c)+b^3(c-a)+c^3(a-b)$

$$=a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)$$

               [arranging according to powers of a]

$$=a^3(b-c)-a(b-c)(b^2+bc+c^2)+bc(b-c)(b+c)$$

$$=(b-c)\{ a^3-a(b^2+bc+c^2)+bc(b+c) \}$$

Now arrange the exp. within $\{$ $\}$ according to powers of b etc. etc.

Or thus (By Remainder Theorem)

Proceeding as in Art. 95, we easily get the factors $a-b$, $b-c$, and $c-a$.

Now the given exp. is of fourth degree, therefore there is one more factor of the first degree. The most general symmetric exp. of the first degree in a, b, c is $k(a+b+c)$.

Hence we have :—

$$a^3(b-c)+b^3(c-a)+c^3(a-b)$$
$$\equiv k(a-b)(b-c)(c-a)(a+b+c).$$

Put $a=0$, $b=1$, $c=-1$ and get the value of k.

6. Either solve by the method of Art. 94 or see solved Example (*ii*) of Art. 85.

7. If Remainder Theorem is to be applied, remember that the most general symmetric expression of the second degree in a, b, c is $k(a^2+b^2+c^2)+l(ab+bc+ca)$.

Hence the given exp. is equal to

$$(a-b)(b-c)(c-a)\{ k(a^2+b^2+c^2)+l(ab+bc+ca)\}$$

To get the values of k and l, put (*i*) $a=0$, $b=1$, $c=-1$ (*ii*) $a=0$, $b=1$, $c=2$

(or any other two independent sets of values of a, b and c), and solve the two resulting equations.

11. No new method is involved.

Given Exp. $=a^2(b+c)+a(b^2+c^2+2ab)+bc(b+c)$
[arranging according to powers of a]
$$=a^2(b+c)+a(b+c)^2+bc(b+c),\text{ etc.}$$

13. Given Exp. $=\{ a+(b+c)\} \{ a(b+c)+bc\} -abc$
$$=a^2(b+c)+a(b+c)^2+abc+bc(b+c)-abc$$
$$=a^2(b+c)+a(b+c)^2+bc(b+c),\text{ etc.}$$

16. Put $a+b=x$, $b+c=y$, $c+a=z$

So that $(a+b)-(b+c)=x-y$, *i.e.*, $a-c=x-y$
$$\text{or } c-a=-(x-y)$$

Similarly, $a-b=-(y-x)$ and $b-c=-(z-x)$

Substituting these values, the given exp. becomes

$$=-(z-x)(y^2)-(x-y)(z^2)-(y-z)(x)^2$$
$$=-\{x^2(y-z)+y^2(z-x)+z^2(x-y)\}. \text{ etc.}$$

17. (*i*) Splitting $3abc$ into three parts $abc+abc+abc$, we have :—

Given Exp.$= \{ ab(a+b)+abc \} + \{ bc(b+c)+abc \}$
$$+ \{ ca(c+a)+abc \}$$
$$=ab(a+b+c)+bc(b+c+a)+ca(c+a+b)$$
$$=(a+b+c)(ab+bc+ca)$$

(*ii*) Open the brackets and bring the exp. to the form of the exp. of part (*i*).

19. Combine x^3 with $x^2(y+z)$ and similarly y^3+z^3 with the other two parts respectively.

22. $(b+c)(b-c)^3=(b+c)(b-c)(b-c)^2=(b^2-c^2)(b-c)^2$
$$=(b^2-c^2)(b^2+c^2-2bc)$$
$$=(b^2-c^2)(b^2+c^2)-2bc(b^2-c^2)$$
$$=b^4-c^4-2bc(b^2-c^2)$$

We get similar results for the other two parts.

$\therefore$ The whole Exp.$=-2 \{ ab(a^2-b^2)+bc(b^2-c^2)$
$$+ca(c^2-a^2) \}$$ [The other terms cancel out]
$$\text{etc.}$$

CHAPTER XVI

HIGHEST COMMON FACTOR   (H. C. F.)

97. Definition. The *highest common factor* of two or more algebraical expressions is the *expression of highest degree* which divides each of them without remainder.

98. H. C. F. by Factors.

From the definition of the last article we easily get the following

Rule.

(i) *Put each expression in the form of factors.*

(ii) *Find the H. C. F. of the numerical coefficients by any rule of Arithmetic.*

(iii) *Select all different factors with their highest possible powers, common to all the expressions.*

Then, the product of the numerical H. C. F. found in step (ii) and the factors selected in step (iii) is the required H.C.F.

99. To find the **H. C. F.** of monomials (Simple Expressions) we start with step (ii) of the rule of the last article, for such expressions are naturally in the form of factors.

Example. Find the H. C. F. of $8a^3b^6c$, $12a^7b^4d$ and $20a^5b^5d^2$.

Solution. H. C.F. of 8, 12 and 20 is 4.

"a" occurs in all the expressions and its highest power, common to all the expressions is a^3,

"b" occurs in all the expressions and its highest power, common to all the expressions is b^4.

Neither "c" nor "d" is present in *all* the expressions.

$\therefore$ Reqd. H. C. F. $= 4a^3b^4$.

EXERCISE 62

Find the H. C. F. of :—

1. ab, ac.
2. b^2c, bc^2.
3. x^3y, x^2z.
4. x^3y^3, xy^2.
5. $3a^2b^4c$, $4a^5b^2$.
6. $4ab^4$, $6a^2b^3c$.
7. $6a^2bc$, $9abc^2$.
8. $8x^2y^3z^4$, $12x^3yz$.
9. $20x^4y^3z^2$, $35x^2y^2z^3$.
10. $21a^3b^7c^5$, $35a^5b^4c^8$.

---

11. $18a^2b^3c^3$, $30a^3b^2c^2$, $42a^4b^4$.
12. $35x^2y^3z$, $42x^3zy^2$, $30xy^2z^3$.
13. $15x^5y^3z^7$, $60x^3y^7z^6$, $25x^4y^5z^2$.
14. $50ab^2z^6$, $100a^3b_2^4$, $125a^4b^5$.

---

100. H. C. F. of Compound Expressions by Factors.

The rule has already been given in Art. 98.

If the compound expressions are given in the form of factors we save the first step of the process as in the case of monomials. [See next Exercise Q. 1 to 12].

EXERCISE 63

Find the H. C. F. of :—

1. $ab(a+b)$, $b(a-b)$. [Solved]
2. $ab(a-b)$, $b^2(a-b)$.   3. $a^2b^2(a^2+b^2)$, $ab(a^2+l^2)$.
4. $a(a+b)^2$, $ab(a+b)$.   5. $ab^2(a+b)(a-b)$, $a(a-b)$.
6. $a^2(a-b)^2$, $a(a-b)^3$.   7. $x^3(x+y)^2$, $y^3(x+y)^3$.
8. $15xyz(x+y)^2$, $20y^2z^2(x-y)^2$.
9. $12(a^2+b^2)$, $18(a+b)^2$. [Hint]
10. $20(a+b)^2$, $21(a-b)^2$.   11. $ab(a+b)^2$, $bc(b+c)^2$.
12. $24a^4(a+b)$, $30a^2(a^2+b^2)$.

———

13. a^2+ab, a^3-b^3. [Solved]
14. $4(x+1)^2$ and $6(x^2-1)$.   15. $2a^2-2ab$, a^3-a^2b.
16. $6a^2-9ab$, $4a^2-9b^2$.   17. x^2+y^2, x^4+y^4.
18. $3a^3b-3ab^3$, $6a^5b^2-6a^2b^5$.
19. $2a^2b-6ab^2$, $2a^2-18b^2$.   20. $x^2-2xy+y^2$, $(x-y)^3$
21. $2x^3+2a^2x$, $4x^4-4a^4$.
22. a^3+8b^3, $a^2+ab-2b^2$. [Hint]
23. $4(a^2+3a+2)$, $6(a^2-4)$.   24. a^2-5a+6, a^2+a-6.
25. $a^2-18a+81$, a^2+a-90.
26. $2a^2-7a+3$, $3a^2-7a-6$.
27. $3a^2-2a-1$, $3a^2+4a+1$.
28. $4x^3y-4xy^3$, $8x^4y-8xy^4$.
29. $2(a^6-a^2b^4)$, $6(a^7b-ab^7)$.   30. x^2+y^2, $x^4+x^2y^2+y^4$.
31. $12(x^6-y^6)$, $16(x^4-y^4)$.   32. a^4-4, a^3+a^2+2a+2.
33. a^3-ax^2, a^2-ax, a^2x-ax^2.

34. a^2-b^2, a^3-b^3, a^4-b^4.

35. $15(x^3+1)$, $20(x^4+x^2+1)$, $25(x^4-x^3+x^2)$.

36. $3a^4+8a^3+4a^2$, $3a^5+11a^4+6a^2$, $3a^4-16a^3-12a^2$.

SOLUTIONS & HINTS—EXERCISE 63

1. I Exp.$=ab(a+b)$

   II Exp.$=b(a-b)$.

Both are already in the form of factors and cannot be further factorised.

The factor a is present in the I Exp., but not in II Exp. [Here a beginner might commit a mistake. He might think that "a" is present in the II Exp. But note carefully that that "a" *is not a factor* : the factor is $a-b$ and not 'a'.]

The factor b is present in both the expressions.

Again, neither $(a+b)$ nor $(a-b)$ is common to the two expressions.

Hence H C. F.$=b$.

9. a^2+b^2 cannot be factorised ; also it is not a factor of $(a+b)^2$. [Some beginners even commit the ridiculous mistake of supposing a^2+b^2 equal to $(a+b)^2$].

13. I Exp.$=a^2+ab=a(a+b)$

   II Exp$=a^2-b^2=(a+b)(a-b)$

The only factor common to both is obviously $(a+b)$.

∴ H. C. F.$=a+b$.

22. II Exp $=a^2+ab-2b^2=(a+2b)(a-b)$ [∵ the numbers whose sum is 1 and product -2 are 2 and -1]

---

101. H. C. F. by Successive Division.

The method is analogous to that used in Arithmetic. For example, to find the H. C. F. of x^2+3x+2 and x^2+4x+3 we have the following process :—

$$x^3+3x+2) \; x^2+4x+3 \; (\; 1$$
$$\underline{x^2+3x+2}$$
$$x+1) \; x^2+3x+2 \; (x+2$$
$$\underline{x^2+x}$$
$$2x+2$$
$$\underline{2x+2}$$
$$\times$$

H. C. F. is the *last divisior, viz.,* $x+1$.

However, in many cases certain modifications are necessary, which we propose to explain through one example each. It may be remarked that the *examples chosen for this purpose are very simple and meant only to teach the different rules, all of which taken together constitute the complete method.*

Examples.

Group 1 *To find the H. C. F. of two expressions having no monomial factors, (i.e., factors consisting of one term only).*

Find the H. C. F. of :—

(i) $x^2+x-6, \; x^2+7x-18$ (ii) $x^2+2x-3, \; 2x^2+3x-5.$

(iii) $2x^2-x-1, \; 3x^2-x-2$

[*It will be noticed that none of the expressions given in the above examples is divisible by a one-term factor. The rules that we are going to enunciate are true only if this condition is satisfied.*]

Solutions.

(i) To begin with, we proceed as usual :—

$$x^2+x-6 \;) \; x^2+7x-18 \; (\; 1$$
$$\underline{x^2+ \; x-6}$$
$$6x-12$$

Examining the remainder we find that the *largest monomial* which can exactly divide it is 6. Remember the remainder *must be divided* by it.

Rule 1 *Every remainder must be divided by the largest possible monomial.*

Observing this rule we have the remaining process as follows :—

$$6 \,)\, 6x-12$$
$$x-2 \,)\, x^2+x-6 \,(\, x+3$$
$$x^2-2x$$
$$\overline{\qquad 3x-6}$$
$$3x-6$$
$$\overline{\qquad \times}$$

H. C. F.$=x-2$. **Ans.**

(ii) $\qquad x^2+2x-3)\, 2x^2+3x-5 \,(2$
$$2x^2+4x-6$$
$$\overline{\quad -\ x+1}$$

The sign of the first term of the remainder is negative. This is generally not desirable. Therefore we change the signs of all terms of the remainder (this is equivalent to dividing the remainder by -1).

Rule 2. *If the first term of a remainder is negative, change the signs of all its terms.*

Observing this rule we have the remaining process as follows :—

$$-\,)-x+1$$
$$x-1)\, x^2+2x-3 \,(\, x+3$$
$$x^2-x$$
$$\overline{\qquad 3x-3}$$
$$3x-3$$
$$\overline{\qquad \times}$$

H. C. F.$=x-1$. **Ans.**

(iii) $\qquad 2x^2-x-1 \,)\, 3x^2-x-2 \,($

As the expressions stand we cannot begin to divide without using a fractional quotient. To remove this difficulty we may multiply the dividend by a suitable number. The smallest number which can serve our purpose is 2.

Rule 3. *At any stage, whenever the division is not possible without using a fractional quotient, we may multiply the dividend by a suitable number so that the division is possible without using a fractional quotient.*

Using this rule we have the following process :—

$$3x^3 - x - 2$$
$$2$$

$$2x^2 - x - 1 \;) \; \overline{6x^2 - 2x - 4} \; (\; 3$$
$$\underline{6x^2 - 3x - 3}$$
$$x - 1 \;) \; 2x^2 - x - 1 \; (\; 2x + 1$$
$$\underline{2x^2 - 2x}$$
$$x - 1$$
$$\underline{x - 1}$$
$$\times$$

H. C. F. $= x - 1$ **Ans.**

Group 2. *H. C. F. of two expressions having monomial factors.*

Find the H. C. F. of :—

(i) $x^2 - 9x + 20$, $2x^3 - 2x^2 - 40x$.

(ii) $4x^3 - 20x^2 + 24x$, $6x^3 + 6x^2 - 36x$.

Solutions.

(i) The first Exp. has no monomial factor while the largest monomial factor of the second Exp is $2x$. We must divide the second expression by $2x$.

Rule 4. *When one of the given expressions has a monomial factor (which is the largest possible one), it must be removed by division.*

Hence, the following process :—

$$2x \;) \; 2x^3 - 2x^2 - 40x$$
$$x^2 - x - 20 \;) \; x^2 - 9x + 20 \; (\; 1$$
$$\underline{x^2 - x - 20}$$
$$-8x + 40$$

[The remainder can be divided by 8. Also the sign of the first term is negative. Hence we divide the remainder by 8 and also change the signs of its terms (Rules 1 and 2), or say, we divide by -8]

$$-8 \;) \; -8x + 40$$
$$x - 5 \;) \; x^2 - x - 20 \; (\; x + 4$$
$$\underline{x^2 - 5x}$$
$$4x - 20$$
$$\underline{4x - 20}$$
$$\times$$

H. C. F. $= x - 5$. **Ans.**

(*ii*) Both the expressions have monomial factors (4*x* and 6*x* respectively). We remove these factors by disision and find the H. C. F. of the quotients obtained. The required H. C. F. is equal to this H. C. F. × the H. C. F. of 4*x* and 6*x*.

Rules 5 *If both the given expressions have monomial factors (taking the largest possible ones) remove them by division and find the H. C. F. of the quotient obtained ; then multiply this H. C. F. by the H. C. F. of the monomial factors, thus getting the required H. C. F.*

According to the rule we have the following process :—

$$4x \,) \, \dfrac{4x^3 - 20x^2 + 24x}{x^2 - 5x + 6} \qquad\qquad 6x \,) \, \dfrac{6x^3 + 6x^2 - 36x}{x^2 + x - 6}$$

$$
\begin{array}{l}
x^2 - 5x + 6 \,)x^2 + \; x - 6 \, (\; 1 \\
\qquad\quad x^2 - 5x + 6 \\
\qquad\quad \overline{6 \,) \, 6x - 12} \\
\qquad\qquad\qquad x - 2 \,) \; x^2 - 5x + 6 \, (\; x - 3 \\
\qquad\qquad\qquad\quad x^2 - 2x \\
\qquad\qquad\qquad\quad \overline{-3x + 6} \\
\qquad\qquad\qquad\quad -3x + 6 \\
\qquad\qquad\qquad\quad \overline{\times}
\end{array}
$$

H. C. F. of 4*x* and 6*x* (the monomial factors of the given expressions)$= 2x$.

∴ Reqd. H. C F.$= 2x(x - 2) = 2x^2 - 4x$. **Ans.**

Note. Rule 4 is only a particular case of Rule 5, for in the former case one of the two monomial factors is 1 and therefore their H. C. F. is also 1, and therefore multiplication at the end is not necessary. However, we have given two rules for the sake of clearness

CONSOLIDATED METHOD

The rules stated and illustrated in the last article, taken together, constitute the complete method for finding the H C F of two expressions by successive division. They are re-stated here in one place for ready reference :—

General Method *Successive division to be performed as in Arithmetic. The last divisor is the reqd. H C. F.*

Group 1. (**When the two expressions have no monomial factors**).

Rule 1. *Every remainder must be divided by the largest possible monomial.*

Rule 2. *If the first term of a remainder is negative, change the signs of all its terms.*

Rule 3. *At any stage, whenever the division is not possible without using a fractional quotient, multiply the dividend by a suitable number so that the division is possible without using a fractional quotient.*

Group 2. (**When the fractions have monomial factors**).

Rule 4. *When one of the given expressions has a monomial factor (which is the largest possible one), it must be removed by division.*

Rules 5. *If both the expressions have monomial factors (taking the largest possible ones) remove them by division and find the H. C. F. of the quotients obtained ; then multiply this H. C. F. by the H. C. F. of the monomial factors, thus getting the required H. C. F.*

102. We may briefly give the proofs for the rules of the last article : — [*The particular references are from the corresponding examples of that Article.*]

Proof for the General Method of Successive Divisions. As in Arithmetic.

Proof for rule 1. The required **H. C. F.** is the **H. C. F.** of the remainder under consideration and the divisor which gave that remainder [$6x-12$ and x^2+x-6]. Since the monomial factor [6] or any one of its factors (2 or 3) is not a factor of the divisor [x^2+x-6], therefore the removal of the monomial factor [6] cannot affect the result.

Proof for rule 2. This can only change the sign of our result. But in Algebra, if an expression is an exact divisor of another, its negative is also an exact divisor of the same. We change the signs simply because it is not desirable that the first term of our result, (or, in fact, of any divisor in the process) should be negative.

Proof for Rule 3. This is the converse of Rule 1, for there we remove a monomial factor [6] while here we introduce one [2]. Hence the proof is the same.

Proof for Rule 4 As that for Rule 1.

Proof for Rule 5 By this rule we, in a way, factorise the two expressions, find the highest common factor of the monomial factors [$4x$ and $6x$] and the other factors [x^2-5x+6 and x^2+x-6] separately and multiply them. [See Art 98]

103. The following example is meant to illustrate most of the above rules. Also, its solution will be presented in two forms, first the usual form of successive division which, though lengthy, is clear to the beginner, and second the concise form involving the same processes, which, though apparently confusing, is very convenient when properly learnt and requires much less space

Example. Find the H. C. F. of $15x^3-x^2+3x-2$ and $20x^3-3x^2+3x-2$.

Solution.

Form 1

$$20x^3-3x^2+3x-2$$
$$3 \qquad\qquad\qquad\qquad \textit{Multiply by 3. [\textbf{Rule 3}]}$$

$$15x^3-x^2+2x-2)\overline{60x^3-9x^2+9x-6}(4$$
$$\underline{60x^3-4x^2+12x-8}$$
$$-)\underline{-5x^2-\ \ 3x+2}\ \textit{Change signs. [\textbf{Rule 2}]}$$
$$5x^2+\ 3x-2)\ 15x^3-x^2+3x-2(3x-2$$
$$\underline{15x^3+9x^2-6x}$$
$$-10x^2+9x-2$$
$$\underline{-10x^2-6x+4}$$
$$3)\ \underline{15x-6}\ \textit{Divide}$$
$$\underline{5x-2}\ \textit{by 3}$$
$$\textit{[\textbf{Rule 1}]}$$

$$5x-2)5x^2+3x-2(x+1$$
$$\underline{5x^2-2x}$$
$$\underline{5x-2}$$
$$5x-2$$
$$\overline{\qquad\times\qquad}$$

H. C. F.$=5x-2$. **Ans.**

Form 2.

$$
\begin{array}{c|ll}
3x & 15x^3 - x^2 + 3x - 2 & 20x^3 - 3x^2 + 3x - 2 \\
   & \underline{15x^3 + 9x^2 - 6x} & 3 \\
   & -10x^2 + 9x - 2 & \underline{60x^3 - 9x^2 + 9x - 6} \quad 4 \\
-2 & \underline{-10x^2 - 6x + 4} & 60x^3 - 4x^2 + 12x - 8 \\
   & 3)\underline{15x - 6} & -)-\underline{5x^2 - 3x + 2} \quad x \\
   & 5x - 2 & 5x^2 + 3x - 2 \\
   & & \underline{5x^2 - 2x} \\
   & & 5x - 2 \\
   & & 5x - 2 \quad 1 \\
   & & \times
\end{array}
$$

H. C. F. $= 5x - 2$. **Ans.**

Note Form 2 is easily understood when compared with Form 1.

EXERCISE 64

Find the H C. F. of :—

Set I (Involving the use of not more than one rule)

1. $x^2 + 3x - 4$ and $x^3 + 5x^2 + 3x - 9$.
2. $x^2 - 4x + 3$ and $4x^3 - 9x^2 - 15x + 18$.
3. $a^3 + a^2 - 2$ and $a^3 + 2a^2 - 3$.
4. $x^3 + 2x^2 - 13x + 10$ and $x^3 + x^2 - 10x + 8$.
5. $2x^3 - 5x^2 + 11x + 7$ and $4x^3 - 11x^2 + 25x + 7$.
6. $x^3 + 6x^2 + 11x + 6$ and $x^3 + 8x^2 + 17x + 10$.
7. $3x^3 + 10x^2 + 7x - 2$ and $3x^3 + 13x^2 + 17x + 6$. [*P.U. 1916*]
8. $x^3 - 5x^2 - 95x + 40$ and $x^3 - 6x^2 - 86x + 35$.
9. $4x^3 - 3x^2 - 24x - 9$ and $8x^3 - 2x^2 - 53x - 39$.
10. $x^3 - x^2 - 5x - 3$ and $x^3 - 4x^2 - 11x - 6$.
11. $4x^3 - 5x + 6$ and $4x^3 - 8x^2 + 7x - 2$.
12. $2x^3 + 4x^2 - 3x - 6$ and $3x^3 + 6x^2 - 4x - 8$.

———

Set II (involving the use of more than one rule)

13. $3a^3 - 13a^2 + 23a - 21$ and $6a^3 + a^2 - 44a + 21$.
14. $4x^4 + 4x^3 - 3x^2 + x - 1$ and $3x^4 + 5x^3 - 2x^2 - 3x + 1$

15. $x^4 - 39x - 22$ and $11x^4 - 39x^3 - 8$.

16. $8x^4 + 3x + 10$ and $10x^4 + 3x^3 + 8$.   [P. U. 1927]

17. $20x^4 + x^2 - 1$ and $25x^4 + 5x^3 - x - 1$.

18. $3x^4 - 3x^3 - 2x^2 - x - 1$ and $9x^4 - 3x^3 - x - 1$.

19. $x^4 - 2x^3 - 5x^2 + 2x + 1$ and $x^4 - 3x^3 - 3x^2 + 6x + 2$.

20. $2x^4 - 7x^3 - 4x^2 + x - 4$ and $3x^4 - 11x^3 - 2x^2 - 4x - 16$.

———

21. $6a^4 - 13a^3 + 6a^2$ and $8a^4 - 36a^3 + 54a^2 - 27a$.   [Solved]

22. $4a^4 - 14a^2 + 2a^2 + 12a$ and $9a^5 - 33a^4 + 6a^3 + 36a^2$.

23. $x^4 - x^3 - x^2 - 2x$ and $x^3 - 2x^2 + 3x - 6$.

24. $x^5 + x^4 - 7x^2 + 5x$ and $x^4 - 9x^2 - 30x - 25$

25. $30x^5 - 5x^4 - 175x^3$ and $42x^5 - 49x^4 - 224x^3 + 210x^2$.

———

Set III (Harder Examples)

26. $2x^3y - 3x^2y^2 + y^4$ and $3x^4 - 5x^3y + 2y^4$.   [Solved]

27. $4x^4y - 4x^2y^3$ and $6x^4y^2 - 6xy^5$

28. $2x^3 + 5x^2y + 5xy^3 + y^3$ and $2x^3 - 7x^2y + 5xy^2 - y^3$.

29. $2x^3 + 3x^2y - 9xy^2$ and $6x^3y - 17x^2y^2 + 14xy^3 - 3y^4$.

30. $x^5 + y^5$ and $x^7 + y^7$.

———

31. $2x^3 + x^2 - x - 2$ and $6x^3 - 4x^2 + 2x - 4$.

32. $a^4 + 3a^2 + 6a + 35$ and $a^4 + 2a^3 - 5a^2 + 26i + 21$.

33. $2a^5 - 11a^2 - 9$ and $4a^5 + 11a^4 + 81$.

34. $4a^5 + 16a^4 + 44a^2 - 24a$ and $2a^5 - 6a^3 + 2a$.

35. $a^5 - a^3 - 4a^2 - 3a - 2$ and $2a^6 - a^4 - a^3 - 3a^2$.

36. $2x^5 - 4x^4 + 8x^3 - 12x^2 + 6x$ and $3x^5 - 3x^4 - 6x^3 + 9x^2 - 3x$.

———

37. $2x^3 + 7x^2 - 5x - 4$, $x^3 + 8x^2 + 11x - 20$,

and $2x^3 + 10x^2 + 49x + 20$.   [Hint]

38. $x^3 - 2x^2 + 1$, $2x^3 + x^2 + 4x - 7$, and $x^4 - x^2 + x - 1$.

———

39. Find the expression of highest degree which can divide $2x^4-2x^3+x^2+3x-6$ and $4x^4-2x^3+3x-9$ exactly.

[*Hint*]

40 Find the expression of highest degree which divides $2x^3-7x^2-8x-30$ and $2x^3+9x^2+16x+27$, leaving the remainders 5 and 6 respectively.

41. What value of x can make both the expressions $x^3+7x^2+17x+15$ and $x^3+8x^2+19x+12$ vanish ?

[*Hint*]

42. For what value of x will the values of the expressions $x^3-10x^2+26x-8$ and $x^3-9x^2+23x-12$ be zero each?

[*Hint*]

———

SOLUTIONS AND HINTS—EXERCISE 64

21.   I Exp.$=6a^4-13a^3+6a^2=a^2(6a^2-13a+6)$.   [*Rule 5*]

     II Exp.$=8a^4-36a^3+54a^2-27a=a(8a^3-36a^2+54a-27)$

                                             [*Rule 5*]

$$8a^3-36a^2+54a-27$$

$$\begin{array}{r} 3 \\ \hline 6a^2-13a+6)\overline{24a^3-108a^2+162a-81}(4a \\ 24a^3-52a^2+24a \\ \hline -56a^2+138a-81 \end{array}$$

               3   *Multiply by 3.* [*Rule 3*]

$$\begin{array}{r} -168a^2+414a-243)(-28 \\ -168a^2+364a-168 \\ \hline 25)\overline{50a-75} \end{array}$$ *Divided by 25.* [*Rule 1*]

$$\begin{array}{r} 2a-3)\overline{6a^2-13a+6}(3a-2 \\ 6a^2-9a \\ \hline -4a+6 \\ -4a+6 \\ \hline \times \end{array}$$

H. C. F. of a^2 and a $=a$

H. C. F. of the other factors $=2a-3$

$\therefore$ Reqd. H. C. F. $=a(2a-3)$.   **Ans.**

Multiply by 3. [*Rule 3*]

26. I Exp. $=2x^3y-3x^2y^2+y^4=y(2x^3-3x^2y+y^3)$.   [Rule 4]
     II Exp. $=3x^4-5x^3y+2y^4$.

$2x^3-3x^2y+0+y^3)3x^4-5x^3y+0+0+2y^4$
$\qquad\qquad\qquad 2 \qquad\qquad\qquad\qquad\qquad$ Multiply by 2. [Rule 3]

$\qquad\qquad 6x^4-10x^3y+0+0+4y^4(3x$
$\qquad\qquad 6x^4-9x^3y+0+3xy^3$
$\qquad\qquad \overline{\qquad\qquad}$
$\qquad\qquad -y)-x^3y+0-3xy^3+4y^4$ Divide by $-y$.
$\qquad\qquad\qquad x^3-0+3xy^2-4y^3 \qquad$ [Rule 1 & 2]
$\qquad\qquad\qquad 2 \qquad\qquad\qquad\qquad$ Multiply by 2.

$\qquad\qquad\qquad 2x^3-0+6xy^2-8y^3(1 \qquad\qquad$ [Rule 3]
$\qquad\qquad\qquad 2x^3-3x^2y+0-y^3$
$\qquad\qquad\qquad 3y)3x^2y+6xy-9y^3 \qquad$ Divide by $3y$.
$\qquad\qquad\qquad\quad x^2+2xy-3y^2 \qquad\qquad\qquad$ [Rule 1]

$x^2+2xy-3y^2)2x^3-3x^2y+0+y^3(2x-7y$
$\qquad\quad 2x^3+4x^2y-6xy^2$
$\qquad\quad \overline{\qquad\qquad\qquad}$
$\qquad\quad -7x^2y+6xy^2+y$
$\qquad\quad -7x^2y-14xy^2+21y^3$
$\qquad\qquad 20y^2)20xy^2-20y^3$ Divide by $20y^2$. [Rule 1]
$\qquad\qquad x-y)x^2+2xy-3y^2(x+3y$
$\qquad\qquad\quad x^2-xy$
$\qquad\qquad\quad \overline{\qquad\qquad}$
$\qquad\qquad\qquad 3xy-3y^2$
$\qquad\qquad\qquad 3xy-3y^2$
$\qquad\qquad\qquad \overline{\qquad\quad}$
$\qquad\qquad\qquad\quad \times$

Only the first. Exp. has a monomial factor

$\therefore$ Reqd. H. C. F $=x-y$. Ans. [See Rule 4].

37. First find the H. C. F. of any two expressions ; then find the H. C. F. of this H. C. F. and the third expression.

39. The Reqd. expression is the H. C. F. of the given expressions. [See Def. of H. C. F.]

41. "To make an expression vanish" means "to reduce its value to zero." Find the H. C. F. of the given expressions. If this H. C. F. becomes zero, the expressions will also become zero each [$\because$ H. C. F. is a factor of each Exp.] Hence put H. C. F. equal to zero and solve the resulting equation for x.

CHAPTER XVII

LOWEST COMMON MULTIPLE [L. C. M.]

104. Definition. The *Lowest Common Multiple* (L.C.M.) of two or more given expressions is an expression of the lowest degree which is exactly divisible by each of them.

Suppose the given expressions are a and b. Now ab, a^2b, ab^2, etc., are all divisible by a and b. but of these ab is of the lowest degree. Therefore the L C. M. is ab.

105. The L C M. of Simple expressions can be found by inspection.

Thus, L. C. M. of x^2, x^5, x^3 and x is x^5, for this is the *lowest* * *power of x divisible by each of them.* Note carefully that this lowest power is *the highest power among the given powers.*

Again, to find the L. C. M. of $6x^4y^3$, $12xy^2$ and $16x^3y^5$ we find the L. C. M. of numerical coefficients (6, 12 and 16). Then, as remarked above, we select the highest powers of x and y occuring in the given expressions. The product of the numerical L C. M. and these powers is the required L C. M. Therefore required L. C. M. $= 48x^4y^5$.

Hence the following

Rule. *First put down the L. C. M. of the numerical coefficients and then take each letter raised to its highest power occuring in the given expressions.*

EXERCISE 65

Find the L. C. M. of :—

1. ab, a^2.

2. a^2b^2, ab^3.

3. ab^4, a^4b.

4. ab, a^3b^3.

5. $2a^2$, $3ab^5$.

6. $4a^4b^4$, $6a^6b^6$.

7. $12ab$, $8cd$.

8. $2ab$, $3bc$, $4ca$.

9. $3x$, $4y$, $5z$.

10. $2a^2b$, $4b^2c$, $6c^2a$.

---

* The next lower power is x^4, but it is not divisible by x^5, which is one of the given expressions.

11. ab^2, b^3c^4, c^5a^6.   12. $7x^4y$, $8xy^5$, $2x^3y^3$.

13. $35x^2y^3z$, $42y^2z^3x$, $30z^2x^3y$.

14. $66x^2yz^4$, $44xy^3z^2$, $24x^3yz^3$.

---

106. L. C. M. of Compound Expression by Factors.

It is easy to see that if the compound expressions be put into the form of factors, the rule for finding their L. C M. will be the same as that of the last article with the word "letter" changed into "factor".

If the compound expressions be already in the form of factors we save the preliminary step of factorisation.

EXERCISE 66

Find the L. C. M. of :—

1. $2(a+b)$ and $(a+b)^3$.   [Solved]

2. $3(a-b)^2$ and $6(a-b)$.   3. $a^2(a-b)$ and $a^3(a+b)$.

4. $4a^3(a+b)^2$ and $6a^2b(a+b)$.

5. $8ab(a-b)$ and $12bc(b-c)$.

6. $10a^2b^2(a^2+b^2)$ and $12a(a+b)$.

---

7. a^4 and a^8+a^2.   [Solved].   8. $3b^3$ and $12b^2+4b$.

9. $4(a^2+4a)$ and $a^2+9a+20$.

10. $2(a-b)^2$ and $4(a^2-b^2)$.

11. $6(x^2+xy)$ and $6y(x^3+y^3)$.   [Solved].

12. $2(x^2-y^2)$ and $3(x^3-y^3)$.

13. $8abc(a^2-b^2)$, $12a^2(a^3+b^3)$.

14. x^2+5x+6, $2x(x^2+14x+24)$.

15. x^2-3x+2, x^3-x.   16 x^3-5x^2+4x, x^2-6x+8.

---

17. $2x^3-2x^2-12x$, x^3+x-2, x^2-4x+3.   [Solved].

18. $4x^3+4x^2-80x$, $2x^3-20x^2+48x$, x^3-x-30.

19. $2a^2 + 3a + 1,\ 4a^2 + 13a + 4,\ 6a^2 + 18a + 12.$

20. $6a^2 - 2a - 28,\ 3a^2 - 13a + 14,\ a^3 - 4a.$

———

21. $10(x^2 - y^2)^2,\ 15(x^3 - y^3),\ 20(x^2y - xy^2).$ [Solved]

22. $4(a^2 - 9),\ 8(a^3 + 27),\ 12(a^5 - 27a^2).$

23. $2x^2 - 2ax,\ 4x^2 - 4a^2,\ 6x^3 - 6a^3,\ (x^3 + a^3)^2.$

24. $a^3 + b^3,\ a^3 - b^3,\ a^4 + a^2b^2 + b^4,\ a^2 - ab + b^2.$

———

SOLUTIONS & HINTS—EXERCISE 66

1.   I Exp. $= 2(a + b)$
    II Exp. $= (a + b)^2.$

$\begin{cases} \text{L. C. M. of numerical coefficients (2 and 1)} = 2 \\ \text{The highest power of the factor } a + b \text{ is } (a + b)^2 \\ \text{There is no other factor.} \end{cases}$

∴   L. C. M. $= 2(a + b)^2$

7.   I Exp. $= a^4$

    II Exp. $= a^3 + a^2 = a^2(a + 1)$

$\begin{cases} \text{The highest power of } a \text{ is } a^4 \\ \text{The highest power of the factor } a + 1 \text{ is } (a + 1) \end{cases}$

∴   L. C. M. $= a^4(a + 1) = a^5 + a^4$

11.   I Exp. $= 6(x^2 + xy) = 6x(x + y)$
    II Exp. $= 6y(x^3 + y^3) = 6y(x + y)(x^2 - xy + y^2)$

∴   L. C. M. $= 6xy(x + y)(x^2 - xy + y^2) = 6xy(x^3 + y^3)$

17.   I Exp. $= 2x^3 - 2x^2 - 12x = 2x(x^2 - x - 6) = 2x(x - 3)(x + 2)$
    II Exp. $= x^2 + x - 2 = (x + 2)(x - 1)$

III Exp $= x^2 - 4x + 3 = (x - 3)(x - 1)$

∴ L. C. M. $= 2x(x - 3)(x + 2)(x - 1).$

21.   I Exp. $= 10(x^2 - y^2)^2 = 10\{ (x + y)(x - y) \}^2$
$= 10(x + y)^2(x - y)^2$

   II Exp. $= 15(x^3 - y^3) = 15(x - y)(x^2 + xy + y^2)$

   III Exp. $= 20(x^2y - xy^2) = 20xy(x - y)$

∴   L. C. M. of numerical coefficients (10, 15, 20) $= 60$

The different factors are $x+y$, $x-y$, x^2+xy+y^2, x and y.
Taking their highest powers we have :—

L. C. M. $=60xy(x+y)^2(x-y)^2(x^2+xy+y^2)$.

107. To prove that the product of two expressions is equal to the product of their H. C. F. and L. C. M.

Let A and B be the two expressions, H their H. C. F. and L their L. C. M. Then we have to show that $A \times B = H \times L$.

Proof :—

Divide A and B by H [Two expressions are always divisible by their H. C. F.]

Let the respective quotients be m and n. Clearly m and n have no common factor.

$\therefore$ $A = mH$

$B = nH$

Also, L. C. M. of A and B $= mnH$.

$\therefore$ $L = mnH$.

Now, $A \times B = mH \times nH = mnH^2$

and $H \times L = H \times mnH = mnH^2$

$\therefore$ $A \times B = H \times L$

108. L. C. M. by H. C. F.

From the last article,

$$H \times L = A \times B$$

$$\therefore L = \frac{A \times B}{H} = \frac{A}{H} \times B \quad \text{or} \quad \frac{B}{H} \times A$$

Hence the following :

Rule. *The L. C. M. of two expressions is equal to their product divided by their H. C. F.*

Or

To find the L. C. M. of two expressions, divide one of them by their H. C. F. and multiply the quotient by the other.

EXERCISE 67

Find the L. C. M. of :—

1. $8x^3+2x^2-11x+4$ and $8x^3+14x^2+18x-8$. [*Solved*]

2. $4x^3 - 9x^2 - 15x + 18$ and $x^2 - 4x + 3$.

3. $a^3 + 5a^2 + 7a + 2$ and $4a^2 + 24a + 32$.

4. $a^3 - 6a^2 + 11a - 6$ and $a^3 - 9a^2 + 26a - 24$.

5. $3a^3 - 21a - 18$ and $a^3 + 8a^2 + 17a + 10$.

6. $2x^3 + 6x^2 - 50x + 42$ and $2x^3 - 9x^2 + 10x - 3$.

7. $a^3 - 9a^2 + 26a - 24$ and $a^3 - 12a^2 + 47a - 60$.

8. $6a^3 + a^2 - 5a - 2$ and $6a^3 + 5a^2 - 3a - 2$.

9. $6a^3 - 19a^2 - 9a + 36$ and $12a^3 - 35a^2 - 23a + 60$.

10. $3a^3 - 23a^2 + 43a - 8$ and $a^4 - 5a^3 - 6a^2 + 35a - 7$.

11. $2a^3 - 5a - 39$ and $a^4 - 21a - 18$.

12. $2a^3 + 4a^2 - 6a + 40$ and $3a^4 - 34a^2 + 51a - 20$.

13. $6x^4 - 5x^2 - 6$ and $8x^3 + 6x^2 - 12x - 9$.

14. $2x^4 + x^3 - 14x^2 - 4x + 15$ and $4x^4 - 29x^2 + 25$.

15. $x^4 + x^3y + xy^3 + y^4$ and $x^4 + x^2y^2 + y^4$.

16. $20x^4 - 3x^3y + y^4$ and $64x^4 - 8xy^3 + 5y^4$.

---

17. $x^3 - x^2 - 4x + 4$, $x^3 - 2x^2 - x + 2$ and $x^3 + 2x^2 - x - 2$.

[*Solved*]

18. $a^3 + 2a^2 - 3a$, $a^3 + 3a^2 - a - 3$ and $a^3 + 4a^2 + a - 6$.

19. $a^3 + 5a^2 + 10a$, $a^3 - 19a - 30$ and $a^3 - 15a - 50$.

20. $3a^3 - 7a^2b + 5ab^2 - b^3$, $a^2b + 8ab^2 - 3a^3 - b^3$
and $3a^3 + 5a^2b + ab^2 - b^3$.

---

21. Find the expression of lowest degree exactly divisible by $a^3 - 9a^2 + 26a - 24$ and $a^3 - 6a^2 + 11a - 6$. [*Hint*]

22. The H. C. F. of two expressions is $a - 7$ and their L. C. M. is $a^3 - 10a^2 + 11a + 70$. One of the expressions is $a^2 - 5a - 14$, find the other. [*Hint*]

23. The L. C. M. of two expressions is $30a^4 + 13a^3 - 11a^2 - 7a - 1$ and H. C. F. $5a^2 - 2a - 1$. If one of the expressions be $10a^3 + a^2 - 4a - 1$, find the other.

SOLUTIONS & HINTS—EXERCISE 67

1. First we find the H. C. F. of the two expressions.

$$
\begin{array}{r|l|l|r}
3x & 3x^3+2x^2-11x+4 & 3x^3+14x^2+13x-\ 8 & 1 \\
 & 3x^3+6x^2-\ 3x & 3x^3+\ 2x^2-11x+\ 4 & \\
\cline{2-3}
-4 & -4x^2-\ 8x+4 & 12)12x^2+24x-12 & \\
 & -4x^2-\ 8x+4 & x^2+\ 2x-\ 1 & \\
\cline{2-3}
 & \times & & \\
\end{array}
$$

H. C. F. $=x^2+2x-1$

$\therefore$ L. C. M. $=\dfrac{(3x^3+2x^2-11x+4)}{(x^2+2x-1)}\times(3x^3+14x^2+18x-8)$

$=(3x-4)(3x^3+14x^2+13x-8).]$

Note. If we divide the 2nd expression by the H. C. F., the quotient is $3x+8$, so that L. C. M. is also $=(3x+8)\times(3x^3+2x^2-11x+4)$.

17. [To find the L. C. M. of three expressions by H. C. F., we first find the L C. M. of any two of them and then the L. C. M. of the L. C. M. and the third expression.

But the following method is more convenient and is applicable in most of the cases :—

Find the H. C. F. of any two expressions. With the help of this H. C. F. factorise those expressions. See if one or more of the factors of these two expressions are also the factors of the third expression. If so, factorise it. otherwise consider it to be a single factor. Then find the L. C. M. by Art. 106.]

We shall use the second method, which is easily applicable to the question in hand.

First we find the H. C. F. of the first two expressions.

$$
\begin{array}{r|l|l|r}
x & x^3-\ x^2-4x+4 & x^3-2x^2-\ x+2 & 1 \\
 & x^3-3x^2+2x & x^3-\ x^2-4x+4 & \\
\cline{2-3}
2 & 2x^2-6x+4 & -)-x^2+3x-2 & \\
 & 2x^2-6x+4 & x^2-3x+2 & \\
\cline{2-3}
 & \times & & \\
\end{array}
$$

H. C. F. of the 1st two expressions $=x^2-3x+2.$

This is a factor of both of them ; the other factors can be found by division. Thus we have :—

I Exp.$=(x^2-3x+2)(x+2)=(x-2)(x-1)(x+2)$ ...... (i)

II Exp.$=(x^2-3x+2)(x+1)=(x-2)(x-1)(x+1)$ ......(ii)

The different factors of these expressions are $x-1$, $x+1$, $x-2$, $x+2$. We try to find if any one of them is a factor of the third expression. We succeed in the first attempt : $x-1$ is a factor.

$\therefore$ III Exp.$=(x-1)(x^2+3x+2)=(x-1)(x+1)(x+2)$...(iii)

From (i), (ii) and (iii) we have, by Art. 106,

Reqd. L. C. M.$=(x-1)(x+1)(x-2)(x+2)$.

21.   The reqd. exp. is their L. C. M. [by def.]

22.   The other Exp.$=\dfrac{\text{H. C. F.} \times \text{L. C. M.}}{\text{The given Exp.}}$. [See Art. 107.]

TEST PAPERS—SET 3
(CHAPTERS VIII TO XVII)
Paper 1.   (Ex. 68)

1.   Solve the equation $\frac{1}{6}(x+1)+\frac{1}{4}(x-1)-\frac{1}{8}(3x-7)=1$.

2.   A number consists of two digits, the left-hand digit being double of the right-hand digit. If 36 be subtracted from the number, the digits are reversed. Find the number.

3.   If $x-\dfrac{1}{x}$ be equal to $\sqrt{3}$, evaluate $x^4+\dfrac{1}{x^4}$.

4.   Factorise a^4+64 and a^4-64.

5.   Find the H. C. F. of x^5-xy^2 and $x^3+x^2y+xy+y^3$.

6.   Find the L. C. M. of a^3-5a^2+9a-9 and a^3+a^2-8a+9.

Paper 2.   (Ex. 69)

1.   Solve the equations $\frac{1}{4}(x+1)=\frac{1}{3}(y-1)$, $\frac{1}{2}(x-1)=\frac{1}{7}(y+8)$.

2.   A person goes a distance by car at the rate of 36 m. p. h. and returns by tonga at 8 m. p. h. If total time taken be 3 hrs. 40 mts., find the distance.

3.   Prove that $x^2+y^2+z^2-xy-yz-zx$ can never be negative.

4.   Factorise $a^{12}-1$.

5. Find the H. C. F. of $x^3 + 8y^3$ and $x^3 + x^2y - 2xy^2$

6. Find the L. C. M. of
$$x^3 - 15x^2y + 48xy^2 + 64y^3 \text{ and } x^2 - 10xy + 16y^2.$$

Paper 3. (Ex 70)

1. If $\frac{1}{3}\left(\frac{x}{2} - 2\right) - 2(x - 30) - \frac{1}{7}(x - 6)$ be equal to -7,

evaluate $x^2 - 8x + 16$.

2. A sum of Rs. 6400 was lent out at simple interest, partly at 5 p. c. and partly at $6\frac{1}{2}$ p. c. per annum. The total annual interest amounted to Rs. 380 How much was lent out at each rate ?

3. Find the continued product of
$$(x + y)(x - y)(x^2 - xy + y^2)(x^2 + xy + y^2)(x^6 + y^6).$$

4. Factorise $6x^4 + x^3y - 12x^2y^2$.

5. Find the H. C. F. of
$$a^3 - b^3 - c^3 - 3abc \text{ and } a^2 - b^2 - c^2 - 2bc.$$

6. Find the L. C. M. of
$$x^4 - 5x^3 + 20x - 16 \text{ and } x^4 - 2x^3 - 3x^2 + 8x - 4.$$

Paper 4. (Ex. 71)

1. Solve for x, y and z :–
$$\frac{y - z}{12} = \frac{y - x}{8} = \frac{b}{4} \quad z - x, \; y + z - zx = 1$$

2. If 16 pounds of tea and 24 pounds of coffee together cost Rs. 29, and a pound of tea cost 4 annas more than a pound of coffee, find the cost of each per pound.

3. Find the continued product of
$$(x + y + z)(x + y - z)(u + v - x)(z + x - y).$$

4. Factorise $x^6 - 9x^3 + 8$.

5. Find the H. C. F. of
$$3a^4 + 8a^3 + 4a^2, \; 3a^5 + 11a^4 + 6a^3 \text{ and } 3a^4 - 16a^3 - 12a^2.$$

6. Find the L. C. M. of
$$a^3 - 3a^2b + 3ab^2 - 2b^3, \; a^3 - a^2b - ab^2 - 2b^3 \text{ and } a^4 + a^2b^2 + b^4.$$

Paper 5.   (Ex. 72)

1. Solve the equation

$$\frac{1}{2} - \frac{1}{2}\left(3x - \frac{2x-5}{10}\right) - \tfrac{1}{12}(2x-57) + \tfrac{5}{6} = 0.$$

2. Divide 180 into 4 parts such that if the first be increased by 2, the second diminished by 2, the third multiplied by 2 and the fourth divided by 2, the result in each case may be the same.

3. If $a^2 + \dfrac{1}{a^2}$ is equal to unity, evaluate $a^3 + \dfrac{1}{a^3}$.

4. Factorise (i) $9a^4 + 2a^2 + 1$.
$\qquad$ (ii) $(a^2 - 4)(a+1)(a+5) - 28$.

5. Find the L. C. M. of $a^3 - b^3$, $a^2 - ab + b^2$, $a^3 + b^3$.

6. Find the expression of the highest degree which can divide $20a^4 - 3a^3 + 1$ and $64a^4 - 8a + 5$ exactly.

Paper 6.   (Ex. 73)

1. Solve for x and y :—

$$\tfrac{1}{2}\left(\frac{2}{x} - \frac{3}{y}\right) = 6\tfrac{1}{2}, \quad 1\tfrac{1}{3}\left(\frac{2}{x} + \frac{8}{y}\right) + \tfrac{1}{3} = 0.$$

2. The denominator of a fraction exceeds the numerator by unity. If 9 be added to the numerator and 6 to the denominator, the resulting fraction is equal to the reciprocal of the original one. Find the fraction.

3. Simplify :—
$\qquad$ $(x+2y)^3 + (2x-y)^3 + 3(x+2y)(2x-y)(3x+y)$.

4. Factorise (i) $9a^2 - 24ab + 16b^2 - 6a + 8b$.
$\qquad$ (ii) $ab(a-b) + bc(b-c) + ca(c-a)$.

5. Find the L. C. M. of
$\qquad$ $x^4 + x^2y^2 + y^4$, $x^3y + y^4$ and $(x^2 - xy)^3$.

6. Find the expression of the lowest degree which is exactly divisible by $3a^3 - 5a^2 + 5a - 2$ and $2a^4 - 2a^3 + 8a^2 - a + 1$.

(Paper 7. Ex. 74)

1. Solve the equation

$$\frac{4x+15}{1\frac{2}{3}} - \frac{12\frac{1}{2}-x}{2\frac{1}{2}} + \frac{10\frac{1}{4}+\frac{x}{2}}{3} = 0.$$

2. Ten years ago a father's age was three times the sum of the ages of his two children, but ten years hence his age will be equal to the sum of their ages. Find his present age, and the sum of the present ages of his children.

3. If $a+b+c=0$, prove that $a^3+b^3+c^3=3abc$. Hence factorise $(x-y)^3+(y-z)^3+(z-x)^3$.

4. For what values of a and b will the expression $2x^3+ax^2+bx+6$ be exactly divisible by x^2-3x+2 ?

5. Find the L. C. M. of
$$a^2-a-2,\ 2a^3-3a^2-2a,\ 2a^4+3a^3+a^2.$$

6. For what value of x will
$$x^3-3x-2 \text{ and } x^3-x^2-4 \text{ both vanish ?}$$

Paper 8. (Ex. 75)

1. Solve for x, y and z :—

$$9x-8y+3z=x-2y+z=0,\ \frac{x}{15}+\frac{y}{10}+\frac{z}{6}=1\frac{1}{2}.$$

2. If A were to give Rs. 20 to B they would have equal sums of money, but if B were to give Rs. 40 to A, the money of A would be double that of B. Find the money each has.

3. If $a^2+b^2+c^2=15$, $a+b+c=5$,

evaluate $a^3+b^3+c^3-3abc$.

4. Factorise (i) $x^3-8y^3-6xy-1$. (ii) $x^3+x^2-14x-24$.

5. Find the L. C. M. of $4a^4(a^3-b^3)$, $6a^2b^2(a^3-3a^2b+3ab^2-b^3)$, and $8a^3b(a^3-a^2b-ab^2+b^3)$.

6. The H. C. F. of two expressions is x^2-5x+3 and L. C. M. is $x^4-14x^3+68x^2-127x+60$. If one of the expressions be $x^3-9x^2+23x-12$, find the other.

CHAPTER XVIII
FRACTIONS

109 The principles involved in the *simplification, addi- tion, subtraction, multiplication* and *division* of fractions in Algebra are the same as in Arithmetic. We therefore proceed at once with examples of graded difficulty, giving sample solutions and hints as usual. *Wherever necessary the student has been warned against probable mistakes. A careful study of such notes will be found very useful.*

EXERCISE 76

Reduce to lowest terms :—

1. $\dfrac{4ab}{6abc}$.   [*Solved*].

2. $\dfrac{5xy}{10xz}$.

3. $\dfrac{8ayx}{6ax}$.

4. $\dfrac{15abc}{25b}$.

5. $\dfrac{14a^3b^2}{-21ab^5}$.   [*Solved*].

6. $\dfrac{-21x^4y}{28x^2yz^2}$.

7. $\dfrac{-24mn^2p}{-30m^2np^2}$.

8. $\dfrac{5abc^2}{-5a^3b^2c}$.

9. $\dfrac{-8xy^3z^4}{16x^2y^2z}$.

10. $\dfrac{m^2n^3pq}{m^3n^2p^4}$.

11. $\dfrac{-30a^2x^4y}{-50a^3x^2y^5}$.

12. $\dfrac{-76k^3p^5m^4}{114k^4p^3m^2}$.

Simplify :—

13. $\dfrac{-4a^2b}{6c^2d} \times \dfrac{c^3d^3}{a^2b^2}$.   [*Solved*]

14. $\dfrac{36a^3bc}{8ab^3} \times \dfrac{12ab^2}{-18bc^3}$.

15. $\dfrac{45x^2yz^3}{-a^3bc} \times \dfrac{-a^3x}{5y^2z}$.

16. $\dfrac{-21a^2b^4}{18ax^2y} \times \dfrac{36x^2c}{-30ac^4}$.

17. $\dfrac{-6a^2bc^2}{32a^3b^3c} \div \dfrac{-12ac}{32b^2x}$.   [*Hint*]

18. $\dfrac{14m^2p}{34x^2y} \times \dfrac{y^4z}{p^2n} \div \dfrac{-21m^2yv^3}{51pyz}$.

19. $\dfrac{14d^3a^4}{a^2bc} \times \dfrac{27c^2}{81d^3a^3} \div \dfrac{8c}{-3b^2}$.

20. $\dfrac{3xy^2a^3}{-ab^4} \div \dfrac{ax}{pb^2} \div \dfrac{3ap}{-b^2}$.

Simplify :—

21. $\dfrac{a^2-ax}{ax}$ [Solved]

22. $\dfrac{ab-ac}{a^3}$

23. $\dfrac{a^3-a^2}{-a^3}.$

24. $\dfrac{b^3-c^3}{b^2-bc}$ [Solved]

25. $\dfrac{x^2-y^3}{x^3-y^3}.$

26. $\dfrac{x^3+y^3}{x^3+xy}.$

27. $\dfrac{a^3+b^3}{a^2-ab+b^2}.$

28. $\dfrac{x^2+xy+y^2}{x^3-y^3}$

29. $\dfrac{a^4-b^4}{b^2-a^2}.$

30. $\dfrac{1-a^2}{a^3-1}.$

31. $\dfrac{x^4-1}{x^3-1}.$

32. $\dfrac{a^4-1}{a^6-1}.$

33. $\dfrac{x^6+y^6}{x^4-x^2y^2+y^4}.$

34. $\dfrac{a^4+a^2b^2+b^4}{a^6-b^6}$

35. $\dfrac{a^2-4a-21}{a^2-5a-14}.$

36. $\dfrac{6a^2-11a+8}{3a^2-18a+4}$

37. $\dfrac{x^4+4}{x^2+2x+2}.$

38. $\dfrac{a^4+10a^2+9}{a^3+4a^2+8a}$

39. $\dfrac{x^2-y^2+2yz-z^2}{y^2-z^2-x^2+2zx}.$

40. $\dfrac{a^3+a^2+a+1}{a^3+a^2+8a+8}.$

Simplify :—

41. $\dfrac{x^3+6x^2-2x-7}{x^3-32x+85}.$ [Solved]

42. $\dfrac{x^3-5x^2+7x-3}{x^3-3x+2}$

43. $\dfrac{5a^3+2a^2-15a-6}{7a^3-4a^2-21a+12}.$

44. $\dfrac{6a^3+a^2-5a-2}{6a^3+5a^2-8a-2}$

45. $\dfrac{4a^2-5ab+b^2}{3a^3-3a^2b+ab^2-b^3}$

46. $\dfrac{x^3+3x^2y+3xy^2+2y^3}{x^3-x^2y-xy^2-2y^3}$

47. $\dfrac{a^4+a^2-12a+21}{a^4-15a+14}$

48. $\dfrac{20x^4-8x^3y+y^4}{64x^4-8xy^3+5y^4}$

Simplify :—

49. $\dfrac{a^3-b^3}{a^2-2ab+b^2} \times \dfrac{a^2-b^2}{a^2+ab}$. [Solved]

50. $\dfrac{1-4a^3}{1+a} \times \dfrac{a^2-1}{(a-1)^2}$

51. $\dfrac{16x^2-9y^2}{x^3-4x} \times \dfrac{x^2-2x}{4x-3y} \times \dfrac{x+2}{4x+3y}$

52. $\dfrac{x^3+5x^2+6r}{x^3-x} \times \dfrac{x^3-2x^2-3x}{x^2-9} \times \dfrac{1}{x}$

53. $\dfrac{4a^2+10a+4}{a^2-4} \times \dfrac{2c^2+8a}{2a^2+9a+4} \times \dfrac{a-2}{4a}$.

54. $\dfrac{4a^2+26a+30}{4a^2-9} \div \dfrac{4a^2+22a+10}{4a^3-a}$

55. $\dfrac{2a^2-28a-30}{a^3-4a^2-45a} \div \dfrac{2a^2-24a-90}{a^3-6a^2-27a}$

56. $\dfrac{a^3-6a^2+36a}{a-7} \div \dfrac{a^4+216a}{a^2-a-42}$

57. $\dfrac{x^3-x^2-2x}{x^2+2x-8} \times \dfrac{x^2-x-20}{x^3-25x} \div \dfrac{x^2+x}{x^3+5x^2}$.

58. $\dfrac{x^3-25x}{x^3-x} \times \dfrac{x^2-8x-9}{x^3-17x^2+72x} \div \dfrac{x^2+4x-5}{x^3-9x^2+8x}$.

59 $\dfrac{a^4-8a}{a^3-a^2-2a} \times \dfrac{a^2+2a+1}{a^2-4a-5} \div \dfrac{2a^2+4a+8}{a-5}$.

60. $\dfrac{x^3+y^3}{(x-y)^2+3xy} - \dfrac{(x+y)^2-3xy}{x^3-y^3} \times \dfrac{xy}{x^2-y^2}$. [Hint]

61. $\dfrac{4a-a^3}{a^2-1a^4} - \dfrac{(1-2a)^2+2a}{2-5a+2a^2} \times \dfrac{1+8a^3}{(2-a)^3}$

62. $\dfrac{a^4-b^4}{(a+b)^3-3ab(a+b)} - \dfrac{(a+b)^2-4ab}{(a+b)^2-3ab} \div \dfrac{(a+b)^2-2ab}{a-b}$

Find the value of :—

63 $\dfrac{x}{2}+\dfrac{3x}{4}+\dfrac{5x}{6}$ [Solved] **64.** $\dfrac{a}{4}+\dfrac{3a}{8}+\dfrac{7a}{16}$

65. $\dfrac{3a}{4} - \dfrac{5a}{6} + \dfrac{a}{12}$.

66. $\dfrac{2x}{5} - \dfrac{3x}{10} - \dfrac{4x}{15}$

67. $\dfrac{5}{6x} + \dfrac{7}{2x} - \dfrac{8}{3x}$. [Solved]

68. $\dfrac{5}{8a} - \dfrac{3}{a} - \dfrac{1}{6a}$.

69. $-\dfrac{3a}{10x} - \dfrac{a}{5x} + \dfrac{5a}{12x}$.

70. $\dfrac{1}{ab} + \dfrac{1}{bc} + \dfrac{1}{ca}$.

71. $\dfrac{a}{bc} + \dfrac{b}{ca} + \dfrac{c}{ab}$.

72. $\dfrac{x}{2yz} + \dfrac{y}{3zx} + \dfrac{z}{4xy}$.

———

Find the value of :—

73. $\dfrac{a}{2} - \dfrac{a+3}{5} + \dfrac{a+7}{10} - \tfrac{1}{8}$. [Solved]

74. $\dfrac{a}{4} + \dfrac{a-5}{6} + \dfrac{2a}{3} - \tfrac{1}{3}$.

75. $\dfrac{5a}{12} - \dfrac{a+2}{6} + \dfrac{2a-3}{9} + \tfrac{2}{3}$.

76. $\dfrac{3x-5y}{4} + \dfrac{2x}{8} - \dfrac{y+z}{3} + \dfrac{x+y+z}{12}$.

77. $\dfrac{x-y}{y} + \dfrac{x+y}{x} - \dfrac{x^2-y^2}{2xy}$ [Solved]

78. $\dfrac{x-y}{xy} + \dfrac{y-z}{yz} + \dfrac{z-x}{zx}$.

79 $\dfrac{2a-3b}{ab} + \dfrac{3a-2c}{ac} + \dfrac{5}{a}$.

80. $\dfrac{x^2-yz}{yz} - \dfrac{xz-y^2}{xz} - \dfrac{xy-z^2}{xy}$.

81. $\dfrac{2}{ab} - \dfrac{3b^2-a^2}{ab^3} + \dfrac{ab+b^2}{a^2b^2}$. [Hint]

82. $\dfrac{1}{x^3} - \dfrac{x-y}{x^2y} + \dfrac{x^2-y^2}{x^3y^2}$.

———

Find the value of :—

83. $\dfrac{1}{a-5} - \dfrac{1}{a-4}$. [Solved]

84. $\dfrac{1}{a+2} + \dfrac{1}{a+3}$.

85. $\dfrac{3}{a-6} - \dfrac{1}{a+2}$.

86. $\dfrac{c}{a-b} - \dfrac{c}{a+b}$.

87. $\dfrac{1}{a+b} + \dfrac{2b}{a^2-b^2}$.   [*Solved*]   **88.** $\dfrac{y}{x-y} - \dfrac{y^2}{x^2-y^2}$.

89. $\dfrac{3}{a-3} + \dfrac{2a}{a^2-9}$ **90.** $\dfrac{x+y}{x-2y} - \dfrac{x^2+2y^2}{x^2-4y^2}$.

91. $\dfrac{2a^3}{a^2-b^2} - \dfrac{2a^2}{a^2+ab}$ [*Hint*]   **92.** $\dfrac{a^2}{a-a^3} - \dfrac{a}{1+a^2}$.

———

93. $\dfrac{a}{a-x} + \dfrac{3a}{a+x} - \dfrac{2ax}{a^2-x^2}$.   [*Solved*]

94. $\dfrac{2y}{x+y} - \dfrac{3x}{x-y} + \dfrac{x^2+y^2}{x^2-y^2}$ **95.** $\dfrac{x-y}{u} + \dfrac{2x}{x-y} - \dfrac{x^3+x^2y}{x^2y-y^3}$.

96. $\dfrac{a^3+ab\,x-bxy}{a(a^2-y^2)} - \dfrac{bx}{a(a+y)} - \dfrac{a}{2(a-y)}$.

———

97. $\dfrac{1}{a^3-1} + \dfrac{a+1}{a^2+a+1}$.   [*Hint*]   **98.** $\dfrac{1}{a^3+1} - \dfrac{1+a}{a^2-a+1}$.

99. $\dfrac{x+y}{x^2+xy+y^2} - \dfrac{x-y}{x^2-xy+y^2}$.   **100.** $\dfrac{a^2-ab+b^2}{a-b} - \dfrac{a^2+ab+b}{a+b}$.

———

101. $\dfrac{1-x^2}{1-x} - \dfrac{1-x^3}{1-x^2}$.   [*Hint*]   **102.** $\dfrac{1-x^3}{1-x} + \dfrac{1+x^3}{1+x}$.

103. $\dfrac{1+x+x^2}{1-x^3} + \dfrac{1-x+x^2}{1+x^3}$.   **104.** $\dfrac{a-b}{a^3-b^3} - \dfrac{a+b}{a^3+b^3}$.

———

105. $\dfrac{1}{a^2-3a+2} - \dfrac{2}{a^2-4a+3} + \dfrac{1}{a^2-5a+4}$.   [*Solved*]

106. $\dfrac{1}{a^2-4a+3} - \dfrac{1}{a^2-3a+2} + \dfrac{1}{a^2-5a+6}$.

107. $\dfrac{1}{a^2-7a+12} - \dfrac{2}{a^2-6a+8} + \dfrac{1}{a^2-5a+6}$.

108. $\dfrac{1}{a^2-7a+12} + \dfrac{2}{a^2-4a+8} - \dfrac{3}{a^2-5a+4}$.

109. $-\dfrac{a-b}{a^2-ab+b^2}+\dfrac{1}{a+b}+\dfrac{ab}{a^3+b^3}$. [Hint]

110. $\dfrac{1}{a-b}+\dfrac{a-b}{a^2+ab+b^2}+\dfrac{ab-2a^2}{a^3-b^3}$.

111. $\dfrac{a+1}{a^2+a+1}+\dfrac{a-1}{a^2-a+1}+\dfrac{2}{a^4+a^2+1}$.

Find the value of :—

112. $\dfrac{1}{a-b}-\dfrac{1}{a+b}-\dfrac{2b}{a^2+b^2}-\dfrac{4b^3}{a^4+b^4}$. [Solved]

113. $\dfrac{a}{a+b}+\dfrac{ab}{a^2-b^2}-\dfrac{a^2}{a^2+b^2}$.

114. $\dfrac{1}{1+x}+\dfrac{1}{1-x}+\dfrac{2}{1+x^2}-\dfrac{4}{1+x^4}$.

115. $\dfrac{1}{1+a}+\dfrac{2}{1+a^2}+\dfrac{4}{1+a^4}+\dfrac{8}{1+a^8}+\dfrac{1}{1-a}$. [Hint]

116. $\dfrac{2x}{x^4-x^2+1}-\dfrac{1}{x^2-x+1}+\dfrac{1}{x^2+x+1}$.

117. $\dfrac{1}{a+1}-\dfrac{1}{(a+1)(a+2)}+\dfrac{1}{(a+1)(a+2)(a+3)}$.

118. $\dfrac{x+3}{x-4}-\dfrac{x+4}{x-3}-\dfrac{8}{x^2-16}$.

119. $\dfrac{1}{a-1}+\dfrac{2}{a+1}-\dfrac{3a-2}{a^2-1}-\dfrac{1}{(1+a)^2}$.

120. $\dfrac{1}{x-8y}+\dfrac{3}{x+y}-\dfrac{1}{x+8y}-\dfrac{8}{x-y}$. [Hint]

121. $\dfrac{1}{x-2y}-\dfrac{4}{x-y}+\dfrac{6}{x}-\dfrac{4}{x+y}+\dfrac{1}{x+2y}$.

122. $\dfrac{1}{1-a}-\dfrac{1}{1+a}-\dfrac{2a}{1+a^2}+\dfrac{2a^2}{(1-a)(1+a^2)}-\dfrac{2a^2}{(1+a)(1+a^2)}$
$-\dfrac{8a^3}{1-a^4}$. [Hint]

Simplify :—

123. $\dfrac{a}{x+a} + \dfrac{2x}{x-a} + \dfrac{a(3x-a)}{a^2-x^2}$.   [Hint]

124. $\dfrac{2y-x}{z-y} - \dfrac{3z(x-y)}{y^2-z^2} + \dfrac{y-2x}{y+z}$.

125. $\dfrac{1}{x+y} + \dfrac{4y}{x^2-y^2} + \dfrac{1}{y-x} - \dfrac{2y}{x^2+y^2}$.

126. $\dfrac{5}{3a-3} + \dfrac{3a-1}{1-a^2} + \dfrac{1}{2a+2} + \dfrac{5a-13}{6a^2-6}$.

SOLUTIONS & HINTS—EXERCISE 76

1. Given fraction $= \dfrac{4ab}{6abc}$

Divide 4 and 6 by 2 each, getting 2 and 3 respectively ; cancel a with a and b with b.

$\therefore$ The fraction $= \dfrac{2}{3c}$.

5. Given fraction $= \dfrac{14a^3b^2}{-21ab^5}$.

$\dfrac{+}{-}$ gives $-$

Divide 14 and 21 by 7 each, getting 2 and 3 respectively. Cancelling a^3 and a we get a^{3-1} (i. e., a^2) in the numerator. Cancelling b^2 and b^5 we get b^{5-2} (or b^3) in the denominator.

$\therefore$ The fraction $= -\dfrac{2a^2}{3b^3}$.

13. Given Exp. $= \dfrac{-4a^2b}{6c^2d} \times \dfrac{c^3d^3}{a^2b^2}$

$= \dfrac{-4a^2bc^3d^3}{6a^2b^2c^2d}$ [re-arranging factors]

$= -\dfrac{2cd^2}{3b}$ [cancelling as before]

17. Change ÷ into × and invert the second fraction.

23. $\dfrac{a^2 - ax}{ax} = \dfrac{a(a-x)}{ax} = \dfrac{a-x}{x}$ [Cancelling a]

Important Note. We cannot cancel ax with ax, because in the numerator ax is not a factor. *We can only cancel a factor of the numerator with an equal factor of the denominator. Hence, before cancelling we must factorise the numerator and denominator.*

24. $\dfrac{b^2 - c^2}{c^2 - bc} = \dfrac{(b+c)(b-c)}{b(b-c)} = \dfrac{b+c}{b}$ [cancelling $(b-c)$]

Note. The student may again be reminded that b^2 cannot be cancelled with b^2, nor can the cancellation of c^2 and c give c in the numerator. [Why ?]

41. Given Fraction $= \dfrac{x^3 + 6x^2 - 2x - 7}{x^3 - 32x + 35}$.

We may factorise the numerator and the denominator by the help of their H. C. F.

$$x^3 + 0 - 32x + 35 \overline{)\ a^3 + 6x^2 - 2x - 7\ (\ 1}$$
$$\underline{x^3 + 0\ \ - 32x + 35}$$
$$6\ \overline{)\ 6x^2 + 30x - 42}$$
$$\overline{x^2 + 5x - 7}\ \overline{)\ x^3 + 0\ \ - 32x + 35\ (\ x - 5}$$

H.C.F. $= x^2 + 5x - 7$.
Dividing the numerator and the
denominator by this H.C.F. we get
the other factors, so that :—

$$x^3 + 5x^2 - 7x$$
$$\overline{\ \ -5x^2 - 25x + 35}$$
$$-5x^2 - 25x + 35$$
$$\overline{\times}$$

Given Fraction $= \dfrac{(x^2 + 5x - 7)(x + 1)}{(x^2 + 5x - 7)(x - 5)} = \dfrac{x + 1}{x - 5}$

[Cancelling the **H. C. F.**]

49. Given Exp. $= \dfrac{a^3 - b^3}{a^2 - 2ab + b^2} \times \dfrac{a^2 - b^2}{a^2 + ab}$

$= \dfrac{(a-b)(a^2 + ab + b^2)}{(a-b)^2} \times \dfrac{(a+b)(a-b)}{a(a+b)}$

$= \dfrac{a^2 + ab + b^2}{a}$ [Cancelling common factors]

60. $(x-y)^2+3xy=x^2+y^2-2xy+3xy=(x^2+y^2+xy)$, which cannot be further factorised.

Similarly $(x+y)^2-3xy=(x^2+y^2-xy)$.

63. Given Exp. $= \dfrac{x}{2}+\dfrac{3x}{4}+\dfrac{5x}{6}$

$\qquad\qquad = \dfrac{6x+9x+10x}{12}$

$$[\because \quad \text{L. C. M. of 2, 4 and } 6=12]$$

$\qquad\qquad = \dfrac{25x}{12}.$

Note. The addition has been performed exactly as in Arithmetic.

67. Given Exp. $= \dfrac{5}{6x}+\dfrac{7}{2x}-\dfrac{8}{3x}$

$\qquad\qquad = \dfrac{5+21-16}{6x}$

$$[\because \cdot \text{ L. C. M. of } 6x, 2x \text{ and } 3x=6x]$$

$\qquad\qquad = \dfrac{10}{6x}=\dfrac{5}{3x}.$

73. Given Exp. $= \dfrac{a}{2}-\dfrac{a+3}{5}+\dfrac{a+7}{10}-\dfrac{1}{2}$

$\qquad\qquad = \dfrac{5a-2(a+3)+(a+7)-5}{10}$

$\qquad\qquad = \dfrac{5a-2a-6+a+7-5}{10}=\dfrac{4a-4}{10}$

$\qquad\qquad = \dfrac{4(a-1)}{10}=\dfrac{2(a-1)}{5}.$

Note the use of brackets in the second step ; it is important.

77 Given Exp. $= \dfrac{x-y}{y}+\dfrac{x+y}{x}-\dfrac{x^2-y^2}{2xy}$

$\qquad\qquad = \dfrac{2x(x-y)+2y(x+y)-(x^2-y^2)}{2xy}$

$$[\textit{Note the use of brackets}]$$

$$= \frac{2x^2 - 2xy + 2xy + 2y^2 - x^3 + y^3}{2xy}$$

$$= \frac{x^2 + 3y^2}{2xy}.$$

81. The third fraction, $\dfrac{ab + b^2}{a^2b^2}$, can be simplified.

This simplification should be done before addition.

$$\left[\frac{ab + b^2}{a^2b^2} = \frac{b(a+b)}{a^2b^2} = \frac{a+b}{a^2b} \right]$$

83. Given Exp. $= \dfrac{1}{a-5} - \dfrac{1}{a-4}$

$$= \frac{(a-4) - (a-5)}{(a-5)(a-4)} = \frac{a - 4 - a + 5}{(a-5)(a-4)}$$

$$= \frac{1}{(a-5)(a-4)}.$$

Note. $a-5$ and $a-4$ cannot be factorised. Hence their L. C. M. is $(a-5)(a-4)$.

Note the use of brackets in the numerator in the second step.

87. Given Exp. $= \dfrac{1}{a+b} + \dfrac{2b}{a^2 - b^2}$

$$= \frac{1}{(a+b)} + \frac{2b}{(a+b)(a-b)}$$

$$= \frac{(a-b) + 2b}{(a+b)(a-b)} = \frac{(a+b)}{(a+b)(a-b)} = \frac{1}{a-b}.$$

91. The second fraction can be simplified, because a is common to the numerator and the denominator.

93. Given Exp. $= \dfrac{a}{a-x} + \dfrac{3a}{a+x} - \dfrac{2ax}{(a+x)(a-x)}$

$$= \frac{a(a+x) + 3a(a-x) - 2ax}{(a+x)(a-x)}$$

$$= \frac{a^2 + ax + 3a^2 - 3ax - 2ax}{(a+x)(a-x)}$$

$$= \frac{4a^2-4ax}{(a+x)(a-x)} = \frac{4a(a-x)}{(a+x)(a-x)} = \frac{4a}{a+x}.$$

97. $a^3-1=(a-1)(a^2+a+1)$ and a^2+a+1 cannot be factorised.

L. C. M. of the denominators $=(a-1)(a^2+a+1)$.

101. Simplify the two fractions before subtracting.

$$\left[\text{First}\quad \text{fraction} = \frac{(1+x)(1-x)}{(1-x)} = 1+x, \right.$$

$$\left. \text{Second}\quad \text{fraction} = \frac{(1-x)(1+x+x^2)}{(1-x)(1+x)} = \frac{1+x+x^2}{(1+x)} \right]$$

105. Given Exp. $= \dfrac{1}{(a-1)(a-2)} - \dfrac{2}{(a-1)(a-3)} + \dfrac{1}{(a-1)(a-4)}$

$$= \frac{(a-3)(a-4) - 2(a-2)(a-4) + (a-2)(a-3)}{(a-1)(a-2)(a-3)(a-4)}$$

$$= \frac{a^2-7a+12 - 2a^2+12a-16 + a^2-5a+6}{\text{Same denr.}}$$

$$= \frac{2}{(a-1)(a-2)(a-3)(a-4)}$$

[Simplifying the numr.]

109. L. C. M. of denominators $=(a+b)(a^2-ab+b^2)$.

112. Given Exp. $= \dfrac{1}{a-b} - \dfrac{1}{a+b} - \dfrac{2b}{a^2+b^2} - \dfrac{4b^3}{a^4+b^4}.$

The first two fractions are together

$$= \frac{1}{a-b} - \frac{1}{a+b} = \frac{a+b-a+b}{(a-b)(a+b)} = \frac{2b}{a^2-b^2}.$$

This result together with the third fraction

$$= \frac{2b}{a^2-b^2} - \frac{2b}{a^2+b^2} = \frac{2b(a^2+b^2)-2b(a^2-b^2)}{(a^2-b^2)(a^2+b^2)} = \frac{4b^3}{a^4-b^4}.$$

This result together with the fourth fraction

$$= \frac{4b^3}{a^4-b^4} - \frac{4b^3}{a^4+b^4} = \frac{4b^3(a^4+b^4)-4b^3(a^4-b^4)}{(a^4-b^4)(a^4+b^4)}$$

$$= \frac{8b^7}{a^8-b^8}, \quad \text{which is the reqd. result.}$$

115. Write down the last fraction as the first, and then proceed as in Q. 112, adding successively.

120. Combine the first and the third fractions together and similarly the second and the fourth; then add the two results.

122. Combine the first, the second, the third and the last fractions as in Q. 112, adding successively.

Combine the remaining two fractions separately.

Add the two results.

123. The denominators are $x+a$, $x-a$ and $(a+x)(a-x)$.

Now $x-a$ and $a-x$ differ only in sign, therefore it is advisable to write both as $x-a$ or both as $a-x$, making corresponding compensations. For this purpose, either write

$$\frac{2x}{x-a} \quad \text{as} \quad -\frac{2x}{a-x}, \quad \text{or} \quad \frac{a(3x-a)}{a^2-x^2} \quad \text{as} \quad -\frac{a(3x-a)}{x^2-a^2}.$$

---

110. Some expressions are best simplified when put in **Cyclic Order.** The student will easily pick up the method from the solutions of the next exercise.

The following results, which have already been established in Chapter XV, should be carefully revised and committed to memory. They will be found very helpful in solving the examples of the next exercise.

1. $a^2(b-c)+b^2(c-a)+c^2(a-b) = -(a-b)(b-c)(c-a)$
2. $ab(a-b)+bc(b-c)+ca(c-a) = -(a-b)(b-c)(c-a)$
3. $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2) = (a-b)(b-c)(c-a)$
4. $a^3(b-c)+b^3(c-a)+c^3(a-b) = -(a-b)(b-c)(c-a) \times (a+b+c)$
5. $a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3) = (a-b)(b-c)(c-a) \times (a+b+c)$
6. $a^4(b-c)+b^4(c-a)+c^4(a-b) = -(a-b)(b-c)(c-a) \times (a^2+b^2+c^2+ab+bc+ca)$
7. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2) = -(a-b)(b-c)(c-a) \times (ab+bc+ca)$

9. Values of a and c as in Q. 5.

$$\therefore \quad \text{L. H. S.} = \frac{a^2+b^2}{a^2-b^2} = \frac{b^2k^2+b^2}{b^2k^2-b^2}$$

$$= \frac{b^2(k^2+1)}{b^2(k^2-1)} = \frac{k^2+1}{k^2-1}.$$

and R. H. S. $= \frac{c^2+d^2}{c^2-d^2} = \frac{d^2k^2+d^2}{d^2k^2-d^2}$

$$= \frac{d^2(k^2+1)}{d^2(k^2-1)} = \frac{k^2+1}{k^2-1}$$

$\because$ the two results are equal,

$\therefore$ L. H. S. = R. H. S.

13. Values of a and c as in Q. 5.

$$\text{L. H. S.} = \frac{(a-c)^2}{(b-d)^2} = \left(\frac{c-a}{b-d}\right)^2 = \left(\frac{bk-dk}{b-d}\right)^2$$

$$= \left\{\frac{k(b-d)}{(b-d)}\right\}^2 = k^2$$

and R. H S. $= \frac{a^2}{b^2} = \frac{b^2k^2}{b^2} = k^2$.

$\because$ The two results are equal,

$\therefore$ L. H. S. = R. H. S.

15. It is required to prove that $\frac{(a+c)^3}{(b+d)^3} = \frac{a^3+c^3}{b^3+d^3}$.

23. a, b, c, d, are in proportion, therefore $\frac{a}{b} = \frac{c}{d} = k$

(suppose)

$$\therefore \quad \begin{matrix} a=bk \\ c=dk \end{matrix} \Big\} \qquad\qquad\qquad \ldots(i)$$

Now, L. H. S. $= abcd\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)$

$$= \frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}$$

10. $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}.$

11. $\dfrac{x^4}{(x-y)(x-z)}+\dfrac{y^4}{(y-z)(y-x)}+\dfrac{z^4}{(z-x)(z-y)}.$

12. $\dfrac{a(b+c)}{(a-b)(a-c)}+\dfrac{b(c+a)}{(b-c)(b-a)}+\dfrac{c(a+b)}{(c-a)(c-b)}.$

13. $\dfrac{a(b^2+bc+c^2)}{(a-b)(a-c)}+\dfrac{b(c^2+ca+a^2)}{(b-c)(b-a)}+\dfrac{c(a^2+ab+b^2)}{(c-a)(c-b)}.$ [Hint]

14. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}+\dfrac{1}{c(c-a)(c-b)}.$ [Hint]

15. $\dfrac{1}{bc(a-b)(a-c)}+\dfrac{1}{ca(b-c)(b-a)}+\dfrac{1}{ab(c-a)(c-b)}.$

16. $\dfrac{a}{bc(a-b)(a-c)}+\dfrac{b}{ca(b-c)(b-a)}+\dfrac{c}{ab(c-a)(c-b)}.$

17. $\dfrac{a^2+bc}{(a-b)(a-c)}+\dfrac{b^2+ca}{(b-c)(b-a)}+\dfrac{c^2+ab}{(c-a)(c-b)}.$ [Hint]

18. $\dfrac{y+z-x}{(x-y)(x-z)}+\dfrac{z+x-y}{(y-z)(y-x)}+\dfrac{x+y-z}{(z-x)(z-y)}.$

19. $\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+\dfrac{(x-c)(x-a)}{(b-c)(b-a)}+\dfrac{(x-a)(x-b)}{(c-a)(c-b)}.$ [Hint]

20. $\dfrac{1}{(a-b)(a-c)(x-a)}+\dfrac{1}{(b-c)(b-a)(x-b)}$

$$+\dfrac{1}{(c-a)(c-b)(x-c)}.$$ [Hint]

21. $\dfrac{a}{(a-b)(a-c)(x-a)}+$ two similar terms

22. $\dfrac{a^2}{(a-b)(a-c)(x+a)}+$ two similar terms.

23. $\dfrac{bc}{a(a^2-b^2)(a^2-c^2)}+$ two similar terms. [Hint]

SOLUTIONS & HINTS—EXERCISE 77

1. Given Exp. $= \dfrac{1}{(a-b)(a-c)} + \dfrac{1}{(b-a)(b-c)} + \dfrac{1}{(c-a)(c-b)}$

$= \dfrac{1}{-(a-b)(c-a)} + \dfrac{1}{-(a-b)(b-c)} + \dfrac{1}{-(c-a)(b-c)}$

          [Putting the denominators in cyclic order]

$= -\left\{ \dfrac{1}{(c-a)(a-b)} + \dfrac{1}{(a-b)(b-c)} + \dfrac{1}{(b-c)(c-a)} \right\}$

$= -\left\{ \dfrac{(b-c)+(c-a)+(a-b)}{(a-b)(b-c)(c-a)} \right\}$

$= -\dfrac{0}{(a-b)(b-c)(c-a)} = -0 = 0.$

4. Proceeding as before, we get numerator equal to

$(y+z)(y-z)+(z+x)(z-x)+(x+y)(x-y)$

     $= y^2 - z^2 + z^2 - x^2 + x^2 - y^2 = 0.$

6. Proceeding as usual, the first term in the numerator $= (1+a)(b-c) = (b-c) + a(b-c)$. Similarly split the other two terms into two parts each. Combine the first parts and the second parts separately.

8. $\dfrac{a^2}{(a-b)(a-c)} + \dfrac{b^2}{(b-a)(b-c)} + \dfrac{c^2}{(c-a)(c-b)}$

$= \dfrac{a^2}{-(a-b)(c-a)} + \dfrac{b^2}{-(a-b)(b-c)} + \dfrac{c^2}{-(c-a)(b-c)}$

$= -\left\{ \dfrac{a^2}{(c-a)(a-b)} + \dfrac{b^2}{(a-b)(b-c)} + \dfrac{c^2}{(b-c)(c-a)} \right\}$

$= -\left\{ \dfrac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{(a-b)(b-c)(c-a)} \right\}$

$= -\left\{ \dfrac{-(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)} \right\}$ [Using result 1, of Art. 110]

$= -\{-1\} = 1.$ **Ans.**

13. Proceeding as usual, the first term of the numerator $= a(b^2+bc+c^2)(b-c) = a(b^3-c^3)$. We have two other similar terms   [Apply Result 5 of Art. 110.]

14.. Putting the denominators in cyclic order, we have their L. C. M. equal to $abc(a-b)(b-c)(c-a)$.

17. Proceeding as usual, the first term of the numerator $=(a^2+bc)(b-c)=a^2(b-c)+bc(b-c)$. Similarly split the other terms into two parts each. Combine the first parts and the second parts separately, applying Results 1 and 2 of Art. 110.

19. Proceeding as usual, the first term of the numerator $=(x-b)(x-c)(b-c)=\{x^2-(b+c)x+bc\}(b-c)$

$=x^2(b-c)-x(b^2-c^2)+bc(b-c)$. Similarly, split the other two terms into three parts each. Combine the first parts, the second parts, the third parts separately.

20. Arranging the denominators in cyclic order, we get their L. C. M. $=(a-b)(b-c)(c-a)(x-a)(x-b)(x-c)$.

Adding up, we have first term in the numerator $=(b-c)(x-b)(x-c)$, as in the last question.

23. Proceeding as usual, the numerator

$=b^2c^2(b^2-c^2)+c^2a^2(c^2-a^2)+a^2b^2(a^2-b^2)$
$=yz(y-z)+zx(z-x)+xy(x-y)$

[Where $x=a^2$, $y=b^2$, $z=c^2$]

$=-(x-y)(y-z)(z-x)$
$=-(a^2-b^2)(b^2-c^2)(c^2-a^2)$.

[re-writing the values of x, y, z]

111. Miscellaneous Fractions.

We now propose to consider fractions of more complicated types than those already discussed. Methods are quite analogous to those used in Arithmetic and therefore need not be discussed in general terms. The solutions and hints to the next exercise will suffice to help the student through this exercise.

EXERCISE 78

Simplify :—

1. $\dfrac{\dfrac{a}{b}-\dfrac{x}{y}}{\dfrac{a}{b}+\dfrac{x}{y}}$. [*Solved*]

2. $\dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{a}{b}+\dfrac{b}{a}}$.

3. $\dfrac{1-\dfrac{1}{x}}{x-\dfrac{1}{x}}$.

4. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{1}{a}-\dfrac{1}{b}}$.

5. $\dfrac{1+\dfrac{c}{a}}{\dfrac{b}{a}-1}$.

6. $\dfrac{1}{x+\dfrac{y}{z}}$.

7. $\dfrac{1-\dfrac{b^2}{a^2}}{1+\dfrac{b^2}{a^2}}$.

8. $\dfrac{b+\dfrac{a^2}{b}}{b-\dfrac{a^4}{b^3}}$.

9. $\dfrac{a+\dfrac{6}{a}+5}{1+\dfrac{6}{a}+\dfrac{8}{a^2}}$. [Solved]

10. $\dfrac{\dfrac{3}{x}+\dfrac{x}{3}-2}{\dfrac{x}{6}+\dfrac{1}{2}-\dfrac{3}{x}}$.

11. $\dfrac{\dfrac{2}{a}-\dfrac{4}{a^2}-\dfrac{6}{a^3}}{\dfrac{9}{a^2}-1}$.

12. $\dfrac{\dfrac{6a^2-3a-18}{12}}{\dfrac{a^2}{a^2}-3}$.

13. $\dfrac{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}{1-\dfrac{x^2+y^2}{(x+y)^2}}$. [Solved]

14. $\dfrac{\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}}{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}$.

15. $\dfrac{\dfrac{x+y}{x-y}-\dfrac{x^2+y^2}{x^2-y^2}}{\dfrac{x}{x+y}+\dfrac{y}{x-y}}$.

16. $\dfrac{1-\dfrac{2xy}{x^2+y^2}}{\dfrac{x^3-y^3}{x-y}-3xy}$.

17. $\dfrac{\dfrac{a}{a+1}+\dfrac{a}{a-1}}{\dfrac{2}{a^2-1}}\times\dfrac{4a-\dfrac{1}{a}}{2+\dfrac{1}{a}}$.

18. $\dfrac{1+\dfrac{y-z}{y+z}}{1-\dfrac{y-z}{y+z}} \times \dfrac{1+\dfrac{z-x}{z+x}}{1-\dfrac{z-x}{z+x}} \times \dfrac{1+\dfrac{x-y}{x+y}}{1-\dfrac{x-y}{x+y}}.$

19. $\dfrac{1+\dfrac{a-b}{a+b}}{1-\dfrac{a-b}{a+b}} \div \dfrac{1+\dfrac{a^2-b^2}{a^2+b^2}}{1-\dfrac{a^2-b^2}{a^2+b^2}}.$

20. $\dfrac{\dfrac{a^3}{b^3}-\dfrac{b^3}{a^3}}{\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\left(\dfrac{a}{b}+\dfrac{b}{a}-1\right)} \div \dfrac{\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{ab}}{\dfrac{1}{b}-\dfrac{1}{a}}.$

21. $\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{x}}}.$ [Solved]

22. $\dfrac{1}{x-\dfrac{1}{x-\dfrac{1}{x}}}.$

23. $\dfrac{1}{a^2-\dfrac{a^3-1}{a+\dfrac{1}{a+1}}}$

24. $\dfrac{x-2}{x-2-\dfrac{x}{x-\dfrac{x-1}{x-2}}}.$

25. $\left\{ a^3-\dfrac{1}{a^3} -8\left(a-\dfrac{1}{a}\right)\right\} \div \left(a-\dfrac{1}{a}\right) -a^2-\dfrac{1}{a^2}.$

26. $\left\{\dfrac{2}{x}-\dfrac{1}{a+x}+\dfrac{1}{a-x}\right\} \div \left(\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right)\times\dfrac{2x}{a}.$

27. $\left(\dfrac{x+y}{x^2-xy+y^2}-\dfrac{x-y}{x^2+xy+y^2}\right) \div \left(\dfrac{x^3+y^3}{x^3-y^3}-\dfrac{x^3-y^3}{x^3+y^3}\right)\times\dfrac{y}{x-y}.$

28. $\left(\dfrac{a^2}{b}+\dfrac{b^2}{a}\right)\left(\dfrac{1}{b^2-a^2}\right)-\dfrac{b}{a^2+ab}+\dfrac{a}{ab-b^2}.$

29. $\dfrac{1+\dfrac{1}{k}}{\dfrac{1}{k}} \times \dfrac{\dfrac{1}{k}}{k^2+\dfrac{1}{k}} \div \dfrac{\dfrac{1}{k}}{k-1+\dfrac{1}{k}}.$

30. $\dfrac{\dfrac{a}{b}+\dfrac{b}{a}-1}{\dfrac{a^2}{b^2}+\dfrac{a}{b}+1} \times \dfrac{1+\dfrac{b}{a}}{a-b} \div \dfrac{1+\dfrac{b^3}{a^3}}{\dfrac{a^2}{b}-\dfrac{b^2}{a}}.$

SOLUTIONS & HINTS—EXERCISE 78

1. $\dfrac{\dfrac{a}{b}-\dfrac{x}{y}}{\dfrac{a}{b}+\dfrac{x}{y}}=\dfrac{\dfrac{ay-bx}{by}}{\dfrac{ay+bx}{by}}=\dfrac{ay-bx}{by}\times\dfrac{by}{ay+bx}=\dfrac{ay-bx}{ay+bx}.$

Or thus :—

To get rid of fractions in the numerator and the denominator we multiply each by by and get at once the given fraction $=\dfrac{ay-bx}{ay+bx}$.

9. Given fraction $=\dfrac{a+\dfrac{6}{a}+5}{1+\dfrac{6}{a}+\dfrac{8}{a^2}}=\dfrac{\dfrac{a^2+6+5a}{a}}{\dfrac{a^2+6a+8}{a^2}}$

$=\dfrac{a^2+5a+6}{a}\times\dfrac{a^2}{a^2+6a+8}$

$=\dfrac{(a+2)(a+3)}{a}\times\dfrac{a^2}{(a+2)(a+4)}=\dfrac{a(a+3)}{a+4}.$

13. Numerator $=\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}=\dfrac{(x+y)^2-(x-y)^2}{(x-y)(x+y)}$

$=\dfrac{4xy}{(x-y)(x+y)}.$

Denominator $=1-\dfrac{x^2+y^2}{(x+y)^2}=\dfrac{(x+y)^2-(x^2+y^2)}{(x+y)^2}=\dfrac{2xy}{(x+y)^2}.$

$\therefore$ Given fraction $=\dfrac{4xy}{(x-y)(x+y)}\times\dfrac{(x+y)^2}{2xy}=\dfrac{2(x+y)}{x-y}.$ **Ans.**

21. $$\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{x}}}=\cfrac{1}{1-\cfrac{1}{\cfrac{x+1}{x}}}=\cfrac{1}{1-\cfrac{1\times x}{x+1}}$$

$$=\cfrac{1}{\cfrac{(x+1)-x}{x+1}}=\cfrac{1}{\cfrac{1}{x+1}}=\frac{1\times(x+1)}{1}=x+1.$$

CHAPTER XIX

INVOLUTION & EVOLUTION

Square Root

112. Definition. Involution is the general name for multiplying an expression by itself so as to find the second, third, fourth, or any other power.

The following cases have already been dealt with :—

1. The square of a binomial [Art. 65.]
2. The square of a multinomial [Art. 68, Formula 6.]
3. The cube of a binomial [Art. 72.]

Here we propose to discuss two cases more :

1. Any power of a simple expression.
2. Any power of a binomial.

113. It is evident from the Rule of Signs that :—

(i) *Even power of any quantity is always positive.*

(ii) *Any odd power of a quantity has the same sign as the quantity itself.*

114. To prove that $(x^m)^n = x^{mn}$, where m and n are positive integers.

Proof. $(x^m)^n = x^m \times x^m \times x^m \times x^m \ldots\ldots\ldots\ldots n$ factors.

$= x^{m+m+m+m+\ldots\ldots\ldots\ldots n \text{ terms}}$

[The Index Law, Art. 23]

$= x^{mn}.$

115. **Any* power of a simple expression** may now be obtained by the following rule :—

(i) *First put down proper sign by Art. 113.*

(ii) *Next write down the required power of the numerical coefficient, calculating it by Arithmetic.*

(iii) *Next put down the required power of each literal factor by Art. 114.*

For example, let us find the value of $\left(-\dfrac{2a^2b^3}{3c^5}\right)^7$.

The quantity within bracket is negative and the power is 7, which is odd, therefore the result must be negative.

Also, in the *numerator* :—

$$2^7 = 2 \times 2 \times 2 \times \ldots 7 \text{ factors} = 128.$$

$$(a^2)^7 = a^{2\times7} = a^{14}.$$

$$(b^3)^7 = b^{3\times7} = b^{21}.$$

And, in the *denominator* :—

$$3^7 = 3 \times 3 \times 3 \times \ldots 7 \text{ factors} = 2187.$$

$$(c^5)^7 = c^{5\times7} = c^{15}.$$

$$\therefore \text{ Given Exp.} = -\frac{128a^{14}b^{21}}{2187c^{35}}.$$

Note that the numerator and the denominator have been operated upon separately.

EXERCISE 79

Write down the square of :—

1. $3ab^3$.

2. $-4a^3b$.

3. $5a^2b^2$.

4. $-6d^5b^4$.

5. $\dfrac{2}{3}x^2y^3$.

6. $-\dfrac{3}{4}x^6y^8$.

7. $-\dfrac{3xy^2z^5}{5a^4}$.

8. $\dfrac{5a^{10}b^{12}}{7c^{15}d^{20}}$.

---

*By " any power " we mean " any integral power."

Write down the cube of :—

9. $2xy^2$.

10. $-3x^2y^3$.

11. $4x^4y^5$.

12. $-\dfrac{1}{5}x^2y^5z^7$.

13. $-\dfrac{6a^4b^6}{7c^3d^7}$.

14. $\dfrac{10a^{10}b^{15}c^{20}}{3x^3y^{12}}$.

_____

15. $(3x^2y^3)^4$.

16. $(-4a^2b^5)^5$

17. $\left(\dfrac{1}{2a^3b^5}\right)^6$

18. $\left(-\dfrac{3a^5}{b^6}\right)^7$.

19. $\left(-\dfrac{2x^3}{y^4}\right)^9$

20. $\left(\dfrac{x^4}{-y^5z^6}\right)^{10}$.

_____

116. Any power of a binomial.

We already know that :—

$(a+b)^2=a^2+2ab+b^2$.

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$

Multiplying the last result by $a+b$, we get :—

$(a+b)^4=a^4+4a^6b+6a^2b^2+4ab^3+b^4$

Similarly :—

$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

$(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$,

etc., etc.

Any one of these results can be written down directly by the rule of the well-known *binomial theorem*. The general statement of this theorem is apt to confuse the student at this stage. We shall explain its application in a particular form through a particular example, and for this purpose we choose the expansion of $(a+b)^6$, which has already been obtained above by ordinary multiplication. This is how we get the different terms in this expansion :—

1st term $[a^6]$=a raised to the given power (6).

2nd term $[6a^5b]$. Coefficient=given power (6).

Index of $a =$ index of a in the 1st term decreased by 1.
Index of $b = 1$.

3rd term $[15a^4b^2]$.

Coefficient

$= \dfrac{\text{coefficient of 2nd term} \times \text{index of } a \text{ in the same term}}{\text{number of terms already written down}}$

$= \dfrac{6 \times 5}{2} = 15.$

Index of $a =$ index of a in the 2nd term decreased by 1.
Index of $b =$ index of b in the 2nd term increased by 1.

4th term $[20a^3b^3]$.

Coefficient

$= \dfrac{\text{coefficient of 3rd term} \times \text{index of } a \text{ in the same term}}{\text{number of terms already written down}} = \dfrac{15 \times 4}{3}$

$= 20.$

Index of $a =$ index of a in the 3rd term decreased by 1.
Index of $b =$ index of b in the 3rd term increased by 1,

etc. etc.

The method outlined above will become clearer by studying another example. Let us expand $(a+b)^7$.

1st term $= a$ raised to the given power $(7) = a^7$.

2nd term :—

$\left. \begin{array}{l} \text{Coefficient } = \text{given power} = 7 \\ \text{Index of } a = 7 - 1 = 6 \\ \text{Index of } b = 1 \end{array} \right\} 7a^6b.$

3rd term :—

$\left. \begin{array}{l} \text{Coefficient} = \dfrac{\text{coeff. of 2nd term} \times \text{index of } a \text{ in the same}}{\text{number of terms already written down}} \\ \qquad = \dfrac{7 \times 6}{2} = 21. \\ \text{Index of } a = 6 - 1 = 5. \\ \text{Index of } b = 1 + 1 = 2. \end{array} \right\} 21a^5b^2$

4th term :—

$$\text{Coefficient} = \frac{\text{coeff. of 3rd term} \times \text{index of } a \text{ in the same}}{\text{number of terms already written down}}$$

$$= \frac{21 \times 5}{3} = 35.$$

Index of $a = 5 - 1 = 4.$

Index of $b = 2 + 1 = 3.$

$\Big\}\ 35a^4b^3$

5th term :—

Coefficient $= \dfrac{35 \times 4}{4} = 35.$

Index of $a = 4 - 1 = 3.$

Index of $b = 3 + 1 = 4.$

$\Big\}\ 35a^3b^4$

6th term $= \dfrac{35 \times 3}{5} a^2b^5 = 21a^2b^5$

7th term $= \dfrac{21 \times 2}{6} ab^6 = 7ab^6$

8th term $= \dfrac{7 \times 1}{7} b^7 = b^7$

Thus we have :—

$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7.$

Note. If the sign of b is negative, the terms in the expansion will be alternately positive and negative, the first term being positive, thus :—

$(a-b)^7 = a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7$

A careful study of the above expansions will reveal the following facts, consideration of which helps a lot in writing down an expansion and saves about half the labour needed otherwise :—

(i) The number of terms in an expansion is greater than the index of the binomial by one. For example, in the expansion of $(a+b)^5$, the number of terms $= 5 + 1 = 6.$

(ii) The first and the last terms in the expansion are respectively a and b, each raised to the same power as the binomial.

(iii) In each successive term, the index of a decreases and that of b increases by 1.

. (iv) The sum of the indices of a and b in any term is equal to the index of the binomial.

(v) *The coefficients of the terms, equidistant from the two ends, are equal.* For example, in the expansion of $(a+b)^6$, coefficient of second term from left=coefficient of 2nd term from right=6. Similarly, third coeff. from left=third coeff. from right=15. Clearly, when we have written down the first four terms in this expansion, the remaining terms may be written down by bearing in mind the facts stated above.

EXERCISE 80

Expand :—

1. $(2a-3b)^5$. [*Solved*]
2. $(a+1)^5$.
3. $(1-a)^6$.
4. $(2x-1)^5$.
5. $(1-3m)^6$.
6. $(1-a)^{10}$.
7. $(3x-2y)^5$.
8. $(3a+2)^6$.
9. $\left(x+\dfrac{1}{x}\right)^5$.
10. $\left(a-\dfrac{1}{a}\right)^7$

Simplify :—

11. $(a+1)^4-(a-1)^4$.
12. $(a+1)^5+(a-1)^5$.
13. $(p+q)^6-(p-q)^6$.
14. $(3x+4y)^4-(3x-4y)^4$.

15. Find the coefficient of x^2 in the expansion of $(2x+3)^3(x-1)^4$. [*Solved*]

16. Find the coefficient of x^2 in the expansion of $(2x-1)^2(3x-1)^3$.

17. Find the coefficient of x^3 in the expansion of $(2x-1)^5(x+1)^2$.

18. Find the coefficient of x^4 in the expansion of $\left(x+\dfrac{1}{x}\right)^5(x-1)^3$.

SOLUTIONS—EXERCISE 80

1. $(2a-3b)^5=(x-y)^5$ [where $x=2a, y=3b$]

$$=x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5$$

$$=(2a)^5-5(2a)^4(3b)+10(2a)^3(3b)^2-10(2a)^2(3b)^3$$
$$+5(2a)(3b)^4-(3b)^5$$

[Restoring the values of x and y]

$$=32a^5-5\times16a^4\times3b+10\times8a^3\times9b^2$$
$$-10\times4a^2\times27b^3+5\times2a\times81b^4-243b^5$$

$$=32a^5-240a^4b+720a^3b^2-1080a^2b^3+810ab^4$$
$$-243b^5.$$

15. Given Exp.$=(2x+3)^3(x-1)^4$.

Now $(2x+3)^3=(a+b)^3$ [where $a=2x$ and $b=3$]

$$=a^3+3a^2b+3ab^2+b^3$$

$$=(2x)^3+3(2x)^2(3)+3(2x)(3)^2+(3)^3$$

[Restoring the values of a and b]

$$=8x^3+3\times4x^2\times8+3\times2x\times9+27$$

$$=8x^3+36x^2+54x+27$$

and $(x-1)^4=x^4-4x^3.1+6x^2.1^2-4x.1^3+1^4$

$$=x^4-4x^3+6x^2-4x+1.$$

∴ Given Exp.$=(8x^3+36x^2+54x+27)$
$$(x^4-4x^3+6x^2-4x+1).$$

In the product, terms containing x^2 are obtained by multiplying $36x^2$ by $+1$, $54x$ by $-4x$, and 27 by $6x^2$ only.

Hence, the required coefficients$=(36)(1)+(54)(-4)+(27)(6)$
$$=36-216+162=-18.$$

117. **Evolution** is the reverse of Involution. It is the operation of finding the *root* of a given expression, that is, a quantity which being multiplied by itself the requisite number of times produces the given expression.

118. By the Rule of signs we see that :—

(*i*) any *even* root of a *positive* quantity may be *either positive or negative* (∵ the even power of a quantity, whether positive or negative, is always positive).

(ii) *no negative* quantity can have an *even* root ($\because$ even power of no quantity can be negative).

(iii) every *odd* root of a quantity has the same sign as the quantity itself ($\because$ every odd power of a quantity has the same sign as the quantity itself).

119. In Art. 114 we have shown that :—

$(x^m)^{\frac{1}{n}} = x^{mn}$.

Hence we have $\sqrt[n]{x^{mn}} = x^m = x^{mn \div n}$.

Replacing mn by a, we get $\sqrt[n]{x^a} = x^{a \div n}$.

Thus, *the nth root of a power is obtained by dividing the index of the power by n.*

120 It is now easy to write down the **rule for extracting any proposed root of a simple expression :—**

(i) *First put down proper sign by Art. 118.*

(ii) *Next write down the required root of the numerical coefficient, calculating it by Arithmetic (when it is not possible to get the exact root, we may leave our result in root form.)*

(iii) *Next put down the required root of each literal factor by Art. 119.*

Note. In the case of a fraction, the numerator and the denominator should be operated separately.

For examples, see solutions to the **next exercise**

EXERCISE 81

Write down the square root of the following expressions :—

1. $9x^2y^6$ [*Solved*]   2. $4x^4y^2$.

3. $16a^8b^{10}$ 4. $25a^4b^2c^6$

5. $\dfrac{36}{x^{36}}$.   6. $\dfrac{x^{12}b^8}{25y^{14}}$

6. $-64a^{20}$ [*Solved*]   8. $\dfrac{-125}{x^{16}}$

Write down the cube root of the following expressions :—

9. $-8a^6b^{15}$. [*Solved*]   10. $27a^6b^3c^9$.

11. $-\dfrac{x^{12}y^{18}}{125}$.   12. $\dfrac{-8y^{24}}{729y^{30}}$.

Write down the value of the each of the following expressions :—

13. $\sqrt[4]{16a^8b^{12}}$. [*Solved*]   14. $\sqrt[5]{-243a^{10}b^{15}}$.

15. $\sqrt[6]{64x^6y^{18}z^{36}}$.   16. $\sqrt[8]{-256a^{16}b^{40}}$.

17. $\sqrt[7]{\dfrac{128}{x^{56}y^{49}}}$.   18. $\sqrt[9]{-\dfrac{x^{18}}{y^{45}z^{63}}}$.

SOLUTIONS—EXERCISE 81

1. Sign $= +$ or $-$, because root is second (even) and given quantity is $+$ ve. [Art. 118]

Sq. root of $9 = 8$.

" " $x^2 = x^{2 \div 2} = x^1 = x$.

" " $y^6 = y^{6 \div 2} = y^3$.

$\therefore$ Reqd. Sq. Root $= +3xy^3$ or $-3xy^3$.

7. No negative quantity can have an even root. [Art. 118]

But sq. root means second root, which is even.

$\therefore$ Sq. root of the given quantity (which is negative) is impossible.

9. The given quantity is negative and we have to find the third root (odd); therefore the result is negative. [Art. 118]

Cube root of $8 = 2$.

" " $b^{15} = b^{15 \div 3} = b^5$.

" " $a^6 = a^{6 \div 3} = a^2$.

$\therefore$ Reqd. cube root $= -2a^2b^5$.

13.   4th root of $16 = 2$.

,,   ,, $a^8 = a^{8 \div 4} = a^2$

,,   ,, $b^{12} = b^{12 \div 4} = b^3$.

$\therefore$ Reqd. Root $= \pm 2a^2b^3$.

---

121   Square Root of a Compound Expression.

There are three main methods :—

Method 1.   By the use of the formulas :—

$a^2 + 2ab + b^2 = (a+b)^2$.

$a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = (a+b+c)^2$.

etc. etc

Method 2.   By factorisation.

Method 3.   General Method analogous to that in Arithmetic.

The student has already had some practice in putting an expression in the form of a perfect square by the use of formulas [Chapter XIII, Ex. 40], and has also learnt different methods of factorisation [Chapter XIV]. Therefore general discussion of Methods 1 and 2 does not seem necessary. However, sufficient number of examples in Ex. 82 have been solved by those methods to guide the student.

Method 3 is discussed in the next article.

122.   General Method for extracting the square root of compound expression.

We know that the square root of $a^2 + 2ab + b^2$ is $a+b$; so that we have to discover a method by which $a+b$ can be derived from the expression $a^2 + 2ab + b^2$.

We first arrange the expression according to descending powers of a.

Obviously, the first term of the reqd. sq. root (viz. a) is equal to the square root of the first term (a^2) of the given expression.

$$
\begin{array}{r|l}
 & \overline{a+b} \\
\hline
a & a^2 + 2ab + b^2 \\
 & a^2 \\
\hline
2a+b & 2ab + b^2 \\
 & 2ab + b^2 \\
\hline
 & \times \\
\end{array}
$$

Subtracting the square of t... first term of the root, i.e., a^2, from the given expression, we get the remainder $2ab+b^2$.

The first term of the remainder is $2ab$. If this be divided by twice the first term of the square root, that is, by $2a$, the quotient is b which is the second term of the sq. root. [$2a$ is called the *trial divisor.*]

Since the remainder is $2ab+b^2$ or $(a+b)b$, the complete divisor (*viz.* $2a+b$) is obtained by adding the second term of the sq. root, namely b, to the trial divisor.

If we multiply $2a+b$ by b and subtract the product from the remainder $2ab+b^2$, no remainder is left. This completes the work.

The same method applies to expressions which consist of more than three terms.

Example. Find the square root of $9a^4+22a^2+9-12a^3-12a$.

Solution :—

$$
\begin{array}{r|l}
 & 3a^2-2a+3 \\
3a^2 & \overline{9a^4-12a^3+22a^2-12a+9} \\
 & 9a^4 \\
\hline
6a^2-2a & \quad -12a^3+22a^2 \\
 & \quad -12a^3+4a^2 \\
\hline
6a^2-4a+3 & \qquad\qquad 18a^2-12a+9 \\
 & \qquad\qquad 18a^2-12a+9 \\
\hline
 & \qquad\qquad\qquad \times
\end{array}
$$

Reqd. sq. root $=3a^2-2a+3$. **Ans.**

Explanation :—

The expression is first arranged in descending powers of a.
The first term of the exp. is $9a^4$. Its sq. root is $3a^2$. This is put down as the first term of the reqd. root and also as the first divisor. Subtracting $9a^4$ from the given exp. we get the remainder $-12a^3+22a^2-12a+9$.

$3a^2$ is doubled (giving $6a^2$) and put down as the first term of the second divisor. The first term of the remainder, i.e., $-12a^3$, is divided by $6a^2$, which gives $-2a$ as quotient. This quotient is put down as the second term of the reqd. root, as well as the second term of the second divisor.

The second divisor is multiplied by $-2a$ and subtracted from the remainder, which gives $18a^2-12a+9$ as the second remainder.

The part of the sq. root obtained so far, $i.e.$, $3a^2-2a$ is doubled giving $6a^2-4a$ and put down as a part of the third divisor. The first term of the second remainder, $i.e.$, $18a^2$ is divided by the first term of the third divisor, $viz.$, $6a^2$, giving a quotient 3. This is put down as the third term of the reqd. root as well as the third term of the third divisor.

Now the third divisor is $6a^2-4a+3$, which multiplied by 3 gives $18a^2-12a+9$, which is equal to the second remainder, so that subtraction gives next remainder equal to 0. The process of extraction of sq. root, therefore, stops and we get $3a^2-2a+3$ as our result.

Note. For more complicated cases see Solutions and Hints to the next exercise.

· EXERCISE 82

Find the square root of :—

1. $4x^2+12xy+9y^2$. [Solved]   2. $x^2+4xy+4y^2$.

3. $a^2-10ab+25b^2$.   4. $81a^2-18ab+b^2$.

5. $x^4+8x^2y^2+16y^4$.   6. a^6-6a^3+9.

---

7. $\dfrac{x^2}{y^2}-\dfrac{4x}{y}+4$. [Solved]   8. $\dfrac{x^2}{4}-3x+9$.

9. $x^2+\dfrac{1}{x^2}-2$.   10. $\dfrac{x^2}{y^2}+\dfrac{y^2}{4x^2}+1$

11. $\dfrac{x^2}{4y^2}+\dfrac{2x}{y}+4$.   12. $\dfrac{4x^4}{y^4}+\dfrac{y^4}{16x^4}-1$

---

13. $4a^2+9b^2+16c^2+12ab-16ac-24bc$. [Solved]

14. $x^2+y^2+z^2-2xy-2xz+2yz$.

15. $x^2+4y^2+25z^2-4xy-20yz+10zx$.

16. $x^2+\dfrac{y^2}{16}+4+\dfrac{xy}{2}-4x-y$.

17. $9 + \dfrac{b^2}{9} + c^2 - 2b - 6c + \dfrac{2bc}{3}$.

———————

18. $(x^2 - x - 6)(x^2 + x - 12)(x^2 + 6x + 8)$.   [*Solved*]

19. $(x^2 - 2x - 15)(x^2 - 3x - 10)(x^2 + 5x + 6)$.

20. $(6x^2 - x - 2)(4x^2 - 1)(6x^2 - 7x + 2)$.

21. $(2x^2 - 3xy - 2y^2)(6x^2 + xy - y^2)(3x^2 - 7xy + 2y^2)$.

22. (i) $(a^2 + ab + b^2)(a^2 - ab + b^2)(a^4 + a^2b^2 + b^4)$.   [*Hint*]

    (ii) $(2x - 3)(8x^3 - 4x^2 - 10x - 3)$.   [*Hint*]

———————

23. $(x + 1)(x + 2)(x + 3)(x + 4) + 1$.   [*Hint*]

24. $(x + 2)(x + 4)(x + 6)(x + 8) + 16$.

25. $(x + 1)(x + 3)(x - 4)(x - 6) + 49$.

26. $m(2m + 1)(m - 2)(2m - 3) + 1$.

———————

*[*See Foot Note*]

27. $x^2 + \dfrac{1}{x^2} - 8\left(x + \dfrac{1}{x}\right) + 18$.   [*Solved*]

28. $x^2 + \dfrac{1}{x^2} - 6\left(x + \dfrac{1}{x}\right) + 11$.

29. $4\left(a^2 + \dfrac{1}{a^2}\right) + 12\left(a + \dfrac{1}{a}\right) + 17$.   [*Hint*]

30. $9\left(a^2 + \dfrac{1}{a^2}\right) - 6\left(a + \dfrac{1}{a}\right) + 19$.

31. $16\left(a^2 + \dfrac{1}{a^2}\right) + 8\left(a - \dfrac{1}{a}\right) - 31$.   [*Hint*]

32. $25\left(x^2 + \dfrac{1}{x^2}\right) - 80\left(x - \dfrac{1}{x}\right) - 41$.

33. $\left(x + \dfrac{1}{x}\right)^2 - 4\left(x - \dfrac{1}{x}\right)$.   [*Hint*]

———————

*Questions 27 to 45 admit of being solved by the General Method as well. Hence, if desired, they may be attempted after Q. 70.

34. $\left(a+\dfrac{1}{a}\right)^2 +6\left(a-\dfrac{1}{a}\right)+5.$

35. $\left(a-\dfrac{1}{d}\right)^2 -8\left(a+\dfrac{1}{a}\right)+20.$ **[Hint]**

36. $\left(x-\dfrac{1}{x}\right)^2 -10\left(-x+\dfrac{1}{x}\right)+29.$

37. $x^4+\dfrac{1}{x^4} +4\left(x^2+\dfrac{1}{x^2}\right)+6.$ **[Hint]**

38. $x^4+\dfrac{1}{x^4} -6\left(x^3-\dfrac{1}{x^2}\right)+7.$

39. $x^4+\dfrac{1}{x^4} -4\left(x-\dfrac{1}{x}\right)^2-2.$ **[Hint]**

40. $x^4+\dfrac{1}{x^4} +8\left(x+\dfrac{1}{x}\right)^2+2.$

41. $\left(a^2+\dfrac{1}{a^2}\right)^2 +4\left(a+\dfrac{1}{a}\right)^2-4.$

42. $\left(m^2+\dfrac{1}{m^2}\right)^2 -12\left(m-\dfrac{1}{m}\right)^2+12.$

---

43. $\left(x^4+\dfrac{1}{x^4}\right)^2 -4\left(x^4-\dfrac{1}{x^4}\right).$

44. $x^8+\dfrac{1}{x^8} +10\left(x^4+\dfrac{1}{x^4}\right)+27.$

45. $\left(m^{12}+\dfrac{1}{m^{12}}\right)^2 -2\left(m^6-\dfrac{1}{m^6}\right)^2-3.$

---

46. $5x^2-2x+1-4x^3+4x^4.$ **[Solved]**

47. $x^4+1+2x+2x^3+3x^2.$

48. $4a^4+25-12a^3-30a+29a^2.$

49. $9a^4-12a^3-2a^2+4a+1.$

50. $1-4a+6a^2-4a^3+a^4.$ **[Note]**

51. $4-4x-7x^2+4x^3+4x^4.$

52. $x^4 - 4x^3 + 8x + 4$.

53. $x^4 + 4a^4 - 2ax^3 - 4a^3x + 5a^2x^2$.

54. $25x^4 + 16a^4 - 30ax^3 - 24a^3x + 49a^2x^2$.

55. $a^6 - 12a^5 + 60a^4 - 160a^3 + 240a^2 - 192a + 64$.

56. $m^6 - 22m^4 + 34m^3 + 121m^2 - 374m + 289$.

57. $x^6 + y^6 + 4x^5y + 4xy^5 - 10x^3y^3$.

58. $1 - 2x + 3x^2 - 4x^3 + 5x^4 - 4x^5 + 3x^6 - 2x^7 + x^8$.

———

59. $4a^8 + 4a^7 - 6a^6 - 4a^5 + 4a^4$. [Hint]

60. $4x^4 - 40x^5 + 108x^6 - 40x^7 + 4x^8$.

———

61. $x^4 - 3x^3 + x + \frac{1}{9} + \frac{19}{2}x^2$. [Solved]

62. $x^4 - x^3 + 1 + x - \frac{7x^2}{4}$.

63. $a^6 - \frac{3}{2}a^4 + \frac{2}{3}a^3 + \frac{9}{16}a^2 - \frac{1}{2}a + \frac{1}{9}$.

64. $a^6 - \frac{1}{3}a^4 + \frac{2}{7}a^3 + \frac{4}{9}a^2 - \frac{4a}{21} + \frac{1}{49}$.

65. $\frac{x^4}{4} + 4x^2 + \frac{ax^2}{3} + \frac{a^2}{9} - 2x^3 - \frac{4ax}{3}$.

66. $\frac{a^4}{25} - \frac{2a^3x}{15} + \frac{14a^2x^2}{45} - \frac{ax^3}{3} + \frac{x^4}{4}$.

———

67. $x^4 + \frac{4}{x^2} - x^2 + 4x + \frac{1}{4} - \frac{2}{x}$. [Solved]

68. $4x^2 + \frac{1}{x^2} - 6x + 6\frac{1}{4} - \frac{3}{x}$.

69. $x^4 + \frac{16}{x^4} + 8x^2 + \frac{32}{x^2} + 24$.

70. $4x^4 + \frac{49}{x^4} + 20x^2 - \frac{70}{x^2} - 3$.

71. $96 + \frac{64}{x^4} + \frac{128}{x^2} + 82x^2 + 4x^4$.

72. $x^6 + \dfrac{1}{x^6} + 6x^4 + \dfrac{6}{x^4} + 15x^2 + \dfrac{15}{x^2} + 20.$

73. $a^6 + \dfrac{1}{a^6} - 4a^4 + 4\left(a^2 - \dfrac{1}{a^2}\right) + 2.$

74. $4a^2 + 4a - 11 + \dfrac{1}{a^4} + \dfrac{6}{a^3} + \dfrac{7}{a^2} - \dfrac{10}{a}.$

---

75. $\dfrac{x^2}{a^2} + \dfrac{a^2}{x^2} + \dfrac{6x}{a} - \dfrac{6a}{x} + 7.$ [Solved]

76. $\dfrac{a^2}{b^2} + \dfrac{b^2}{a^2} - \dfrac{2a}{b} - \dfrac{2b}{a} + 3.$

77. $\dfrac{9a^2}{b^2} - \dfrac{6a}{5b} + \dfrac{101}{25} - \dfrac{4b}{15a} + \dfrac{4b^2}{9a^2}.$

78. $\dfrac{x^4}{4} + \dfrac{x^3}{y} + \dfrac{x^2}{y^2} - xy - 2 + \dfrac{y^2}{x^2}.$

79. $24 + \dfrac{16y^2}{x^4} - \dfrac{8x^2}{y} + \dfrac{x^4}{y^2} - \dfrac{32y}{x^2}.$

---

80. $(x+y)^4 + x^4 + y^4.$ [Hint]

81. $(a-b)^4 + a^4 + b^4.$

Find the fourth root of :—

82. $16a^4 - 32a^3 + 24a^2 - 8a + 1.$ [Hint]

83. $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4.$

Find an expression, containing no higher power of x than the first, which must be

84. subtracted from $27x^2 - 9x + 2 - 10x^3 + x^4$ to make it a perfect square. [Hint]

85. subtracted from $10x^2 - x + 9 - 12x^3 + 9x^4$ to make it a perfect square.

86. added to $4x^3 - 7x^2 - 6x + 7 + 4x^4$ to make it a perfect square. [Hint]

87. added to $13x^2+6x+1+x^4+6x^3$ to make it a perfect square.

---

88. For what value of a will $9a^4-12a^3+12-13a+22a^2$ be a perfect square ? [*Hint*]

89. What value of x will make $9x^4-12x^3+5x-2x^2+6$, a perfect square ?.

90. For what value of m will the expression $a^4+6a^3+7a^2-6a+m$, be a perfect square ? [*Hint*]

91. For what value of k will $9x^4-12x^3+10x^2-4x+2k$, be a perfect square ?

92. Find the numerical values of a and b if $x^4-8x^3+32ax+8b$ is to be a perfect square. [*Hint*]

---

93. Prove that if 1 be added to the product of any four consecutive integers, the result is a perfect square.

[*Hint*]

94. Prove that if 16 be added to the product of any four consecutive odd numbers, the result is a perfect square.

95. Show that the statement of the last question is true if the word " odd " be replaced by " even."

SOLUTIONS & HINTS—EXERCISE 82

1. Given Exp.$=4x^2+12xy+9y^2$
$$=(2x)^2+2(2x)(3y)+(3y)^2$$
$$=(2x+3y)^2$$

∴ Reqd. Sq. Root$=2x+3y$.

7. Given Exp.$=\dfrac{x^2}{y^2}-\dfrac{4x}{y}+4$
$$=\left(\dfrac{x}{y}\right)^2-2\left(\dfrac{x}{y}\right)(2)+(2)^2$$
$$=\left(\dfrac{x}{y}-2\right)^2$$

$\therefore$ Reqd. Sq. Root $=\dfrac{x}{y}-2.$

13.   Given Exp. $=4a^2+9b^2+16c^2+12ab-16ac-24bc.$

$=(2a)^2+(3b)^2+(-4c)^2+2(2a)(3b)$
$+2(2a)(-4c)+2(3b)(-4c)$

$=(2a+3b-4c)^2$

$\therefore$ Reqd. Sq. Root $=2a+3b-4c.$

18.   Given Exp. $=(x^2-x-6)(x^2+x-12)(x^2+6x+8)$

$=(x-3)(x+2)(x+4)(x-3)(x+4)(x+2)$

[Factorising]

$=(x-3)^2(x+2)^2(x+4)^2$

$\therefore$ Reqd. Sq. Root $=(x-3)(x+2)(x+4).$

22.   (i) Either split the third factor into two factors.

Or

Multiply the first two factors ; the product obtained is
equal to the third factor ; thus the sq. root $=$ the third
factor.

(ii) Obviously, the second factor contains $2x-3$ as a
factor [$\because$ otherwise the sq. root is not exact].
Therefore divide the second factor by $2x-3$ and
factorise the quotient obtained.

23.   Given Exp. $=(x+1)(x+2)(x+3)(x+4)+1$

$=\{(x+1)(x+4)\}\{(x+2)(x+3)\}+1$

$=(x^2+5x+4)(x^2+5x+6)+1$

$=(a+4)(a+6)+1$ [where $a=x^2+5x$]

$=a^2+10a+24+1$

$=a^2+10a+25$

$=(a+5)^2$

$=(x^2+5x+5)^2$ [Restoring the value of a]

$\therefore$ Reqd. Sq. Root $=x^2+5x+5.$

27. Given Exp. $= x^2 + \dfrac{1}{x^2} - 8\left(x + \dfrac{1}{x}\right) + 18$

$$= \left(x + \dfrac{1}{x}\right)^2 - 2 - 8\left(x + \dfrac{1}{x}\right) + 18$$

$$\left[\because \quad x^2 + \dfrac{1}{x^2} = \left(x + \dfrac{1}{x}\right)^2 - 2\right]$$

$$= \left(x + \dfrac{1}{x}\right)^2 - 8\left(x + \dfrac{1}{x}\right) + 16$$

[Combining -2 and 18]

$$= a^2 - 8a + 16 \quad \left[\text{where } a = x + \dfrac{1}{x}\right]$$

$$= (a - 4)^2$$

$$= \left(x + \dfrac{1}{x} - 4\right)^2$$

[Rewriting the value of a]

$\therefore$ Reqd. Sq. Root $= x + \dfrac{1}{x} - 4.$

Or thus :—

[*This method should be studied after finishing the exercise upto Q. 79*].

Given Exp. $= x^2 + \dfrac{1}{x^2} - 8x - \dfrac{8}{x} + 18$

[Opening the bracket]

$$= x^2 - 8x + 18 - \dfrac{8}{x} + \dfrac{1}{x^2}$$

[Arranging according to powers of x]

$$
\begin{array}{r|l}
 & x-4+\dfrac{1}{x} \\[2ex]
x & x^2-8x+18-\dfrac{8}{x}+\dfrac{1}{x^2} \\[2ex]
 & x^2 \\[1ex]
\cline{2-2}
2x-4 & -8x+18 \\
 & -8x+16 \\
\cline{2-2}
2x-8+\dfrac{1}{x} & \qquad 2-\dfrac{8}{x}+\dfrac{1}{x^2} \\[2ex]
 & \qquad 2-\dfrac{8}{x}+\dfrac{1}{x^2} \\[2ex]
\cline{2-2}
 & \times
\end{array}
$$

$\therefore$ Sq. Root $=x-4+\dfrac{1}{x}$.

Or thus :—

Given Exp. $= x^2+\dfrac{1}{x^2}-8x-\dfrac{8}{x}+18$

[opening the bracket]

$$
=\frac{x^4+1-8x^3-8x+18x^2}{x^2}
$$

$$
=\frac{x^4-8x^3+18x^2-8x+1}{x^2}
$$

We find the sq. roots of the numerator and the denominator separately.

$\therefore$ Reqd. Sq. Root

$$
=\frac{x^2-4x+1}{x}
$$

$$
=\frac{x^2}{x}-\frac{4x}{x}+\frac{1}{x}
$$

$$
= x-4+\frac{1}{x}.
$$

$$
\begin{array}{r|l}
 & x^2-4x+1 \\
x^2 & x^4-8x^3+18x^2-8x+1 \\
 & x^4 \\
\cline{2-2}
2x^2-4x & -8x^3+18x^2 \\
 & -8x^3+16x^2 \\
\cline{2-2}
2x^2-8x\cdot 1 & 2x^2-8x+1 \\
 & 2x^2-8x+1 \\
\cline{2-2}
 & \times
\end{array}
$$

Note. Questions 27 to 45 can be solved by any one of the above three methods.

29. $4\left(a^2+\dfrac{1}{a^2}\right)$ may be written as $4\left\{\left(a+\dfrac{1}{a}\right)^2-2\right\}$

or as $4\left(a+\dfrac{1}{a}\right)^2-8.$

31. $16\left(a^2+\dfrac{1}{a^2}\right)$ may be written as $16\left\{\left(a-\dfrac{1}{a}\right)^2+2\right\}$

or as $16\left(a-\dfrac{1}{a}\right)^2+32.$

33. $\left(x+\dfrac{1}{x}\right)^2$ may be written as $\left(x-\dfrac{1}{x}\right)^2+4.$

35. $\left(a-\dfrac{1}{a}\right)^2$ may be written as $\left(a+\dfrac{1}{a}\right)^2-4.$

37. $x^4+\dfrac{1}{x^4}$ may be written as $\left(x^2+\dfrac{1}{x^2}\right)^2-2.$

39. Given Exp. $= x^4+\dfrac{1}{x^4}-4\left(x-\dfrac{1}{x}\right)^2-2$

$\qquad = x^4+\dfrac{1}{x^4}-4\left(x^2+\dfrac{1}{x^2}-2\right)-2$

$\qquad = x^4+\dfrac{1}{x^4}-4\left(x^2+\dfrac{1}{x^2}\right)+8-2.$

Now write $x^4+\dfrac{1}{x^4}$ as $\left(x^2+\dfrac{1}{x^2}\right)^2-2$ and proceed as before.

46. Arranging the exp. in descending powers of x, we have :—

$$
\begin{array}{r|l}
 & \underline{2x^2-x+1} \\
2x^2 & 4x^4-4x^3+5x^2-2x+1 \\
 & 4x^4 \\
 & \overline{\qquad} \\
4x^2-x & \quad -4x^3+5x^2 \\
 & \quad -4x^3+\ x^2 \\
 & \overline{\qquad} \\
4x^2-2x+1 & \qquad\quad 4x^2-2x+1 \\
 & \qquad\quad 4x^2-2x+1 \\
 & \overline{\qquad} \\
 & \qquad\qquad \wedge
\end{array}
$$

$\therefore$ Sq. Root $= 2x^2 - x + 1$.

50. Note. The expression, as it stands, is arranged in ascending powers of a. Therefore no re-arrangement of terms is necessary.

59. Given Exp. $= a^4(4a^4 + 4a^3 - 7a^2 - 4a + 4)$

Sq. root of a^4 is a^2. Find the sq. root of the other factor and multiply it by a^2.

61. Arranging the exp. in descending powers of x we have :—

$$
\begin{array}{r|l}
 & x^2 - \tfrac{3}{2}x - \tfrac{1}{3} \\
x^2 & \overline{x^4 - 3x^3 + 1\tfrac{9}{12}x^2 + x + \tfrac{1}{9}} \\
 & x^4 \\
2x^2 - \tfrac{3}{2}x & \overline{-3x^3 + 1\tfrac{9}{12}x^2} \\
 & -3x^3 + \tfrac{9}{4}x^2 \\
2x^2 - 3x - \tfrac{1}{3} & \overline{-\tfrac{2}{3}x^2 + x + \tfrac{1}{9}} \\
 & -\tfrac{2}{3}x^2 + x + \tfrac{1}{9} \\
 & \overline{\times}
\end{array}
$$

$[-3x^3 \div 2x^2 = -\tfrac{3}{2}x]$

$[-\tfrac{2}{3}x^2 \div 2x^2 = -\tfrac{1}{3}]$

Sq. Root $= x^2 - \tfrac{3}{2}x - \tfrac{1}{3}$.

Or thus :—

Given Exp. $= x^4 - 3x^3 + 1\tfrac{9}{12}x^2 + x + \tfrac{1}{9}$

$$= \frac{36x^4 - 108x^3 + 57x^2 + 36x + 4}{36}$$

Sq. root of the numerator will be found $= 6x^2 - 9x - 2$

Also, sq. root of the denominator $= 6$

$\therefore$ Reqd sq. root $= \dfrac{6x^2 - 9x - 2}{6} = \dfrac{6x^2}{6} - \dfrac{9x}{6} - \dfrac{2}{6}$

$$= x^2 - \tfrac{3}{2}x - \tfrac{1}{3}.$$

[By this method we get rid of fractions in the actual work of extracting the sq. root]

67 In this expression x occurs in the numerator as well as in the denominator. To arrange it according to descending powers of x we first arrange the terms containing x in the numerator as usual, then we write down the constant term (*i.e.*, the term free from x), and then we arrange the remaining terms according

to *ascending powers of x as they occur in the denominator.* The student will learn in the next chapter that this arrangement of the remaining terms is *in fact according to the descending powers of x,* and in keeping with the previous arrangement.

We have :—

$$
\begin{array}{r|l}
 & x^2 - \tfrac{1}{2} + \dfrac{2}{x} \\[2mm]
\hline
x^2 & x^4 - x^2 + 4x + \tfrac{1}{4} - \dfrac{2}{x} + \dfrac{4}{x^2} \\[2mm]
 & \underline{x^4} \\[1mm]
2x^2 - \tfrac{1}{2} & -x^2 + 4x + \tfrac{1}{4} \qquad\qquad [-x^2 \div 2x^2 = -\tfrac{1}{2}] \\[1mm]
 & \underline{-x^2 + 0 + \tfrac{1}{4}} \\[1mm]
2x^2 - 1 + \dfrac{2}{x} & \quad 4x \qquad - \dfrac{2}{x} + \dfrac{4}{x^3} \qquad \left[\; 4x \div 2x^2 = \dfrac{2}{x}\; \right] \\[2mm]
 & \quad \underline{4x \qquad - \dfrac{2}{x} + \dfrac{4}{x^2}} \\[1mm]
 & \qquad\qquad\quad \times
\end{array}
$$

Sq. root $= x^2 - \tfrac{1}{2} + \dfrac{2}{x}$.

Note that at any step we may bring down as many terms as necessary.

Or thus :—

Given Exp. $= x^4 + \dfrac{4}{x^2} - x^2 + 4x + \tfrac{1}{4} - \dfrac{2}{x}$

$$= \frac{4x^6 + 16 - 4x^4 + 16x^3 + x^2 - 8x}{4x^2}$$

It is now easy to arrange the numerator in descending powers of x and extract its square root ($2x^3 - x + 4$).

Also sq. root of the denominator $= 2x$.

$\therefore$ Reqd. sq. root $= \dfrac{2x^3 - x + 4}{2x} = \dfrac{2x^3}{2x} - \dfrac{x}{2x} + \dfrac{4}{2x}$

$$= x^2 - \frac{1}{2} + \frac{2}{x}.$$

75.

$$\dfrac{x}{a}+3-\dfrac{a}{x}$$

$$\dfrac{x}{a}\ \bigg|\ \dfrac{x^2}{a^2}+\dfrac{6x}{a}+7-\dfrac{6a}{x}+\dfrac{a^2}{x^2}$$

$$\dfrac{x^2}{a^2}$$

$$\dfrac{2x}{a}+3\qquad \dfrac{6x}{a}+7\qquad \left[\dfrac{6x}{a}\div\dfrac{2x}{a}=3\right]$$

$$\dfrac{6x}{a}+9$$

$$\dfrac{2x}{a}+6-\dfrac{a}{x}\qquad -2-\dfrac{6a}{x}+\dfrac{a^2}{x^2}\qquad \left[-2\div\dfrac{2x}{a}=-\dfrac{a}{x}\right]$$

$$-2-\dfrac{6a}{x}+\dfrac{a^2}{x^2}$$

$$\times$$

Sq. Root $=\dfrac{x}{a}+3-\dfrac{a}{x}$.

Or Thus :—

Given Exp. $=\dfrac{x^2}{a^2}+\dfrac{a^2}{x^2}+\dfrac{6x}{a}-\dfrac{6a}{x}+7$

$$=\dfrac{x^4+a^4+6ax^3-6a^3x+7a^2x^2}{a^2x^2}.$$

Sq. root of the numerator can be easily found $=x^2+3ax-a^2$.
Also sq. root of the denominator $=ax$.

$\therefore$ Reqd. sq. root $=\dfrac{x^2+3ax-a^2}{ax}=\dfrac{x}{a}+3-\dfrac{a}{x}$.

80.   Given Exp $=(x+y)^4+x^4+y^4$

$$=(x^4+4x^3y+6x^2y^2+4xy^3+y^4)+x^4+y^4$$

$$=2x^4+4x^3y+6x^2y^2+4xy^3+2y^4$$

$$=2(x^4+2x^3y+3x^2y^2+2xy^3+y^4)$$

Sq. root of 2 is $\sqrt{2}$, and sq. root of the exp. within bracket can be easily found. Reqd. sq. root $=$ the product of these two results.

82. Find the sq. root of the given expression and then the square root of this square root.

84. Arrange the expression in descending powers of x and proceed to extract the sq. root by the general method. The last remainder (which contains no higher powers of x than the first) is the reqd. expression.

86. Proceeding as in Q. 84, get the last remainder. This remainder is the expression which must be *subtracted* for the purpose, hence this *remainder with its signs changed* is the exp. which must be *added*, and therefore the reqd. exp.

88. Proceed as in Q. 84; put the remainder equal to zero and solve this equation for a.

90. Proceed as in Q. 84; equate the last remainder to zero and solve your equation for m.

92. Proceed as in Q. 84; equate the coefficient of x in the last remainder to zero and also the constant term to zero. These two equations give the values of a and b respectively.

93. Let the consecutive integers be x, $x+1$, $x+2$ and $x+3$.

Then we have to show that $x(x+1)(x+2)(x+3)+1$ is a perfect square. Proceed as in Q. 23.

CHAPTER XX

INDICES

123. We have already defined a^m (where m is a positive integer) as the product of m factors, each equal to a. Also, a is called the *base* and m the *index*. The plural of 'index' is 'indices'

124. Laws of Indices.

The student has already learnt the following *laws of indices*, when m and n are positive integers :—

Law 1 $a^m \times a^n = a^{m+n}$. [*Fundamental Index Law*]

Law 2. $a^m - a^n \left(\text{or } \dfrac{a^m}{a^n} \right) = a^{m-n}$. [$m > n$]

Law 3. $(a^m)^n = a^{mn}$.

Law 4. $(ab)^n = a^n b^n$.

Law 5. $\left(\dfrac{a}{b}\right) = \dfrac{a^n}{b^n}$.

The proofs are reproduced here for ready reference :—

1. $a^m \times a^n = (a \times a \times a \times ... m \text{ factors}) \times (a \times a \times a ... n \text{ factors})$

$= a \times a \times a \times ....(m+n) \text{ factors}$

$= a^{m+n}$.

2. $\dfrac{a^m}{a^n} = \dfrac{a \times a \times a \times ....m \text{ factors}}{a \times a \times a \times ....n \text{ factors}}$

$= a \times a \times a \times ....(m-n) \text{ factors}$

[after cancelling n factors]

$= a^{m-n}$.

3. $(a^m)^n = a^m \times a^m \times a^m \times ....n \text{ factors}$

$= a^{m+m+m+............n \text{ terms}}$. **[Law 1]**

$= a^{mn}$.

4. $(ab)^n = ab \times ab \times ab \times ....n \text{ factors}$

$= (a \times a \times a \times ....n \text{ factors})$

$\times (b \times b \times b \times ...$ $n \text{ factors})$

$= a^n . b^n$.

5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a}{b} \times \dfrac{a}{b} \times \dfrac{a}{b} ....n \text{ factors}$

$= \dfrac{a \times a \times a \times ....n \text{ factors}}{b \times b \times b \times ....n \text{ factors}}$

$= \dfrac{a^n}{b^n}$.

125 To prove that $a^0 = 1$

Proof $\dfrac{a^n}{a^n} = 1$. [∵ numerator = denominator]

Also, $\dfrac{a^n}{a^n} = a^{n-n}$. **[Law 2]**

$= a^0$

∴ $a^0 = 1$.

126. Fractional and Negative Indices.

When m is fractional or negative, a^m has no meaning for the present.

Now it is important that all indices, whether positive or negative, integral or fractional, should be governed by the same laws which have been established for positive integral indices. We therefore assume the *fundamental index law*, [namely, $a^m \times a^n = a^{m+n}$] to be also true for all negative and fractional values of m and n and thence determine meanings for symbols such as $a^{\frac{p}{q}}$, a^{-n} etc. The symbols so interpreted will also obey the other laws of Art. 124. [Proof of this fact will not be given, as it is beyond the scope of this book.]

The method of finding a meaning for a symbol, as stated above, deserves careful attention. The usual algebraical process is to fix symbols, give them meanings and then prove the rules for their combination. Here the process is reversed : *the symbols and the laws which they obey are given, and from this the meanings of the symbols are to be determined.* This is done in the next three articles.

127. *To find a meaning for $a^{\frac{1}{p}}$, p being a positive integer.*

By the fundamental index law

$$a^{\frac{1}{p}} \times a^{\frac{1}{p}} \times a^{\frac{1}{p}} \times \ldots\ldots\ldots p \text{ factors}$$

$$= a^{\frac{1}{p} + \frac{1}{p} + \ldots\ldots\ldots p \text{ terms}}$$

$$= a^{\frac{1}{p} \times p} = a.$$

Hence $a^{\frac{1}{p}}$ is the pth root of a.

128. To find a meaning for $a^{\frac{p}{q}}$, p and q being positive integers.

By the fundamental index law

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \dots\dots\dots\text{to } q \text{ factors.}$$

$$= a^{\frac{p}{q} + \frac{p}{q} + \frac{p}{q} + \dots\dots\dots q \text{ terms}}$$

$$= a^{\frac{p}{q} \times q} = a^p.$$

i.e., $\left(a^{\frac{p}{q}}\right)^{q} = a^p.$

Taking qth root of both sides, we have :—

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

Hence $a^{\frac{p}{q}}$ is equal to the qth root of a^p.

Besides this, we have *another meaning of* $a^{\frac{p}{q}}$:—

By the fundamental index law

$$a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times \dots\dots\dots\text{to } p \text{ factors}$$

$$= a^{\frac{1}{q} + \frac{1}{q} + \frac{1}{q} + \dots\dots\dots p \text{ terms}}$$

$$= a^{\frac{1}{q} \times p} = a^{\frac{p}{q}}$$

i.e., $\left(a^{\frac{1}{q}}\right)^{p} = a^{\frac{p}{q}}$

Hence $a^{\frac{p}{q}}$ also means the pth power of $a^{\frac{1}{q}}$

  i.e., pth power of the qth-root of a. [Art. 127]

Note. It is now clear that when the index is a fraction, the numerator indicates the power and the denominator the root, and *we may operate upon the base either first by the power and then by the root or first by the root and then by the power.*

For example, $(4)^{\frac{3}{2}}$ may be evaluated in two ways :—

(1) $(4)^{\frac{3}{2}} = (4^{\frac{1}{2}})^3 = $ (second root of 4)^3, i.e., (sq. root of 4)^3

$= (2)^3 = 8$.

(2) $4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}} = (64)^{\frac{1}{2}} = $ sq. root of $64 = 8$.

129. *To find a meaning for* a^{-n}, *where* n *has any positive value*

By the fundamental index law

$a^n \times a^{-n} = a^{n-n} = a^0 = 1$.   [Art. 125]

$\therefore \quad a^{-n} = \dfrac{1}{a^n}$

Hence a^{-n} is the reciprocal of a^n.

Also, clearly, a^n is the reciprocal of a^{-n}. $\left[\because a^n = \dfrac{1}{a^{-n}} \right]$

Note. From the equalities $a^{-n} = \dfrac{1}{a^n}$ and $a^n = \dfrac{1}{a^{-n}}$ it follows

that a factor may be transferred from the numerator of a fraction to the denominator, or from the denominator to the numerator, by changing the sign of its index.

Thus $\dfrac{x^2 y^{-3}}{z^{-4} a^5} = \dfrac{x^2 z^4}{y^3 a^5}$.

130. Here is the summary of all the fore-going articles of this chapter :—

(a) If m is a positive integer, $a^m = a \times a \times a \times \ldots\ldots m$ factors 'a' is the *base* and 'm' the *index*. The plural of 'index' is 'indices.'

(b) For all values of m and n (positive or negative, integral or fractional) :—

$$a^m \times a^n = a^{m+n} \quad \text{[Law 1]}$$

$$a^m \div a^n \text{ or } \frac{a^m}{a^n} = a^{m-n}. \quad \text{[Law 2]}$$

$$(a^m)^n = a^{mn}. \quad \text{[Law 3]}$$

$$(ab)^n = a^n b^n. \quad \text{[Law 4]}$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}. \quad \text{[Law 5]}$$

(c) If p and q are positive integers,

$a^{\frac{1}{p}}$ means the pth root of a.

$a^{\frac{p}{q}}$ means the qth root of a^p, or the pth power of the qth root of a [i.e., $\sqrt[q]{a^p}$ or $(\sqrt[q]{a})^p$].

In particular, $a^{\frac{1}{2}}$ means the 2nd root of a, i.e., the sq. root of a ($=\sqrt{a}$).

(d) $a^0 = 1$.

(e) $a^{-n} = \frac{1}{a^n}$ and $a^n = \frac{1}{a^{-n}}$. [n being any positive number, integral or fractional]. That is to say, a factor may be transferred from the numerator of a fraction to the denominator, or from the denominator to the numerator, by changing the sign of its index.

A thorough mastery of the above facts is essential to work examples in indices. Besides these, the following *extensions of the laws*, which can be easily deduced from the laws given above, are also extremely useful :—

$$x^a \times x^b \times x^c \times \dots\dots\dots = x^{a+b+c+\dots\dots} \quad \text{[Extension of Law 1]}$$

$$\left[\left\{\left[(x^a)^b\right]^c\right\}^d\right] = x^{abcd} \quad \text{[Extension of Law 3]}$$

$$(abcd\dots\dots)^m = a^m.b^m.c^m.d^m\dots\dots \quad \text{[Extension of Law 4]}$$

$$\left(\frac{abc\dots\dots}{xyz\dots\dots}\right)^m = \frac{a^m b^m c^m \dots\dots}{x^m y^m z^m \dots\dots} \quad \text{[Extension of Laws 4 and 5]}$$

$$\frac{a^l.b^m.c^n\dots\dots}{a^p.b^q.c^r\dots\dots} = a^{l-p}.b^{m-q}.c^{n-r}\dots \quad \text{[Extension of Law 2]}$$

EXERCISE 83 (a)

Find the value of :—

1. 3^4. [Solved]

2. 4^5

3. 2^5.

4. 5^4.

5. 3^{-5}. [Solved]

6. 2^{-6}.

7. 4^{-3}.

8. 5^{-4}.

9. $8^{\frac{1}{3}}$. [Solved]

10. $81^{\frac{1}{4}}$.

11. $16^{\frac{1}{2}}$.

12. $1024^{\frac{1}{5}}$

13. $256^{-\frac{1}{4}}$. [Solved]

14. $343^{-\frac{1}{3}}$.

15. $1024^{-\frac{1}{5}}$.

16. $196^{-\frac{1}{2}}$.

17. $27^{\frac{2}{3}}$. [Solved]

18. $25^{\frac{3}{2}}$.

19. $32^{\frac{2}{5}}$.

20. $729^{\frac{5}{6}}$.

21. $27^{-\frac{2}{3}}$. [Hint]

22. $64^{-\frac{1}{3}}$.

23. $32^{-\frac{3}{5}}$.

24. $25^{-\frac{3}{2}}$

25. $\left(\frac{25}{4}\right)^{-\frac{3}{2}}$. [Solved]

26. $\left(\frac{1}{216}\right)^{-\frac{2}{3}}$

27. $\left(\frac{256}{625}\right)^{-\frac{1}{4}}$.

28. $\left(\frac{16}{81}\right)^{-\frac{3}{4}}$

Express with positive indices :—

29. $3a^{-\frac{1}{2}}$ [Solved]

30. $4x^{-\frac{1}{2}}$

31. $\dfrac{1}{5x^{-\frac{1}{2}}}.$

32. $\dfrac{-5}{6a^{-2}}.$ [Hint]

33. $\dfrac{-7y}{z^{-1}}.$

34. $\dfrac{3x^{-2}y}{-z^2t^{-3}}.$

Express with radical signs and positive indices :—

35. $5x^{-\frac{3}{4}}.$ [Solved]

36. $8a^{-\frac{2}{5}}.$

37. $\dfrac{-3a}{4b^{-\frac{3}{7}}}$

38. $\dfrac{x^{-\frac{b}{a}}}{2}$

Remove radical signs and express with positive indices :—

39. $\dfrac{-1}{2\sqrt[5]{x^{-2}}}.$ [Solved]

40. $\dfrac{\sqrt{a^{-3}}}{3}.$

41. $\dfrac{-4\sqrt[3]{a^{-1}}}{5}.$

42 $\dfrac{\sqrt[b]{x^{-a}}}{-2}.$

SOLUTIONS & HINTS—EXERCISE 83 (a)

1. $3^4 = 3 \times 3 \times 3 \times 3 = 81.$

5. $3^{-5} = \dfrac{1}{3 \times 3 \times 3 \times 3 \times 3} = \dfrac{1}{243}.$

9. $8^{\frac{1}{3}} = \sqrt[3]{8} = 2.$

13. $256^{-\frac{1}{4}} = \dfrac{1}{256^{\frac{1}{4}}} = \dfrac{1}{\sqrt[4]{256}} = \frac{1}{4}.$

17. $27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = (3)^2 = 9.$

21. $27^{-\frac{2}{3}} = \dfrac{1}{27^{\frac{2}{3}}}$, etc. {See Q. 17}

25. $\left(\dfrac{25}{4}\right)^{-\frac{3}{2}} = \left(\dfrac{4}{25}\right)^{\frac{3}{2}}$ $\left\{\because a^{-p} = \dfrac{1}{a^p} = \left(\dfrac{1}{a}\right)^p, \right.$

$$= (\sqrt{{}_2{}^4_5})^3$$ that is, *if the base be invert-*
$$= (\tfrac{2}{3})^3$$ *ed, the index changes sign*
$$= \tfrac{2}{3} \times \tfrac{2}{3} \times \tfrac{2}{3} = {}_1{}^8_{25}.$$ *In this question, base* $= {}^2_4{}^5$]

29. $8a^{-\frac{1}{4}} = \dfrac{8}{a^{\frac{1}{4}}}$ [∵ the factor $a^{-\frac{1}{4}}$ is taken from the numerator to the denominator and therefore its index changes sign]

32. Don't try to change the sign of -5. This sign can never be changed. Also note that you have to make the *indices* positive ; -5 is not an index.

35. $5x^{-\frac{3}{4}} = \dfrac{5}{x^{\frac{3}{4}}} = \dfrac{5}{(\sqrt[4]{x})^3}$ or $\dfrac{5}{\sqrt[4]{x^3}}$.

39. $\dfrac{-1}{2\sqrt[5]{x^{-3}}} = \dfrac{-1}{2x^{-\frac{3}{5}}} = \dfrac{-x^{\frac{3}{5}}}{2}.$

EXERCISE 83 (b)

Simplify, expressing your result with radical signs and positive indices :—

1. $a^{\frac{1}{4}} \times a^{\frac{3}{4}}$. [*Solved*]   2. $x^{\frac{2}{3}} \times x^{\frac{5}{6}}$.

3. $x^{-\frac{1}{3}} \times a^{\frac{1}{6}}$.   4. $\sqrt[3]{a} \times \sqrt[4]{a^{-3}}$. [*Hint*]

5. $\sqrt[7]{a^{-5}} \times \sqrt[4]{a^3}$.   6. $(\sqrt[5]{b})^{-2} \times ({}^{10}\sqrt{b})^{-3}$.

---

7. $x^{-\frac{2}{5}} \div x^{-\frac{1}{10}}$ [*Solved*] 8. $x^{\frac{3}{4}} \div x^{-\frac{1}{4}}$.

9. $\dfrac{\sqrt[3]{a^2}}{\sqrt[6]{a^1}}$.   10. $\dfrac{4\sqrt{a}}{6a^{-1}}$.

---

11. $(a^3)^5$. [*Solved*].   12. $(x^2)^6$.

13. $(a^4)^{\frac{3}{8}}$.   14. $(x^{\frac{2}{3}})^{\frac{3}{2}}$.

15. $(a^{-\frac{1}{3}})^{\frac{3}{2}}$.   16. $(a^{\frac{2}{3}})^{-\frac{3}{4}}$.

17. $(\sqrt{a})^{-\frac{1}{3}}$. [*Hint*]   18. $(\sqrt[3]{a})^{\frac{2}{3}}$.

19. $(\sqrt[3]{x^{-2}})^{\frac{5}{4}}$.

20. $\sqrt[6]{x}^{-\frac{2}{3}}$.

21. $\sqrt{(\sqrt[3]{x^{-2}})^{\frac{3}{4}}}$. [*Solved*]

22. $\sqrt[5]{(\sqrt{x^{-3}})^{\frac{5}{3}}}$.

23. $(\sqrt[4]{a^{\frac{2}{3}}})^{-1}$.

24. $(\sqrt{a^{-\frac{4}{5}}})^{\frac{5}{2}}$.

Evaluate :—

25. $2^{3^2} \div (2^3)^2$. [*Solved*]

26. $2^{2^3} + (2^2)^3$.

27. $(3^3)^2 \div 3^{2^3}$.

28. $(2^3)^3 - 2^{3^2}$.

SOLUTIONS & HINTS—EXERCISE 83 (b)

1. $x^{\frac{1}{2}} \times x^{\frac{3}{4}} = x^{\frac{1}{2}+\frac{3}{4}} = x^{\frac{5}{4}} = \sqrt[4]{x^5}$ or $(\sqrt[4]{x})^5$.

4. Given Exp. $= a^{\frac{1}{3}} \times a^{-\frac{3}{4}}$ etc.

7. $x^{-\frac{3}{5}} \div x^{-\frac{1}{10}} = x^{-\frac{3}{5}-(-\frac{1}{10})} = x^{-\frac{3}{5}+\frac{1}{10}} = x^{-\frac{1}{2}}$
$$= \frac{1}{x^{\frac{1}{2}}} = \frac{1}{\sqrt{x}}.$$

11. $(x^3)^5 = x^{3 \times 5} = x^{15}$.

17. $(\sqrt{a})^{-\frac{1}{2}} = (a^{\frac{1}{2}})^{-\frac{1}{2}}$ etc.

21. $\sqrt{(\sqrt[3]{x^{-2}})^{\frac{3}{4}}} = \left\{ (x^{-\frac{2}{3}})^{\frac{3}{4}} \right\}^{\frac{1}{2}}$

[removing all radical signs]
$$= x^{-\frac{2}{3} \times \frac{3}{4} \times \frac{1}{2}} = x^{-\frac{1}{4}} = \frac{1}{x^{\frac{1}{4}}} = \frac{1}{\sqrt[4]{x}}.$$

25. $2^{3^2} = 2^9$ [$\because 3^2 = 9$]
$$(2^3)^2 = 2^{3 \times 2} = 2^6$$
$\therefore$ Given Exp. $= 2^9 \div 2^6 = 2^{9-6} = 2^3 = 8$.

Note carefully the difference between 2^{3^2} *and* $(2^3)^2$. 3^{3^2} means that 2 is raised to the power 3^2, *i.e.*, to the power 9, while $(2^3)^2$ means that the third power of 2 is raised to the power 2.

EXERCISE 83 (c)

Simplify and express with positive indices :—

1. $\left(\sqrt[9]{a^{-4} b^3}\right)^{-3}$. [Solved]

2. $\left(\sqrt[6]{a^{-\frac{2}{3}} y^{\frac{1}{2}}}\right)^{-3}$

3. $\sqrt[4]{\left(\dfrac{16a^2}{b^{-2}}\right)^{-1}}$

4. $\sqrt[3]{\left(\dfrac{27a^3}{8b^{-3}}\right)^{-2}}$

5. $(9b^{-2} \div 16y^4)^{-\frac{1}{2}}$.

6. $\sqrt[8]{a} \cdot \sqrt[3]{a^{-1}}$

7. $\sqrt[4]{x^{-2} y} \times \sqrt[6]{x y^{-3}}$. [Solved]

8. $\sqrt[3]{a^{2b} x^3} \times (a^{\frac{1}{3}} x^{-\frac{1}{2}})^{-2b}$.

9. $\sqrt[6]{a^{-1} \sqrt{b^3}} \div \sqrt[4]{b \cdot \sqrt[3]{a}}$.

10. $(b^{-\frac{1}{2}} \sqrt[3]{y})^{-6} \times y^{-2} \sqrt{b^{-6}}$.

11. $\left(\dfrac{x^{-2} y^3}{x^8 y^{-7}}\right)^{\frac{1}{15}} \times \sqrt[6]{\dfrac{x^{-1} y^{-4}}{x^{-5} y^4}}$. [Solved]

12. $\left(\dfrac{a^{-1} b^{\frac{1}{2}}}{a^{\frac{3}{2}} b^{-2}}\right)^{-6} \div \left(\dfrac{a^2 b^{-2}}{a^{-6} b^4}\right)^{\frac{5}{2}}$

13. $\left(\dfrac{a^{-\frac{1}{3}} x^{\frac{1}{4}}}{x^{-\frac{1}{2}} \sqrt{a}}\right)^4 \div \sqrt[6]{\dfrac{a^{-2}}{x^{-6}}}$.

14. $\left(\dfrac{b^{-6}}{a^{\frac{4}{7}} c^{-2}}\right)^{-\frac{3}{4}} \times \left(\dfrac{b^{\frac{2}{3}} a^{-\frac{1}{7}}}{c^{-\frac{3}{4}}}\right)^2$.

Simplify :—

15. $\left(\dfrac{x^a}{x^b}\right)^{a+b} \times \left(\dfrac{x^b}{x^c}\right)^{b+c} \times \left(\dfrac{x^c}{x^a}\right)^{c+a}$. *[Solved]*

16. $\left(\dfrac{x^a}{x^b}\right)^c \times \left(\dfrac{x^b}{x^c}\right)^a \times \left(\dfrac{x^c}{x^a}\right)^b$.

17. $\left(\dfrac{a^x}{a^y}\right)^{x^2+xy+y^2} \times \left(\dfrac{a^y}{a^z}\right)^{y^2+yz+z^2}$

$$\times \left(\dfrac{a^z}{a^x}\right)^{z^2+zx+x^2}. \quad [Hint]$$

18. $\sqrt[lm]{\dfrac{a^l}{a^m}} \times \sqrt[mn]{\dfrac{a^m}{a^n}} \times \sqrt[nl]{\dfrac{a^n}{a^l}}$.

19. $\left(x^{\frac{1}{a-b}}\right)^{\frac{1}{a-c}} \times$ two similar factors.

20. $\left\{\left(\dfrac{x^a}{x^b}\right)^a \times \left(\dfrac{x^b}{x^c}\right)^b \times \left(\dfrac{x^c}{x^a}\right)^c\right\}^{a+b+c}$

21. $\dfrac{2^{m+1} \cdot 3^{2m-n} \cdot 5^{m+n} \cdot 6^n}{6^m \cdot 10^{n+2} \cdot 15^m}$. *[Solved]*

22. $\dfrac{10^{m-n} \cdot 12^{m+1} \cdot 5^n}{25 \cdot 15^{m-1} \cdot 2^{3m-n+2}}$.

23. $\dfrac{2^{n+1} \cdot 6^{m+1} \cdot 10^{m-n} \cdot 15}{2^{2m+1} \cdot 3^{2m+n} \cdot 25^{m-1} \cdot 15^{-m-n+3}}$.

24. $\dfrac{5^3 \cdot 2^{\frac{1}{4}} \cdot 10^{-\frac{1}{4}}}{15^{\frac{3}{4}} \cdot 6^{-\frac{3}{4}} \cdot 4^{\frac{3}{8}}}$,   **25.** $\dfrac{(256)^{-\frac{1}{3}} \times (144)^{\frac{4}{3}}}{3 \times 2^{\frac{2}{3}} \times 243^{\frac{1}{3}}}$. *[Hint]*

26. $\dfrac{243^{\frac{1}{4}} \cdot 96^{\frac{1}{5}} \cdot 75^{\frac{1}{2}}}{3^{-\frac{1}{20}} \cdot 9}$.

27. $\dfrac{6^{n+2}-32 \cdot 6^n}{6^{n+1}-2 \cdot 6^n}$. *[Solved]*   **28.** $\dfrac{5^{x+5}-25 \times 5^{x+3}}{40 \times 5^{x+2}}$.

29. $\dfrac{3 \cdot 2^{n+1}-4 \cdot 2^{n-1}}{2^n-2^{n-1}}$.   **30.** $\dfrac{3^{n+3}-6 \cdot 3^n}{3^{n+2}+2 \cdot 3^{n+3}}$.

SOLUTIONS & HINTS—EXERCISE 83 (c)

1. $(\sqrt[9]{a^{-4}\,b^3})^{-3} = \left\{(a^{-4}\,b^3)^{\frac{1}{9}}\right\}^{-3} = (a^{-4}\,b^3)^{-\frac{1}{3}}$

$$[\because\quad -3\times\tfrac{1}{9}=-\tfrac{1}{3}]$$

$$=(a^{-4})^{-\frac{1}{3}}\times(b^3)^{-\frac{1}{3}}=a^{\frac{4}{3}}\,b^{-1}=\frac{a^{\frac{4}{3}}}{b}.$$

7. $\sqrt[4]{x^{-2}\,y}\times\sqrt[6]{x\,y^{-3}}=(x^{-2}\,y)^{\frac{1}{4}}\times(x\,y^{-3})^{\frac{1}{6}}$

$$=(x^{-2})^{\frac{1}{4}}\times y^{\frac{1}{4}}\times x^{\frac{1}{6}}\times(y^{-3})^{\frac{1}{6}}=x^{-\frac{1}{2}}\times y^{\frac{1}{4}}\times x^{\frac{1}{6}}\times y^{-\frac{1}{2}}$$

$$=x^{-\frac{1}{2}+\frac{1}{6}}\times y^{\frac{1}{4}-\frac{1}{2}}=x^{-\frac{1}{3}}\cdot y^{-\frac{1}{4}}=\frac{1}{x^{\frac{1}{3}}\cdot y^{\frac{1}{4}}}.$$

11. Given Exp. $=\left(\dfrac{x^{-2}\,y^3}{x^8\,y^{-7}}\right)^{\frac{1}{15}}\times\left(\dfrac{x^{-1}\,y^{-4}}{x^{-5}\,y^4}\right)^{\frac{1}{6}}$

$$=(x^{-2-8}\cdot y^{3+7})^{\frac{1}{15}}\times(x^{-1+5}\cdot y^{-4-4})^{\frac{1}{6}}$$

$$=(x^{-10}\,y^{10})^{\frac{1}{15}}\times(x^4\cdot y^{-8})^{\frac{1}{6}}$$

$$=x^{-\frac{2}{3}}\cdot y^{\frac{2}{3}}\cdot x^{\frac{2}{3}}\cdot y^{-\frac{4}{3}}$$

$$=x^{-\frac{2}{3}+\frac{2}{3}}\cdot y^{\frac{2}{3}-\frac{4}{3}}$$

$$=x^0\,y^{-\frac{2}{3}}=1\times y^{-\frac{2}{3}}=\frac{1}{y^{\frac{2}{3}}}.$$

15 Given Exp. $=\left(\dfrac{x^a}{x^b}\right)^{a+b}\times\left(\dfrac{x^b}{x^c}\right)^{b+c}\times\left(\dfrac{x^c}{x^a}\right)^{c+a}$

$$=(x^{a-b})^{a+b}\times(x^{b-c})^{b+c}\times(x^{c-a})^{c+a}$$

$$=x^{(a-b)\times(a+b)}\times x^{(b-c)(b+c)}\times x^{(c-a)(c+a)}.$$

$$=x^{a^2-b^2}\times x^{b^2-c^2}\times x^{c^2-a^2}.$$

$$=x^{a^2-b^2+b^2-c^2+c^2-a^2}=x^0=1.$$

17. First factor $=(a^{x-y})^{x^2+xy+y^2}=a^{(x-y)(x^2+xy+y^2)}$

$$=a^{x^3-y^3}\quad\text{etc.}$$

21. Given Exp. $= \dfrac{2^{m+1} \cdot 32^{2m-n} \cdot 5^{m+n} \cdot 6^n}{6^m \quad 10^{n+2} \quad 15^m}$

$= \dfrac{2^{m+1} \quad 3^{2m-n} \cdot 5^{n+n} \cdot (2\times3)^n}{(2\times3)^m \cdot (2\times5)^{n+2} \cdot (3\times5)^m}$. [Factorising all bases]

$= \dfrac{2^{m+1} \cdot 3^{2m-n} \cdot 5^{m+n} \cdot 2^n \cdot 3^n}{2^m \quad 3^m \cdot 2^{n+2} \quad 5^{n+2} \cdot 3^m \cdot 5^m}$ [opening brackets]

$= 2^{m+1+n-m-n-2} \quad 3^{2m-n+n-m-m} \cdot 5^{m+n-n-2-m}$

[Combining factors having the same base, by adding the indices in the numerator and subtracting those in the denominator]

$= 2^{-1} \cdot 3^0 \cdot 5^{-2} = \tfrac{1}{2} \times 1 \times \dfrac{1}{5^2} = \tfrac{1}{2} \times \tfrac{1}{25} = \tfrac{1}{50}$.

25. $(256)^{-\frac{1}{3}}$ can be conveniently written as $(2^8)^{-\frac{1}{3}} = 2^{-\frac{8}{3}}$.

Similarly $(144)^{\frac{4}{3}} = (16\times9)^{\frac{4}{3}} = (2^4\times3^2)^{\frac{4}{3}} = 2^{\frac{16}{3}} \times 3^{\frac{8}{3}}$.

27. Given Exp. $= \dfrac{6^{n+2} - 32 \cdot 6^n}{6^{n+1} - 2 \cdot 6^n} = \dfrac{6^n(6^2 - 32)}{6^n(6^1 - 2)}$

[See if you get back the original exp. by opening brackets]

$= \dfrac{6^2 - 32}{6 - 2}$ [Cancelling 6^n]

$= \dfrac{36 - 32}{4} = \tfrac{4}{4} = 1$.

EXERCISE 83 (d)

Multiply :—

1. $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$. [*Solved*]

2. $x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$ by $x^{\frac{1}{4}} + y^{\frac{1}{4}}$.

3. $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + a^{\frac{2}{3}}$ by $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.

4. $x + y + z - x^{\frac{1}{2}}y^{\frac{1}{2}} - y^{\frac{1}{2}}z^{\frac{1}{2}} - z^{\frac{1}{2}}x^{\frac{1}{2}}$ by $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}}$.

5. $3a^{-\frac{1}{6}}+a^{\frac{1}{2}}+2a^{\frac{1}{3}}$ by $a^{\frac{1}{6}}-2$. [Solved]

6. $a^{-\frac{2}{3}}+a^{-\frac{1}{3}}+1$ by $a^{-\frac{1}{3}}-1$.

7. $a^{\frac{4}{3}}-2+a^{-\frac{4}{3}}$ by $a^{\frac{2}{3}}-a^{-\frac{2}{3}}$.

8. $3x^{\frac{2}{3}}-4x^{\frac{1}{3}}-x^{\frac{1}{3}}$ by $-6x^{-\frac{3}{3}}+3x^{\frac{1}{3}}+x^{-\frac{1}{3}}$.

Divide :—

9. $x^{\frac{1}{2}}+3x^{\frac{1}{6}}y^{-\frac{1}{3}}-3x^{\frac{1}{3}}y^{-\frac{1}{6}}-y^{-\frac{1}{2}}$ by $x^{\frac{1}{3}}+y^{-\frac{1}{4}}-2x^{\frac{1}{6}}y^{-\frac{1}{6}}$.

10. $x^{\frac{2}{3}}+x^{\frac{1}{3}}+21x+1$ by $3x^{\frac{1}{3}}+1$.

11. $x^{\frac{2}{3}}-y^{\frac{2}{3}}$ by $x^{\frac{1}{6}}-y^{\frac{1}{6}}$.

12. $64a^{-1}+27b^{-2}$ by $4a^{-\frac{1}{3}}+3b^{-\frac{2}{3}}$.

13. $1-a^{-\frac{1}{2}}+2a^{\frac{3}{4}}+a^{\frac{3}{2}}$ by $1+a^{-\frac{1}{4}}+a^{\frac{3}{4}}$.

14. $x^2y^{-2}+y^2x^{-2}+1$ by $xy^{-1}+yx^{-1}-1$.

15. $a-b+c+3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$ by $a^{\frac{1}{3}}-b^{\frac{1}{3}}+c^{\frac{1}{3}}$.

16. $x^{12}-x^{-12}+6(x^8-x^{-8})+9(x^4-x^{-4})$
$$\text{by } x^7-x^{-6}+3(x^2-x^{-2}). \quad [Hint]$$

Find the square root of :—

17. $x^{\frac{4}{3}}-11x^{\frac{2}{3}}+x^{-2}-14x^{-1}+2x^{-\frac{1}{3}}+49$. [Solved]

18. $x^{\frac{5}{3}}+4x^{\frac{4}{3}}+2x^{\frac{7}{6}}+4x-4x^{\frac{5}{6}}+x^{\frac{2}{3}}$.

19. $a^{\frac{6}{5}}+a^{-\frac{2}{5}}+1-2a^{\frac{2}{5}}-2a^{\frac{3}{5}}+2a^{-\frac{1}{5}}$.

20. $(a+a^{-1})^2-4(a-a^{-1})$. [Hint]

21. $a+a^{-1}-2(a^{\frac{1}{2}}+a^{-\frac{1}{2}})+3$.

SOLUTIONS & HINTS—EXERCISE 83(d)

1. The expressions are already in descending powers of x. Hence we have :—

$$x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$$

$$x^{\frac{1}{3}}-y^{\frac{1}{3}}$$

$$\overline{x \;+x^{\frac{2}{3}}y^{\frac{1}{3}}+x^{\frac{1}{3}}y^{\frac{2}{3}}}$$

$$\frac{-x^{\frac{2}{3}}y^{\frac{1}{3}}-x^{\frac{1}{3}}y^{\frac{2}{3}}-y}{x \qquad\qquad\qquad -y}$$

[Note that $x^{\frac{2}{3}}\times x^{\frac{1}{3}}=x^{\frac{2}{3}+\frac{1}{3}}=x^{1}=x.$ etc. etc.]

<center>*Or Thus :—*</center>

Put $x^{\frac{1}{3}}=a$ and $y^{\frac{1}{3}}=b.$

$\therefore$ I Exp.$=\left(x^{\frac{1}{3}}\right)^{2}+x^{\frac{1}{3}}y^{\frac{1}{3}}+\left(y^{\frac{1}{3}}\right)^{2}=a^{2}+ab+b^{2}$

and II Exp.$=a-b.$

$\therefore$ Reqd. Product$=(a^{2}+ab+b^{2})(a-b)=a^{3}-b^{3}$

$$=\left(x^{\frac{1}{3}}\right)^{3}-\left(y^{\frac{1}{3}}\right)^{3} \qquad \text{[Putting back the}$$
<div align="right">values of a and b]</div>

$$=x-y.$$

5. [Note. When negative indices are involved, the beginner is apt to find some difficulty in arranging an expression according to powers of a letter. For this it should be remembered that -1 is less than 0, -2 is less than -1, -3 is less than -2, and so on.]

Arranging according to descending powers of a we have the following process :—

$$a^{\frac{1}{2}}+2a^{\frac{1}{3}}+3a^{-\frac{1}{6}}$$

$$a^{\frac{1}{6}}-2$$

$$\overline{a^{\frac{2}{3}}+2a^{\frac{1}{2}}+3}$$

$$\frac{-2a^{\frac{1}{2}}-4a^{\frac{1}{3}}-6a^{-\frac{1}{6}}}{a^{\frac{2}{3}}\qquad\;\; -4a^{\frac{1}{3}}+3-6a^{-\frac{1}{6}}}$$

9. Arranging according to descending powers of x we have :—

$$x^{\frac{1}{3}}-2x^{\frac{1}{6}}y^{-\frac{1}{6}}+y^{-\frac{1}{3}}\ )\ x^{\frac{1}{2}}-3x^{\frac{1}{3}}y^{-\frac{1}{6}}+3x^{\frac{1}{6}}y^{-\frac{1}{3}}-y^{-\frac{1}{2}}\ (\ x^{\frac{1}{6}}-y^{-\frac{1}{6}}$$

$$\underline{x^{\frac{1}{2}}-2x^{\frac{1}{3}}y^{-\frac{1}{6}}+\ x^{\frac{1}{6}}y^{-\frac{1}{3}}}$$

$$-x^{\frac{1}{3}}y^{-\frac{1}{6}}+2x^{\frac{1}{6}}y^{-\frac{1}{3}}-y^{-\frac{1}{2}}$$

$$\underline{-x^{\frac{1}{3}}y^{-\frac{1}{6}}+2x^{\frac{1}{6}}y^{-\frac{1}{3}}-y^{-\frac{1}{2}}}$$

$$\times$$

16. Dividend $=x^{12}+6x^{8}+9x^{4}-9x^{-4}-6x^{-8}-x^{-12}$

Divisor $=x^{6}+3x^{2}-3x^{-2}-x^{-6}$

17.

$$x^{\frac{2}{3}}\ \Big)\ x^{\frac{4}{3}}-14x^{\frac{2}{3}}+49+2x^{-\frac{1}{3}}-14x^{-1}+x^{-2}\ \Big(\ x^{\frac{2}{3}}-7+x^{-1}$$

$$\underline{x^{\frac{4}{3}}}$$

$$2x^{\frac{2}{3}}-7\ \Big|\qquad -14x^{\frac{2}{3}}+49$$

$$\underline{-14x^{\frac{2}{3}}+49}$$

$$2x^{\frac{2}{3}}-14+x^{-1}\ \Big|\qquad\qquad 2x^{-\frac{1}{3}}-14x^{-1}+x^{-2}$$

$$\underline{2x^{-\frac{1}{3}}-14x^{-1}+x^{-2}}$$

$$\times$$

20. Either open the brackets and proceed as in Q. 17, or write $(a+a^{-1})^2$ as $(a-a^{-1})^2-4a\times a^{-1}$, that is as $(a-a^{-1})^2-4$.

EXERCISE 83(c)

Find the square of :—

1. $\frac{1}{3}x^{\frac{2}{3}}+x^{-\frac{1}{3}}$. [Solved] **2.** $2x^{\frac{1}{2}}+\frac{1}{2}x^{-\frac{1}{2}}$

3. $x^{m}+x^{n}$. **4.** $x^{-\frac{n}{2}}-nx^{-\frac{n}{2}}$

5. $3a^{\frac{1}{2}}-a^{\frac{1}{2}}-3$ [Solved] **6.** $1-a^{\frac{1}{3}}-a^{-\frac{1}{3}}$

7. $z^{\frac{2}{3}} - z^{\frac{1}{2}} y^{-\frac{1}{4}} + y^{-\frac{1}{2}}$

8. $x^{a} - \frac{1}{3} - x^{-a}$.

_____

Find the cube of :—

9. $x^{\frac{1}{3}} + 3x^{-\frac{1}{3}}$. [*Solved*]

10. $x^{\frac{2}{3}} - x^{-\frac{2}{3}}$.

11. $\cdot a^{n} - a^{-m}$.

12. $a^{-\frac{n}{3}} + \frac{1}{3}$.

_____

Write down the value of :— '

13. $(x^{\frac{1}{4}} - y^{\frac{1}{4}})(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{4}} + y^{\frac{1}{4}})$. [*Solved*]

14. $(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{2}{3}} + y^{\frac{2}{3}})(x^{\frac{1}{3}} - y^{\frac{1}{3}})$.

15. $(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{2}{3}} - x^{\frac{1}{3}} y^{\frac{1}{3}} + y^{\frac{2}{3}})$.

16. $(x^{\frac{1}{3}} - y^{-\frac{1}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}} y^{-\frac{1}{3}} + y^{-\frac{2}{3}})$.

17. $(a^{\frac{1}{3}} - a^{-\frac{1}{3}} - 1)(a^{\frac{2}{3}} + a^{-\frac{2}{3}} + 2 + a^{\frac{1}{3}} - a^{-\frac{1}{3}})$. [*Hint*]

_____

Write down the quotient of :—

18. $a - 9b$ by $a^{\frac{1}{2}} + 8b^{\frac{1}{2}}$. [*Solved*]

19. $x^{2m} - 16$ by $x^{m} - 4$.

20. $x^{3n} + 8$ by $x^{n} + 2$.

21. $1 - 8a^{-3}$ by $1 - 2a^{-1}$.

22. $x - y + z + 3x^{\frac{1}{3}} y^{\frac{1}{3}} z^{\frac{1}{3}}$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}} + z^{\frac{1}{3}}$.

_____

23. If $x = 2^{\frac{1}{3}} + 2^{-\frac{1}{3}}$, prove that $2x^3 - 6x = 5$. [*Solved*]

24. If $x = 3^{\frac{1}{3}} - 3^{-\frac{1}{3}}$, show that $3x^3 + 9x = 8$.

25. If $a = \sqrt[3]{2} - \dfrac{1}{\sqrt[3]{2}}$, show that $2a^3 + 6a - 3 = 0$.

_____

Simplify and express your result with positive indices :—

26. $\dfrac{x + 5 + 6x^{-1}}{1 + 6x^{-1} + 8x^{-2}}$. [*Hint*]

27. $\dfrac{x^{-1} - 2x^{-2} - 3x^{-3}}{9x^{-1} - x}$.

28. $\dfrac{y^{-1}}{x^{-1}+y^{-1}}+\dfrac{y^{-1}}{x^{-1}-y^{-1}}.$ 29. $\left(\dfrac{e^x+e^{-x}}{2}\right)^2-\left(\dfrac{e^x-e^{-x}}{2}\right)^2$

30. $\left(\dfrac{y^{-1}+x^{-1}}{y^{-1}-x^{-1}}\right)^2-\left(\dfrac{y^{-1}-x^{-1}}{y^{-1}-x^{-1}}\right)^2.$

SOLUTIONS & HINTS—EXERCISE 83(c)

1. $\left(\tfrac{1}{2}x^{\frac{2}{3}}-x^{-\frac{1}{3}}\right)^2=\left(\tfrac{1}{2}x^{\frac{2}{3}}\right)^2-2\times\tfrac{1}{2}x^{\frac{2}{3}}\times x^{-\frac{1}{3}}+\left(x^{-\frac{1}{3}}\right)^2$

$$=\tfrac{1}{4}x^{\frac{4}{3}}-x^{\frac{2}{3}-\frac{1}{3}}+x^{-\frac{2}{3}}$$

$$=\tfrac{1}{4}x^{\frac{4}{3}}-x^{\frac{1}{3}}+x^{-\frac{2}{3}}.$$

5. $\left(3a^{\frac{1}{2}}-a^{-\frac{1}{2}}-2\right)^2$

$=\left(3a^{\frac{1}{2}}\right)^2+\left(-a^{-\frac{1}{2}}\right)^2+(-2)^2+2(3a^{\frac{1}{2}})(-a^{-\frac{1}{2}})$

$$+2(3a^{\frac{1}{2}})(-2)+2(-a^{-\frac{1}{2}})(-2)$$

$=9a+a^{-1}+4-6a^0-12a^{\frac{1}{2}}+4a^{-\frac{1}{2}}$

$=9a+a^{-1}+4-6-12a^{\frac{1}{2}}+4a^{-\frac{1}{2}}$ $[\because \quad a^0=1]$

$=9a-12a^{\frac{1}{2}}-2+4a^{-\frac{1}{2}}+a^{-1}.$ [Arranging according to
powers of a]

9. $\left(x^{\frac{1}{3}}+3x^{-\frac{1}{3}}\right)^3=\left(x^{\frac{1}{3}}\right)^3+3\left(x^{\frac{1}{3}}\right)^2\left(3x^{-\frac{1}{3}}\right)+3\left(x^{\frac{1}{3}}\right)\left(3x^{-\frac{1}{3}}\right)^2$

$$+\left(3x^{-\frac{1}{3}}\right)^3$$

$$=x+3\times x^{\frac{2}{3}}\times 3x^{-\frac{1}{3}}+3\times x^{\frac{1}{3}}\times 9x^{-\frac{2}{3}}+27x^{-1}$$

$$=x+9x^{\frac{1}{3}}+27x^{-\frac{1}{3}}+27x^{-1}.$$

13. Given Exp. $=\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)\left(x^{\frac{1}{4}}-y^{\frac{1}{4}}\right)\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)$

[writing last factor first]

$$=\left\{\left(x^{\frac{1}{4}}\right)^2-\left(y^{\frac{1}{4}}\right)^2\right\}\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)$$

[multiplying the first two factors by the formula
$(a+b)(a-b)=a^2-b^2$]

$$= (x^{\frac{1}{2}} - y^{\frac{1}{2}}) (x^{\frac{1}{2}} + y^{\frac{1}{2}})$$
$$= (x^{\frac{1}{2}})^2 - (y^{\frac{1}{2}})^2$$

[Applying the same formula again]

$$= x - y.$$

17. Apply the formula $(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$
$$= x^3+y^3+z^3-3xyz.$$

18. Dividend $= a - 9b = (a^{\frac{1}{2}})^2 - (3b^{\frac{1}{2}})^2 = (a^{\frac{1}{2}} + 3b^{\frac{1}{2}}) (a^{\frac{1}{2}} - 3b^{\frac{1}{2}})$

Divisor $= a^{\frac{1}{2}} + 3b^{\frac{1}{2}}$.

$\therefore$ Reqd. Quotient $= a^{\frac{1}{2}} - 3b^{\frac{1}{2}}$.

23. $x = 2^{\frac{1}{3}} + 2^{-\frac{1}{3}}$ [given]

$\therefore$ $x^3 = (2^{\frac{1}{3}} + 2^{-\frac{1}{3}})^3$

$$= (2^{\frac{1}{3}})^3 + (2^{-\frac{1}{3}})^3 + 3 \times 2^{\frac{1}{3}} \times 2^{-\frac{1}{3}} (2^{\frac{1}{3}} + 2^{-\frac{1}{3}})$$
$$[\because (a+b)^3 = a^3 + b^3 + 3ab(a+b)]$$

$$= 2 + 2^{-1} + 3(x) [\because 2^{\frac{1}{3}} \times 2^{-\frac{1}{3}} = 2^0 = 1$$
$$\text{and } 2^{\frac{1}{3}} + 2^{-\frac{1}{3}} = x \text{ (given)}]$$

i.e., $x^3 = 2 + \frac{1}{2} + 3x = \frac{5}{2} + 3x$

or $2x^3 = 5 + 6x$

or $2x^3 - 6x = 5$.

26. Given Exp. $= \dfrac{x+5+\dfrac{6}{x}}{1+\dfrac{6}{x}+\dfrac{8}{x^2}}$ etc.

_____

EXERCISE 83(f)

Solve the following equations :—

1. $4^{x+2} = 8^{x-3}$ [*Solved*]

2. $3^{x+5} = 9^{x+1}$

3. $8^{2(x-1)} = 4^{x+5}$

4. $2^{x+6} = 512$.

5. $8^{x-3}=1$. [*Hint*]   6. $(\sqrt{5})^x=125$.

7. $(\sqrt[3]{3})^{x+1}=(\sqrt[5]{81})^{x-1}$.   8. $(\sqrt{7})^{3x-2}=(\sqrt[3]{49})^{2x-3}$.

9. $9^x=\dfrac{9}{3^x}$. [*Solved*]   10. $2^{x+5}=\dfrac{4}{2^{-2x}}$.

11. $25^x=\dfrac{5^{2x+3}}{25^{x-1}}$.   . 12. $9.3^{2x-3}=\dfrac{27}{9^x}$.

13. $2^{x+3}=2^{x+5}-6$. [*Solved*]   14. $3^{x+5}=3^{x+3}+2^3$.

15. $4^{x-1}=2^{2x+1}-7$.   16. $125^x=5^{3x-1}+\frac{4}{5}$.

17. $2^{x-2}.3^{x-1}=18$. [*Solved*]   18. $3^{x+5}.5^{x+6}=1125$.

19. $2^{x+3}.3^{x+2}=432$.   20. $7^{x-1}.3^{x-2}=7$.

21. $\left.\begin{array}{l}2^{x+y}=32 \\ 3^{x-y}=3.\end{array}\right\}$[*Hint*]   22. $\begin{array}{l}3^x.9^y=81 \\ 3^x\div9^y=9.\end{array}$

23. $\begin{array}{l}4^x\times8^y=128 \\ 9^x\div27^y=3.\end{array}$ 24. $\begin{array}{l}3^{x+y-2z}=27^{x-y} \\ 4^{3y}=16^{z+x-1} \\ 2^{-4x+2z-y}=1.\end{array}$

25. $\left.\begin{array}{l}2^x+3^y=4 \\ 2^{x+1}+3^{y+1}=11.\end{array}\right\}$[*Solve*]   26. $\begin{array}{l}3^x+4^y=19 \\ 3^{x-1}+4^{y-1}=5.\end{array}$

27. $\left.\begin{array}{l}2^x+5^y=9 \\ 4^x-25^y=-9.\end{array}\right\}$[*Hint*]   28. $\begin{array}{l}7^x-5^y=6 \\ 49^x-25^y=48.\end{array}$

SOLUTIONS & HINTS—EXERCISE 83 (*f*)

1. $4^{x+2}=8^{x-3}$

or $(2^2)^{x+2}=(2^3)^{x-3}$ [reducing both sides to the same base, 2]

or $2^{2x+4}=2^{3x-9}$

$2x+4=3x-9$ [The bases being equal, the indices **must be equal**]

$x=13$.

5. 1 may be written as 8^0.

9. $9^x = \dfrac{9}{3^x}$

or $(3^2)^x = \dfrac{3^2}{8^x}$ · [reducing each power to the same base, 8]

or $3^{2x} = 3^{2-x}$

∴ $2x = 2 - x$ [equating indices]

∴ $x = \frac{2}{3}$.

13. $2^{x+3} = 2^{x+5} - 6$

or $2^{x+5} - 2^{x+3} = 6$ [By transposition]

or $2^{x+3}(2^2 - 1) = 6$

or $2^{x+3} \times 3 = 6$ [∵ $2^2 - 1 = 4 - 1 = 3$]

or $2^{x+3} = 2$

∴ $x + 3 = 1$ [Equating indices]

or $x = -2$.

17. $2^{x-2} \times 3^{x-1} = 18$

or $\dfrac{2^{x-2} \times 3^{x-1}}{18} = 1$ [Dividing both sides by 18]

or $\dfrac{2^{x-2} \times 3^{x-1}}{2^1 \times 3^2} = 1$

or $2^{x-3} \times 3^{x-3} = 1$

or $(2 \times 3)^{x-3} = 1$

or $6^{x-3} = 1 = 6^0$

∴ $x - 3 = 0$ [equating indices]

or $x = 3$

21. $2^{x+y} = 32$ ...(i)

$\quad\;\; 8^{x-y} = 3$ ...(ii)

From (i) $2^{x+y} = 2^5$ ∴ $x + y = 5$ ..(iii)

From (ii) $3^{x-y} = 3^1$ ∴ $x - y = 1$ ...(iv)

Solve equations (iii) and (iv) simultaneously.

25. $2^x + 3^y = 4$ ...(i)

    $2^{x+1} + 3^{y+1} = 11$ ...(ii)

From (ii) $2^x \times 2 = 3^y \times 3 = 11$ ...(iii)

Put $2^x = a$ and $3^y = b$, then :—

  (i) becomes $a + b = 4$ ...(iv)

(iii) becomes $2a + 3b = 11$ ...(v)

Solving (iv) and (v) simultaneously, we get :—

        $a = 1, \quad b = 3$

$\therefore$ $2^x = 1$ and $3^y = 3$.

Of these, the first equation gives $2^x = 2^0$, so that $x = 0$ and the second equation gives $3^y = 3^1$, so that $y = 1$.

27. The second equation may be written as $(2^2)^x - (5^2)^y = -9$ or as $(2^x)^2 - (5^y)^2 = -9$.

Now, if we put $2^x = a$ and $5^y = b$, the given equations become :—

        $a + b = 9$ ...(i)

and $a^2 - b^2 = -9$ ...(ii)

Dividing (ii) by (i) we get $a - b = -1$ ...(iii)

Solve (i) and (iii) simultaneously ; etc.

CHAPTER XXI

SURDS

131. When a root of a number cannot be exactly obtained it is called an **irrational quantity** or a **surd**, e.g., $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt[4]{7}$, etc.

When a root of an algebraical expression cannot be denoted without the use of a fractional index, it is also called an irrational quantity or a surd., e.g.. $\sqrt{a}$, $\sqrt{a^2 + b^2}$, $\sqrt[3]{a^2}$, etc.

Note that $\sqrt{4}$, $\sqrt[3]{27}$ and $\sqrt[4]{625}$ are surds merely in form ; they are not really surds, for $\sqrt{4}=2$, $\sqrt[3]{27}=3$, and $\sqrt[4]{625}=5$.

Quantities which are not surds are called **rational quantities.**

132. The **order** of a surd is indicated by the index of its radical sign.   Thus $\sqrt[3]{5}$ is a surd of the third order ; $\sqrt[n]{a}$ is a surd of the nth order.

Surds of the second order are also called **quadratic surds.** Thus $\sqrt{2}$ and $\sqrt{x}$ are quadratic surds.

133. A rational quantity may be expressed in the form of a surd of *any required order.*   For example :—

$$2=\sqrt{4}=\sqrt[3]{8}=\sqrt[4]{16}=\ldots\ldots\ldots\ldots$$

$$a=\sqrt{a^2}=\sqrt[3]{a^3}=\sqrt[4]{a^4}=\ldots\ldots\ldots\ldots$$

134. **Simplification of Surds.**

A surd can sometimes be expressed as the product of a rational quantity and a surd.   For example :—

$$\sqrt{18}=\sqrt{9\times 2}=\sqrt{9}\times\sqrt{2}=3\sqrt{2}$$

$$\sqrt[3]{24}=\sqrt[3]{8\times 3}=\sqrt[3]{8}\times\sqrt[3]{3}=2\sqrt[3]{3}.$$

A surd so reduced is said to be in its *simplest form,* if the integer under the radical sign is the least possible.

Conversely, a rational coefficient of a surd may be put under the radical sign after due modification.   For example :

$$5\sqrt{2}=\sqrt{5^2}\times\sqrt{2}=\sqrt{5^2\times 2}=\sqrt{50}.$$

135. Surds are said to be **Similar** or **like** when they can be expressed with the same irrational factor.

Thus $3\sqrt{2}$, $7\sqrt{2}$ and $-5\sqrt{2}$ are similar surds, because the irrational factor in each case is $\sqrt{2}$.

$\sqrt{8}$ and $\sqrt{50}$ are also similar surds, for they are equal to $2\sqrt{2}$ and $5\sqrt{2}$ respectively.

136. Addition and Subtraction of Surds.

These operations are possible only when the surds are similar. For a beginner it is better to put the common irrational factor equal to a letter. In this way the method becomes self-evident. For example :—

$$13\sqrt{2}+7\sqrt{2}-5\sqrt{2}=13a+7a-5a \quad [\text{where } a=\sqrt{2}]$$
$$=15a=15\sqrt{2}.$$

But after a little practice the result should be obtained directly, without making any substitution.

137. Multiplication and division of Surds.

If the bases are equal, apply the laws $a^m \times a^n = a^{m+n}$ and $a^m \div a^n = a^{m-n}$ but if the bases are unequal and cannot be made equal reduce the given surds to surds of the same order and apply the laws

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab} \quad \text{and} \quad \sqrt[n]{a} \div \sqrt[n]{b} = \sqrt[n]{\frac{a}{b}}. \quad \text{Solved examples}$$

of the next exercise will make these methods quite clear.

EXERCISE 84

Express in the simplest form :—

1. $\sqrt{75}$. [Solved]
2. $\sqrt{80}$.
3. $\frac{5}{3}\sqrt{54}$.
4. $\frac{1}{4}\sqrt{128}$.
5. $\sqrt[3]{54}$. [Solved]
6. $\sqrt[3]{375}$.
7. $\sqrt[4]{3125}$.
8. $\sqrt[3]{-2187}$.
9. $\sqrt{36a^3b}$. [Solved]
10. $\sqrt{18a^5b^2}$
11. $\sqrt[3]{-108a^4b^3}$.
12. $\sqrt{x^3+2x^2y+xy^2}$.

Express as entire surds :—

13. $6\sqrt{2}$. [Solved]
14. $5\sqrt{7}$.
15. $7\sqrt{3}$.
16. $a^2b\sqrt{ab}$.
17. $\frac{3}{4}\sqrt{32}$.
18. $\frac{m}{n}\sqrt{x}$.

19. $3\sqrt[3]{5}$. [Solved]   20, $\frac{4}{3}\sqrt[3]{3}$.

21. $2\sqrt[5]{9}$.   22, $\frac{a^2}{b}\sqrt[3]{a^4b}$.

Simplify :—

23. $\sqrt{32}+\sqrt{50}$. [Solved] 24. $\sqrt{40}+\sqrt{90}$.
25. $\sqrt{147}-\sqrt{108}-\sqrt{3}$.   26. $\frac{1}{4}\sqrt{75}-\frac{1}{4}\sqrt{48}$.

27. $\sqrt{128}+\sqrt{72}-\sqrt{32}-\sqrt{200}$.

28. $\frac{1}{2}\sqrt{486}-\sqrt{2\frac{7}{2}}+\sqrt{54}$.
29. $\sqrt[3]{320}+\sqrt[3]{625}-\sqrt[3]{1080}$.   30. $\sqrt[3]{128}+\sqrt[3]{432}-\sqrt[3]{1024}$.
31. $\sqrt{50x^5}+\sqrt{72x^5}-\sqrt{128x^5}$. 32. $3\sqrt[3]{8x^4}-2x\sqrt[3]{125x}+\sqrt[3]{27x^4}$.

33. Express 2 as a surd of the fourth order. [Solved]
34. Express 5 as a surd of the third order.
35. Express 4 as a surd of the fifth order.
36. Express a as a surd of the nth order.

37. Express $\sqrt[4]{5}$ as a surd of the 12th order. [Solved]

38. Express $\sqrt[4]{8}$ as a surd of the 20th order.
39. Express $\sqrt[6]{4}$ as a surd of the 24th order.

40. Express $\sqrt[10]{5}$ as a surd of the 50th order.

Reduce to surds of the same order :—

41. $\sqrt[4]{3}$, $\sqrt[6]{5}$. [Solved]   42. $\sqrt{2}$, $\sqrt[3]{5}$.

43. $\sqrt[5]{2}$, $\sqrt[4]{3}$.   44. $\sqrt[6]{6}$, $\sqrt[4]{3}$, $\sqrt[3]{2}$.

Which is greater :—

45. $\sqrt[6]{5}$ or $\sqrt[4]{3}$? [Hint]. 46. $\sqrt{8}$ or $\sqrt[4]{10}$?
47. $\sqrt[3]{5}$ or $\sqrt[4]{6}$.

48. Arrange in ascending order of magnitude :—

$$\sqrt[4]{3}, \ \sqrt[6]{4}, \ \sqrt[8]{7}.$$

49. Arrange in descending order of magnitude :—

$$\sqrt[4]{10}, \ \sqrt[3]{6}, \ \sqrt{3}.$$

50. Arrange in order of magnitude :— $\sqrt[6]{5}, \ \sqrt[4]{3}, \ \sqrt[8]{10}.$

———

Simplify :—

51. $\sqrt{2} \times \sqrt{18}.$ [*Solved*]   **52.** $\sqrt{3} \times \sqrt{75}.$

53. $\sqrt{5} \times \sqrt{12} \times \sqrt{15}.$ **54.** $\sqrt{a^3 b} \times \sqrt{b^3 c} \times \sqrt{c^3 a}.$

———

55. $3\sqrt{2} \times 4\sqrt{7}.$ [*Solved*]   **56.** $3\sqrt{8} \times \sqrt{6}.$

57. $2\sqrt{14} \times 3\sqrt{21}.$ **58.** $3\sqrt{22} \times 5\sqrt{88}.$

———

59. $\sqrt[3]{2} \times \sqrt{5}.$ [*Solved*]   **60.** $\sqrt[4]{3} \times \sqrt[8]{6}.$

61. $\sqrt[6]{3} \times \sqrt[12]{4}.$ **62.** $\sqrt{7} \times \sqrt[3]{7} \times \sqrt[6]{7}.$ [*Solved*]

63. $\sqrt{2} \times \sqrt[4]{2} \times \sqrt[8]{2}.$ **64.** $\sqrt{ab} \cdot \sqrt[6]{ab} \cdot \sqrt[12]{ab}.$

———

65. $16\sqrt{21} \div \sqrt{28}.$ [*Solved*]

66. $4\sqrt{54} \div 2\sqrt{6}.$ **67.** $21\sqrt{384} \div 8\sqrt{98}.$

68. $\sqrt[3]{6} \div \sqrt[4]{3}.$ [*Solved*]   **69.** $\sqrt{15} \div \sqrt[3]{40}.$

70. $\sqrt[4]{5} \div \sqrt[3]{10}.$

———

SOLUTIONS & HINTS—EXERCISE 84

1. $\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5 \times \sqrt{3} = 5\sqrt{3}.$

5. $\sqrt[3]{54} = \sqrt[3]{27 \times 2} = \sqrt[3]{27} \times \sqrt[3]{2} = 3 \times \sqrt[3]{2} = 3\sqrt[3]{2}.$

9. $\sqrt{36a^3 b} = \sqrt{36 \times a^2 \times ab} = \sqrt{36} \times \sqrt{a^2} \times \sqrt{ab} = 6a\sqrt{ab}.$

13. $6\sqrt{2} = \sqrt{36} \times \sqrt{2} = \sqrt{36 \times 2} = \sqrt{72}.$

19. $8\sqrt[3]{5} = \sqrt[3]{27} \times \sqrt[3]{5} = \sqrt[3]{27 \times 5} = \sqrt[3]{135}.$

23. $\sqrt{32}+\sqrt{50}=\sqrt{16\times2}+\sqrt{25\times2}=4\sqrt{2}+5\sqrt{2}=9\sqrt{2}.$

[A beginner may put $\sqrt{2}=x$, so that the given exp. $=4x+5x=9x=9\sqrt{2}$, after re-writing the value of x]

33. $2=(2^4)^{\frac{1}{4}}=(16)^{\frac{1}{4}}=\sqrt[4]{16}.$

37. $\sqrt[4]{5}=5^{\frac{1}{4}}=5^{\frac{3}{12}}=\sqrt[12]{5^3}=\sqrt[12]{125}.$

41. $\sqrt[4]{3}=3^{\frac{1}{4}}=3^{\frac{3}{12}}=\sqrt[12]{3^3}=\sqrt[12]{27}.$

$\sqrt[6]{5}=5^{\frac{1}{6}}=5^{\frac{2}{12}}=\sqrt[12]{5^2}=\sqrt[12]{25}.$

[Note that 12 is the L. C. M. of 4 and 6; we have expressed each of the given surds as a surd of the 12th order.]

45. Reduce the given surds to surds of the same order as in question 41 ; then that surd which has the larger number under the radical sign is greater.

51. $\sqrt{2}\times\sqrt{18}=\sqrt{2\times18}=\sqrt{2\times2\times3\times3}=2\times3=6.$

55. $3\sqrt{2}\times4\sqrt{7}=3\times4\times\sqrt{2\times7}=12\times\sqrt{14}=12\sqrt{14}.$

59. $\sqrt[3]{2}\times\sqrt{5}=2^{\frac{1}{3}}\times5^{\frac{1}{2}}=2^{\frac{2}{6}}\times5^{\frac{3}{6}}$ [reducing the indices to the same denominator]

$$=\sqrt[6]{2^2}\times\sqrt[6]{5^3}$$
$$=\sqrt[6]{4}\times\sqrt[6]{125}=\sqrt[6]{4\times125}=\sqrt[6]{500}.$$

62. $\sqrt{7}\times\sqrt[3]{7}\times\sqrt[6]{7}=7^{\frac{1}{2}}\times7^{\frac{1}{3}}\times7^{\frac{1}{6}}=7^{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}}=7^1=7.$

[Note carefully that in this question we need not reduce the surds to the same order, as in Q 59, for here the bases are the same and therefore we can multiply by the fundamental law of indices].

65. Given Exp.$=\dfrac{16\times\sqrt{21}}{\sqrt{28}}=\dfrac{16\times\sqrt{7}\times\sqrt{3}}{\sqrt{7}\times\sqrt{4}}=\dfrac{16\times\sqrt{3}}{\sqrt{4}}.$

$$=\dfrac{16\times\sqrt{3}}{2}=8\sqrt{3}.$$

68. $\dfrac{\sqrt[3]{6}}{\sqrt[4]{3}}=\dfrac{6^{\frac{1}{3}}}{3^{\frac{1}{4}}}=\dfrac{6^{\frac{4}{12}}}{3^{\frac{3}{12}}}=\dfrac{\sqrt[12]{6^4}}{\sqrt[12]{3^3}}=\sqrt[12]{\dfrac{6\times6\times6\times6}{3\times3\times3}}=\sqrt[12]{48}.$

138. Compound Surds.

An expression which involves two or more simple surds connected by the sign + or − is called a *compound surd*.

Thus $\sqrt{3}$ and $\sqrt{2}$ are *simple surds*, while $\sqrt{3}+\sqrt{2}$ is a compound surd

139. Multiplication of Compound Surds.

Compound surds are multiplied like compound expressions. Solved Examples of the next exercise will make the method quite clear.

EXERCISE 85

Multiply :—

1. $\sqrt{2}+1$ by $\sqrt{2}+3$. [*Solved*]
2. $\sqrt{3}+2$ by $\sqrt{3}-1$.   3. $\sqrt{2}-3$ by $2\sqrt{2}-5$.
4. $\sqrt{3}+7$ by $\sqrt{12}-1$.
5. $4\sqrt{5}-3\sqrt{2}$ by $3\sqrt{5}+4\sqrt{2}$. [*Solved*]
6. $3\sqrt{2}-2\sqrt{3}$ by $3\sqrt{2}-\sqrt{3}$.
7. $3\sqrt{5}-4\sqrt{2}$ by $2\sqrt{5}+3\sqrt{2}$.
8. $5\sqrt{7}-4\sqrt{5}$ by $2\sqrt{7}+\sqrt{20}$.
9. $3\sqrt{2}+\sqrt{3}$ by $2\sqrt{5}+\sqrt{7}$.
10. $2\sqrt{5}+\sqrt{24}$ by $\sqrt{12}-2\sqrt{2}$.

———

Find the value of :—

11. $(3\sqrt{5}+2\sqrt{3})(3\sqrt{5}-2\sqrt{3})$. [*Solved*]
12. $(5\sqrt{2}-2\sqrt{5})(5\sqrt{2}+2\sqrt{5})$.
13. $(2\sqrt{6}+\sqrt{3})(\sqrt{3}-2\sqrt{6})$.
14. $(5\sqrt{7}-\sqrt{80})(4\sqrt{5}+5\sqrt{7})$.

———

15. $(5-3\sqrt{7})^2$. [*Solved*]   16. $(2\sqrt{3}+3\sqrt{2})^2$.
17. $(2\sqrt{6}-3\sqrt{5})^2$.   18. $(4\sqrt{3}+\sqrt{6})^2$.
19. $(\sqrt{a+x}-\sqrt{a-x})^2$.   20. $(2\sqrt{x}-\sqrt{1+4x})^2$.
21. $(3\sqrt{x^2+y^2}-2\sqrt{x^2-y^2})^2$ 22. $(\sqrt{4a^2+1}-\sqrt{4a^2-1})^2$.

———

23. $(\sqrt{2}+\sqrt{3}-\sqrt{5})(\sqrt{2}-\sqrt{3}+\sqrt{5})$. *[Solved]*

24. $(\sqrt{5}+\sqrt{7}+3\sqrt{2})(\sqrt{5}-\sqrt{7}+3\sqrt{2})$.

25. $(4\sqrt{3}-2\sqrt{2}+\sqrt{5})(4\sqrt{3}+2\sqrt{2}-\sqrt{5})$.

26. $(5\sqrt{2}-2\sqrt{5}-1)(1+5\sqrt{2}+2\sqrt{5})$.

Find the continued product of :—

27. $(1+\sqrt{3})(\sqrt{2}+\sqrt{6})(2\sqrt{2}+\sqrt{6})$. *[Solved]*

28. $(\sqrt{3}+\sqrt{5})(8-2\sqrt{15})(\sqrt{5}+\sqrt{3})$.

29. $(\sqrt{2}+\sqrt{3}+\sqrt{5})(-\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}-\sqrt{3}+\sqrt{5})\times$
$(\sqrt{2}+\sqrt{3}-\sqrt{5})$. *[Hint]*

30. $(\sqrt{3}+\sqrt{5}+\sqrt{7})(-\sqrt{3}+\sqrt{5}+\sqrt{7})(\sqrt{3}-\sqrt{5}+\sqrt{7})\times$
$(\sqrt{3}+\sqrt{5}-\sqrt{7})$

SOLUTIONS & HINTS—EXERCISE 85

1. Reqd. Product $=(\sqrt{2}+1)(\sqrt{2}+3)$

$=(\sqrt{2}+1)\times\sqrt{2}+(\sqrt{2}+1)\times3$

$=2+\sqrt{2}+3\sqrt{2}+3$

$=5+4\sqrt{2}$.

5. Reqd. Product $=(4\sqrt{5}-3\sqrt{2})(3\sqrt{5}+4\sqrt{2})$

$=(4\sqrt{5}-3\sqrt{2})\times3\sqrt{5}+(4\sqrt{5}-3\sqrt{2})\times4\sqrt{2}$

$=12\times5-9\sqrt{10}+16\sqrt{10}-12\times2$

$=64-9\sqrt{10}+16\sqrt{10}-24$

$=36+7\sqrt{10}$.

11. Given Exp. $=(3\sqrt{5}+2\sqrt{3})(3\sqrt{5}-2\sqrt{3})$

$=(3\sqrt{5})^2-(2\sqrt{3})^2$ $[\because (a+b)(a-b)=a^2-b^2]$

$=9\times5-4\times3$

$=45-12=33$.

15. $(5-3\sqrt{7})^2=(5)^2+(3\sqrt{7})^2-2\times5\times3\sqrt{7}$
$[\because (a-b)^2=a^2+b^2-2ab]$

$=25+63-30\sqrt{7}$

$=88-30\sqrt{7}$.

23. Given Exp. $=(\sqrt{2}+\sqrt{3}-\sqrt{5})(\sqrt{2}-\sqrt{3}+\sqrt{5})$

$=\{\sqrt{2}+(\sqrt{3}-\sqrt{5})\}\{\sqrt{2}-(\sqrt{3}-\sqrt{5})\}$

$=(\sqrt{2})^2-(\sqrt{3}-\sqrt{5})^2$

$=2-(3+5-2\sqrt{15})=2-(8-2\sqrt{15})$

$=2-8+2\sqrt{15}=-6+2\sqrt{15}.$

27. Given Exp. $=(1+\sqrt{3})(\sqrt{2}+\sqrt{6})(2\sqrt{2}-\sqrt{6})$

$=(1+\sqrt{3})\{2\times2+2\sqrt{12}-\sqrt{12}-6\}$

   [Multiplying the last two factors]

$=(1+\sqrt{3})(4+4\sqrt{3}-2\sqrt{3}-6)$

$=(1+\sqrt{3})(-2+2\sqrt{3})$

$=2(1+\sqrt{3})(-1+\sqrt{3})$

   [Taking out 2 from the second factor]

$=2\{(\sqrt{3})^2-(1)^2\}=2(3-1)=2\times2=4.$

30. Multiply the first two factors and the last two factors separately [See Q. 23]. Then multiply the two products.

---

140. Rationalising Factor. When the product of two irrational expressions is rational, each is said to be the *rationalising factor* of the other.

For example, a rationalising factor of $\sqrt{2}$ is $\sqrt{2}$, for $\sqrt{2}\times\sqrt{2}=2$.

A rationalising factor of $2-\sqrt{3}$ is $2+\sqrt{3}$, because

$(2-\sqrt{3})(2+\sqrt{3})=4-3=1.$

Note. There may be more than one rationalising factor of a given surd, but we should choose that which is the simplest. Thus the simplest rationalising factor of $\sqrt{8}$ is $\sqrt{2}$, although $\sqrt{8}$, $\sqrt{32}$, $\sqrt{72}$, etc., are also rationalising factors.

141. The problems of (i) *division by a surd* and (ii) *rationalising the denominator of a fraction* are identical : we find the simplest rationalising factor of the denominator and multiply both the numerator and the denominator by it.

EXERCISE 86

Give the simplest rationalising factor of :—

1. $\sqrt{3}$.
2. $\sqrt{12}$
3. $\sqrt{20}$.
4. $3\sqrt{a}$.
5. $\sqrt{3}+\sqrt{2}$.
6. $2\sqrt{5}-\sqrt{3}$.
7. $4+3\sqrt{5}$.
8. $3\sqrt{5}-5\sqrt{7}$.

Express with rational denominators :—

9. $\dfrac{5}{\sqrt{2}}$. [*Solved*]
10. $\dfrac{6}{\sqrt{5}}$.

11. $\dfrac{15}{\sqrt{10}}$.
12. $\dfrac{6}{\sqrt{3}}$.

13. $\dfrac{2\sqrt{3}}{3\sqrt{2}}$.
14. $\dfrac{\sqrt{m}}{x\sqrt{n}}$.

Rationalise the denominators of :—

15. $\dfrac{1}{\sqrt{3}-1}$. [*Solved*]
16. $\dfrac{1}{\sqrt{5}-2}$.

17. $\dfrac{1}{\sqrt{6}-\sqrt{5}}$.
18. $\dfrac{-2}{4-3\sqrt{2}}$.

19. $\dfrac{\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.
20. $\dfrac{30}{5\sqrt{3}-3\sqrt{5}}$.

21. $\dfrac{8+\sqrt{2}}{2-\sqrt{2}}$. [*Solved*]
22. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

23. $\dfrac{2\sqrt{5}+\sqrt{3}}{3\sqrt{5}+2\sqrt{3}}$.
24. $\dfrac{\sqrt{7}+\sqrt{2}}{9+2\sqrt{14}}$.

25. $\dfrac{2\sqrt{3}+3\sqrt{2}}{5+2\sqrt{6}}$.
26. $\dfrac{8\sqrt{5}-3\sqrt{2}}{2\sqrt{5}-5\sqrt{2}}$.

Divide :—

27. $\sqrt{5}$ by $\sqrt{5}-\sqrt{3}$. [*Solved*]
28. $\sqrt{2}$ by $\sqrt{5}-\sqrt{2}$.
29. $5\sqrt{5}-2$ by $\sqrt{10}-\sqrt{2}$.
30. $4\sqrt{8}-3\sqrt{2}$ by $\sqrt{3}+\sqrt{2}$.

Express with rational denominator :—

31. $\dfrac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}}$ [Solved] 32. $\dfrac{x^3}{\sqrt{x^2+a^2}+a}$.

33. $\dfrac{\sqrt{a}+\sqrt{a-1}}{\sqrt{a}-\sqrt{a-1}}$. 34. $\dfrac{\sqrt{x^2+y^2}+\sqrt{x^2-y^2}}{\sqrt{x^2+y^2}-\sqrt{x^2-y^2}}$.

35. $\dfrac{\sqrt{2}}{1+\sqrt{3}+\sqrt{2}}$. [Solved] 36. $\dfrac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$.

37. $\dfrac{4}{\sqrt{3}-\sqrt{2}-1}$.

38. $\dfrac{2}{\sqrt{18}+\sqrt{50}-\sqrt{98}}$. [Hint]

39. $\dfrac{3\sqrt{8}}{\sqrt{75}-\sqrt{300}+\sqrt{108}}$.

40. $\dfrac{5}{\sqrt{27}-\sqrt{32}+\sqrt{8}-\sqrt{12}+\sqrt{18}}$.

Simplify :—

41. $\dfrac{3}{5-\sqrt{3}}+\dfrac{2}{5+\sqrt{3}}$. [Solved]

42. $\dfrac{4+\sqrt{5}}{4-\sqrt{5}}+\dfrac{4-\sqrt{5}}{4+\sqrt{5}}$. 43. $\dfrac{4+\sqrt{6}}{4-\sqrt{6}}-\dfrac{4-\sqrt{6}}{4+\sqrt{6}}$.

44. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}+\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

45. $\dfrac{1}{2-\sqrt{3}}-\dfrac{1}{\sqrt{3}+\sqrt{2}}+\dfrac{4}{3-\sqrt{5}}$. [Hint]

46. $\dfrac{8\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$.

47. $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}-\dfrac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}-\dfrac{a-b}{a+b}$. [Hint]

SOLUTIONS & HINTS—EXERCISE 86

9.　$\dfrac{5}{\sqrt{2}} = \dfrac{5 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \dfrac{5\sqrt{2}}{2}$

15.　$\dfrac{1}{\sqrt{3}-1} = \dfrac{1}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1} = \dfrac{\sqrt{3}+1}{3-1} = \dfrac{\sqrt{3}+1}{2}.$

21.　$\dfrac{8+\sqrt{2}}{2-\sqrt{2}} = \dfrac{3+\sqrt{2}}{2-\sqrt{2}} \times \dfrac{2+\sqrt{2}}{2+\sqrt{2}} = \dfrac{6+2\sqrt{2}+3\sqrt{2}+2}{4-2}$

$$= \dfrac{8+5\sqrt{2}}{2}$$

27. Reqd. quotient $= \dfrac{\sqrt{5}}{\sqrt{5}-\sqrt{3}} = \dfrac{\sqrt{5}}{\sqrt{5}-\sqrt{3}} \times \dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$

$$= \dfrac{5+\sqrt{15}}{5-3} = \dfrac{5+\sqrt{15}}{2} = \tfrac{5}{2}+\tfrac{1}{2}\sqrt{15}.$$

31.　$\dfrac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}} = \dfrac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}} \times \dfrac{a+\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}$

$$= \dfrac{a^2+(a^2-x^2)+2a\sqrt{a^2-x^2}}{a^2-(a^2-x^2)}$$

$$= \dfrac{2a^2-x^2+2a\sqrt{a^2-x^2}}{x^2}.$$

35.　$\dfrac{\sqrt{2}}{1+\sqrt{3}+\sqrt{2}} = \dfrac{\sqrt{2}}{(1+\sqrt{3})+\sqrt{2}} \times \dfrac{(1+\sqrt{3})-\sqrt{2}}{(1+\sqrt{3})-\sqrt{2}}$

$$= \dfrac{\sqrt{2}(1+\sqrt{3}-\sqrt{2})}{(1+\sqrt{3})^2-(\sqrt{2})^2} = \dfrac{\sqrt{2}+\sqrt{6}-2}{2+2\sqrt{3}}$$

$$[\because \quad \text{Den.}=1+3+2\sqrt{3}-2=2+2\sqrt{3}]$$

$$= \dfrac{\sqrt{2}+\sqrt{6}-2}{2(1+\sqrt{3})} \times \dfrac{1-\sqrt{3}}{1-\sqrt{3}}$$

$$= \dfrac{\sqrt{2}+\sqrt{6}-2-\sqrt{6}-3\sqrt{2}+2\sqrt{3}}{2(-2)}$$

$$= \dfrac{-2\sqrt{2}-2+2\sqrt{3}}{-4} = \dfrac{\sqrt{2}+1-\sqrt{3}}{2}$$

[Dividing the num. and den by -2]

38. Denominator $= \sqrt{18}+\sqrt{50}-\sqrt{98}=3\sqrt{2}+5\sqrt{2}-7\sqrt{2}$
$= \sqrt{2}$. Rationalise the denominator.

41. $\dfrac{3}{5-\sqrt{3}}+\dfrac{2}{5+\sqrt{3}}=\dfrac{15+8\sqrt{3}+10-2\sqrt{3}}{(5-\sqrt{3})(5+\sqrt{3})}=\dfrac{25+\sqrt{3}}{25-3}$

$$=\frac{25+\sqrt{3}}{22}.$$

45. First fraction $=\dfrac{1}{2-\sqrt{3}}=\dfrac{1}{2-\sqrt{3}}\times\dfrac{2+\sqrt{3}}{2+\sqrt{3}}=\dfrac{2+\sqrt{3}}{1}$

$$=2+\sqrt{3}.$$

Similarly, rationalise the denominators of the other two fractions and combine the three results thus obtained.

47. Add the first two fractions and then add the sum obtained to the third fraction.

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142. Theorem. *The square root of a rational quantity cannot be partly rational and partly a quadratic surd.*

Proof. If possible let $\sqrt{a}=b+\sqrt{c}$, where a and b are rational and $\sqrt{c}$ a true quadratic surd.

Squaring we have :—

$$a=b^2+c+2b\sqrt{c}$$
$$\therefore \quad 2b\sqrt{c}=a-b^2-c$$
$$\therefore \quad \sqrt{c}=\frac{a-b^2-c}{2b},$$

that is, a true surd is equal to a rational quantity, which is impossible. Hence the theorem.

143. Theorem. *If $a+\sqrt{b}=x+\sqrt{y}$, where a and x are both rational and $\sqrt{b}$ and $\sqrt{y}$ are true surds, then will $a=x$ and $b=y$.*

Proof. If a is not equal to x, let a be equal to $x+m$, where m is a rational quantity. Then we have :—

$$x+m+\sqrt{b}=x+\sqrt{y}$$
$$\therefore \quad m+\sqrt{b}=\sqrt{y}$$

that is, sq. root of y is partly rational and partly irrational, which is impossible by the last theorem [Art. 142]

Hence the theorem.

Cor. If $a+\sqrt{b}=x+\sqrt{y}$, then $a-\sqrt{b}=x-\sqrt{y}$.

[For, by the above theorem $a=x$ and $b=y$ and therefore the two sides are equal]

Note. The theorem of Art. 143 is true only when $\sqrt{b}$ and $\sqrt{y}$ are true surds and not surds in form only. Thus $2+\sqrt{9}=3+\sqrt{4}$ [∵ each$=5$], but from this we cannot conclude that $2=3$ and $9=4$.

144. Theorem. If $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, then $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

Proof. $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$ [given]

∴ $a+\sqrt{b}=x+y+2\sqrt{xy}$ [Squaring both sides]

∴ $a=x+y$ and $\sqrt{b}=2\sqrt{xy}$

[Theorem of Art. 143]

∴ $a-\sqrt{b}=x+y-2\sqrt{xy}=(\sqrt{x}-\sqrt{y})^2$

∴ $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

145. *To find the square root of an expression of the form* $a+\sqrt{b}$, *where* $\sqrt{b}$ *is a true surd.*

We shall give three methods by solving a particular

Example. Find the sq. root of $14+6\sqrt{5}$.

Method 1.

Let $\sqrt{14+6\sqrt{5}}=\sqrt{x}+\sqrt{y}$

∴ $14+6\sqrt{5}=x+y+2\sqrt{xy}$ [Squaring both sides]

∴ $x+y=14$ ...(i) ⎱
and $2\sqrt{xy}=6\sqrt{5}$ or $4xy=180$ ...(ii) ⎰ [Art. 143]

Now, $(x-y)^2=(x+y)^2-4xy$ [Formula]

$=(14)^2-180$ · [using results (i) & (ii)]

$=16$.

$x-y=4$............(iii)

(i)+(iii) gives $2x=18$ or $x=9$

(i)−(iii) $2y=10$ or $y=5$

∴ Reqd. sq. root$=\sqrt{9}+\sqrt{5}=3+\sqrt{5}$

Note. After obtaining equations (i) and (ii), which are
$$x+y=14 \text{ and } xy=45$$
we might be able to find the values of x and y by inspection. For, x and y are those quantities whose sum $=14$ and product $=45$.

Method 2

Let $\sqrt{14+6\sqrt{5}}=\sqrt{x}+\sqrt{y}$ ....(i)

Then $\sqrt{14-6\sqrt{5}}=\sqrt{x}-\sqrt{y}$ [Art. 144]

Multiplying the two equations we get :—

$$\sqrt{(14+6\sqrt{5})(14-6\sqrt{5})}=(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$$

i.e., $\sqrt{196-180}=x-y$

i.e., $x-y=4$ ....(ii)

Also squaring (i) and equating rational parts only, we get
$$x+y=14 \text{ ....}(iii) \quad \text{[as in Method 1].}$$

Solving (ii) and (iii) we get $x=9$ and $y=5$, etc.

Method 3 (by inspection)

$14+6\sqrt{5}=14+2\times3\times\sqrt{5}$

$\qquad =14+2\sqrt{45}$ [by taking 3 inside the radical sign, where it becomes 3^2 or 9.]

$\qquad =9+5+2\sqrt{9\times5}$ [The number 9 and 5, whose sum is 14 and product 45, have been found by inspection.]

$\qquad =(3)^2+(\sqrt{5})^2+2\times3\times\sqrt{5}$

$\qquad =(3+\sqrt{5})^2$

$\therefore$ Reqd. sq. root $=3+\sqrt{5}$.

146. *Square root of an expression of the form $a-\sqrt{b}$* is found by supposing it to be equal to $\sqrt{x}-\sqrt{y}$ and proceeding as in Art. 145 (Methods 1 and 2) or by inspection as in the same article (Method 3). For actual illustrations see solutions to the next exercise.

147. *Square root of an expression of the form $\sqrt{a}+\sqrt{b}$ or $\sqrt{a}-\sqrt{b}$.*

Express the given expression as the product of two factors, one of which is of the form $\sqrt{k}$ and the other of the form $p + \sqrt{q}$ or $p - \sqrt{q}$. The sq. root of $\sqrt{k}$ is $\sqrt[4]{k}$ and the sq. root of the other factor can be found by Art. 145 or 146. The required square root is the product of these two square roots.

Example. Find the sq. root of $14\sqrt{2} + 6\sqrt{10}$.

Solution $14\sqrt{2} + 6\sqrt{10} = \sqrt{2}(14 + 6\sqrt{5})$

Sq. root of $\sqrt{2} = \sqrt[4]{2}$

and sq. root of $14 + 6\sqrt{5} = 3 + \sqrt{5}$

[see solved Example, Art. 145]

$\therefore$ Reqd. sq. root $= \sqrt[4]{2}.(3 + \sqrt{5})$.

EXERCISE 87

Find the square root of :—

1. $7 + 4\sqrt{3}$. [Solved]
2. $3 + 2\sqrt{2}$.
3. $5 + 2\sqrt{6}$.
4. $8 + 2\sqrt{15}$.
5. $11 + 4\sqrt{6}$.
6. $30 + 12\sqrt{6}$.

---

7. $14 - 6\sqrt{5}$. [Solved]
8. $7 - 2\sqrt{10}$.
9. $18 - 8\sqrt{5}$.
10. $41 - 24\sqrt{2}$.
11. $18 - 12\sqrt{2}$.
12. $35 - 12\sqrt{6}$.

---

13. $2 - \sqrt{3}$. [Solved]
14. $3 - \sqrt{5}$.
15. $4 + \sqrt{7}$.
16. $6 + \sqrt{35}$.
17. $\dfrac{7}{2} - \dfrac{3}{2}\sqrt{5}$.
18. $\dfrac{9}{4} + \sqrt{5}$.
19. $\dfrac{11}{4} + \sqrt{6}$.
20. $3\dfrac{1}{10} - \sqrt{6}$.

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21. $\sqrt{32} + \sqrt{24}$. [Solved]
22. $\sqrt{27} - \sqrt{24}$.
23. $5\sqrt{5} - \sqrt{120}$.
24. $\sqrt{175} + \sqrt{147}$.
25. $\sqrt{54} - \sqrt{30}$.
26. $\sqrt{180} + \sqrt{55}$.

---

27. If $\sqrt{13+2\sqrt{80}}=\sqrt{x}+\sqrt{y}$, evaluate x and y. [Hint].

28. If $\sqrt{14+8\sqrt{3}}=\sqrt{x}+\sqrt{y}$, evaluate x and y.

29. If $\sqrt{15-10\sqrt{2}}=\sqrt{a}-\sqrt{b}$, evaluate a and b.

30. If $\sqrt{31-10\sqrt{6}}=\sqrt{p}-\sqrt{q}$, find the values of p and q.

SOLUTIONS & HINTS—EXERCISE 87

1. Let $\sqrt{7+4\sqrt{3}}=\sqrt{x}+\sqrt{y}$.

Then $\qquad 7+4\sqrt{3}=x+y+2\sqrt{xy}$ [squaring both sides]

$\therefore \qquad x+y=7\ldots\ldots\ldots(i)$ } [Equating rational and ir-
and $2\sqrt{xy}=4\sqrt{3}\ldots\ldots(ii)$ } rational parts]

From (ii) we get $\sqrt{xy}=2\sqrt{3}$

$\qquad\qquad \therefore \quad xy=12\ldots\ldots(iii)$ [squaring]

From equations (i) and (iii) we note that sum of x and y is 7 and product of x and y is 12. Therefore, clearly, $x=4$, $y=3$.

$\qquad \therefore$ Reqd. sq. root $=\sqrt{4}+\sqrt{3}=2+\sqrt{3}$.

Note. We may say, $x=3$ and $y=4$; these values also give the same result. However, when our result is of the form $\sqrt{x}-\sqrt{y}$ we give larger value to x and smaller to y, so that our result may be positive. It is understood that we have to give a positive result in each case, otherwise sq. root in the above example can also be $-2-\sqrt{3}$.

7. Let $\sqrt{14-6\sqrt{5}}=\sqrt{x}-\sqrt{y}$.

Squaring, $14-6\sqrt{5}=x+y-2\sqrt{xy}$.

Equating rational and irrational parts we have :—

$\qquad\qquad x+y=14\ldots\ldots\ldots\ldots\ldots(i)$

and $2\sqrt{xy}=6\sqrt{5}$ or $\sqrt{xy}=3\sqrt{5}$, $\therefore \quad xy=45\ldots\ldots\ldots(ii)$

Numbers whose sum is 14 and product 45 are 9 and 5.

$\qquad \therefore \quad x=9$ and $y=5$.

$\qquad \therefore$ Reqd. sq. root $=\sqrt{9}-\sqrt{5}=3-\sqrt{5}$.

13. Let $\sqrt{2-\sqrt{3}}=\sqrt{x}-\sqrt{y}\ldots\ldots\ldots(i)$

$\qquad \therefore \quad \sqrt{2+\sqrt{3}}=\sqrt{x}+\sqrt{y}\ldots\ldots\ldots(ii)$

Multiplying, $\sqrt{(2-\sqrt{3})(2+\sqrt{3})}=(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})$

or   1 $=x-y\ldots\ldots\ldots\ldots(iii)$

Also, squaring (i) we have $2-\sqrt{3}=x+y-2\sqrt{xy}$.

Equating rational parts $2=x+y\ldots\ldots\ldots\ldots\ldots(iv)$

Adding (iii) and (iv) $3=2x,$ $\therefore$ $x=\frac{3}{2}$

Subtracting (iii) from (iv) $1=2y,$ $\therefore$ $y=\frac{1}{2}.$

$\therefore$ Reqd. sq. root $=\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}.$

Or Thus :—

Let $\sqrt{2-\sqrt{3}}=\sqrt{x}-\sqrt{y}\ldots\ldots\ldots\ldots(i)$

Squaring $2-\sqrt{3}=x+y-2\sqrt{xy}.$

Equating rational and irrational parts we have :—

$$x+y=2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

and $2\sqrt{xy}=\sqrt{3},$ $\therefore$ $4xy=3\ldots\ldots(iii)$

Now, $(x-y)^2=(x+y)^2-4xy$ [Formula]

$=(2)^2-3$ [using results (ii) and (iii)

$=1$

$\therefore$ $x-y=\sqrt{1}=1\ldots\ldots\ldots\ldots(iv)$

[Taking only positive value]

Solving (ii) and (iv) we get, as before, $x=\frac{3}{2},\ y=\frac{1}{2}$, etc.

21. $\sqrt{32}+\sqrt{24}=4\sqrt{2}+2\sqrt{6}=\sqrt{2}(4+2\sqrt{3}).$

Now sq. root of $\sqrt{2}$ is $\sqrt[4]{2}$.

And sq. root of $4+2\sqrt{3}$ may be found as in **Q. 13,** or thus :—

$4+2\sqrt{3}=3+1+2\sqrt{3}.$

$=(\sqrt{3})^2+(1)^2+2\times\sqrt{3}\times1$

$=(\sqrt{3}+1)^2$

$\therefore$ Sq. root of $4+2\sqrt{3}=\sqrt{3}+1.$

$\therefore$ Reqd. sq. root $=\sqrt[4]{2}(\sqrt{3}+1).$

27   Square both sides, etc. etc.

148. Some questions in surds are best done by the use of certain *devices* instead of obvious methods. This will be clear from the solutions to the next exercise.

EXERCISE 88

1. If $x = 3 + \sqrt{8}$, find the value of $x^3 + x^{-3}$. [*Solved*]
2. If $x = 2 + \sqrt{3}$, find the value of $x^2 + x^{-2}$.

3. If $a = 3 - 2\sqrt{2}$, evaluate $a^2 + \dfrac{1}{a^2}$.

4. If $m = \sqrt{6} - \sqrt{5}$, evaluate $m^2 + m^{-4}$.

---

5. If $k = 2 - \sqrt{5}$, evaluate $\left(k + \dfrac{1}{k}\right)^2$. [*Solved*]

6. If $l = 3 + \sqrt{10}$, evaluate $\left(l - \dfrac{1}{l}\right)^2$.

7. If $a = 1 + \sqrt{2}$, evaluate $\left(a + \dfrac{1}{a}\right)^4$.

8. If $x = 1 - \sqrt{2}$, evaluate $\left(x - \dfrac{1}{x}\right)^3$.

---

9. If $x = 2 - \sqrt{3}$, evaluate $x^4 + \dfrac{1}{x^4}$. [*Hint*]

10. If $a = \sqrt{2} - 1$, evaluate $a^4 + \dfrac{1}{a^4}$.

11. If $m = \sqrt{3} + 2$, evaluate $m^4 + m^{-4}$.

---

12. If $x = 3 - 2\sqrt{2}$, evaluate $x^3 + x^{-3}$ [*Solved*]
13. If $x = 2 - \sqrt{3}$, evaluate $x^3 + x^{-3}$.
14. If $x = 2 + \sqrt{5}$, evaluate $x^3 - x^{-3}$. [*Hint*]
15. If $a = 3 + \sqrt{10}$, evaluate $a^3 - a^{-3}$.
16. If $n = \sqrt{2} + 1$, evaluate $n^3 + n^{-3}$. [*Hint*]
17. If $m = 2 + \sqrt{3}$, evaluate $m^3 - m^{-3}$

---

18. If $x=5+2\sqrt{6}$, evaluate $x^{\frac{1}{2}}+x^{-\frac{1}{2}}$. [Solved]

19. If $x=4-\sqrt{15}$, evaluate $x^{\frac{1}{2}}+x^{-\frac{1}{2}}$.

20. If $a=7+4\sqrt{3}$, evaluate $\sqrt{a}+\dfrac{1}{\sqrt{a}}$.

21. If $a=5-2\sqrt{6}$, evaluate $\sqrt{a}-\dfrac{1}{\sqrt{a}}$. [Hint]

---

22. Evaluate $m^{\frac{1}{2}}-m^{-\frac{1}{2}}$ when (i) $m=4+\sqrt{15}$, (ii) $m=4-\sqrt{15}$

Find the value of :—

23. $(3+\sqrt{8})^2+\dfrac{1}{(3+\sqrt{8})^2}$. [Hint]

24. $(\sqrt{6}-\sqrt{5})^2+(\sqrt{6}-\sqrt{5})^{-2}$. 25. $\left(8+\sqrt{10}-\dfrac{1}{3+\sqrt{10}}\right)^2$.

26. $(\sqrt{3}+2)^4+(\sqrt{3}+2)^{-4}$.   27. $(2-\sqrt{3})^3+(2-\sqrt{3})^{-3}$.

28. $(2+\sqrt{5})^3-(2+\sqrt{5})^{-3}$.

29. $(7+4\sqrt{3})^{\frac{1}{2}}+(7+4\sqrt{3})^{-\frac{1}{2}}$.

30. $(9-4\sqrt{5})^{\frac{1}{2}}-(9-4\sqrt{5})^{-\frac{1}{2}}$.

---

31. If $x=\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$ and $y=\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$, evaluate x^2+y^2. [Solved]

32. If $x=\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ and $y=\dfrac{\sqrt{8}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$, evaluate x^2+y^2.

33. If $a=\dfrac{2-\sqrt{5}}{2+\sqrt{5}}$ and $b=\dfrac{2+\sqrt{5}}{2-\sqrt{5}}$, evaluate a^2-b^2. [Hint]

Find the value of :—

34. $\sqrt{\dfrac{\sqrt{5}+2}{\sqrt{5}-2}}-\sqrt{\dfrac{\sqrt{5}-2}{\sqrt{5}+2}}$. [Solved]

35. $\sqrt{\dfrac{\sqrt{5}+\sqrt{8}}{\sqrt{5}-\sqrt{8}}}+\sqrt{\dfrac{\sqrt{5}-\sqrt{8}}{\sqrt{5}+\sqrt{8}}}$.

36. $\left(\dfrac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}\right)^3+\left(\dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}\right)^3.$

37. $\left(\dfrac{\sqrt{5}+1}{\sqrt{5}-1}\right)^4+\left(\dfrac{\sqrt{5}-1}{\sqrt{5}+1}\right)^4.$

SOLUTIONS & HINTS—EXERCISE 88

1. [The obvious method is : $x^2+x^{-2}=x^2+\dfrac{1}{x^2}$

$$=(3+\sqrt{8})^2+\frac{1}{(3+\sqrt{8})^2},\text{ etc.}$$

But it is rather tedious.]

$x=3+\sqrt{8}.$ [given]

$\therefore\ \dfrac{1}{x}=\dfrac{1}{3+\sqrt{8}}=\dfrac{1}{3+\sqrt{8}}\times\dfrac{3-\sqrt{8}}{3-\sqrt{8}}=\dfrac{3-\sqrt{8}}{9-8}=3-\sqrt{8}$

$\therefore\ x+\dfrac{1}{x}=(3+\sqrt{8})+(3-\sqrt{8})=6$

$\therefore\ \left(x+\dfrac{1}{x}\right)^2=(6)^2$

$\therefore\ x^2+\dfrac{1}{x^2}+2=36$

$\therefore\ x^2+\dfrac{1}{x^2}=36-2=34.$ *Ans.*

5. $k=2-\sqrt{5}$

$\therefore\ \dfrac{1}{k}=\dfrac{1}{2-\sqrt{5}}=\dfrac{1}{2-\sqrt{5}}\times\dfrac{2+\sqrt{5}}{2+\sqrt{5}}=\dfrac{2+\sqrt{5}}{4-5}=\dfrac{2+\sqrt{5}}{-1}$

$=-(2+\sqrt{5})$

$k+\dfrac{1}{k}=(2-\sqrt{5})-(2+\sqrt{5})=-2\sqrt{5}$

$\therefore\ \left(k+\dfrac{1}{k}\right)^2=(-2\sqrt{5})^2=4\times5=20.$

9. First find the value of $x^2+\dfrac{1}{x^2}$ as in Q. 1. Square

once more and get the value of $x^4+\dfrac{1}{x^4}$.

12. $x=3-2\sqrt{2}$.   [given]

$$\therefore \quad \frac{1}{x}=\frac{1}{3-2\sqrt{2}}=\frac{1}{3-2\sqrt{2}}\times\frac{3+2\sqrt{2}}{3+2\sqrt{2}}=\frac{3+2\sqrt{2}}{9-8}=3+2\sqrt{2}.$$

$$\therefore \quad x+\frac{1}{x}=(3-2\sqrt{2})+(3+2\sqrt{2})=6.$$

$$\therefore \quad \left(x+\frac{1}{x}\right)^3=(6)^3$$

$$\therefore \quad x^3+\frac{1}{x^3}+3.x.\frac{1}{x}\left(x+\frac{1}{x}\right)=216$$

$$\therefore \quad x^3+\frac{1}{x^3}+3(6)=216$$

$$\therefore \quad x^3+\frac{1}{x^3}=216-18=198.$$

14. The value of $\dfrac{1}{x}$ will be found equal to $-2+\sqrt{5}$.

$$\therefore \quad x-\frac{1}{x}=(2+\sqrt{5})-(-2+\sqrt{5})=4$$

Cube both sides, etc.

16. $n+\dfrac{1}{n}$ will be found equal to $2\sqrt{2}$.

$$\therefore \quad \left(n+\frac{1}{n}\right)^3=(2\sqrt{2})^3=2\sqrt{2}\times2\sqrt{2}\times2\sqrt{2}=16\sqrt{2}. \quad \text{etc.}$$

18. $x=5+2\sqrt{6}$.   [given]

$$\frac{1}{x}=\frac{1}{5+2\sqrt{6}}=\frac{1}{5+2\sqrt{6}}\times\frac{5-2\sqrt{6}}{5-2\sqrt{6}}=\frac{5-2\sqrt{6}}{25-24}=5-2\sqrt{6}$$

$$\therefore \quad x+\frac{1}{x}=(5+2\sqrt{6})+(5-2\sqrt{6})=10$$

$$\therefore \quad x+\frac{1}{x}+2=10+2. \quad \text{[adding 2 to both sides]}$$

or $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2=12.$ $\left[\because\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2=x+\frac{1}{x}+2\right]$

or $\sqrt{x}+\frac{1}{\sqrt{x}}=\sqrt{12}=2\sqrt{3}.$

21. First get the value of $a+\frac{1}{a}$. Then subtract 2 from both sides.

Remember that $a+\frac{1}{a}-2=\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2$:

Important Note. We get $\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2=8.$ $\therefore$ $\sqrt{a}-\frac{1}{\sqrt{a}}$ $=+\sqrt{8}$ or $-\sqrt{8}$. To decide the sign, we note that $a=5-2\sqrt{6}$ which is le·s than 1 (because $\sqrt{6}>2$, $\therefore$ $2\sqrt{6}>2\times2$ i.e., >4). Therefore $\sqrt{a}$ is also less than 1. Therefore $\frac{1}{\sqrt{a}}$ is greater than 1. Therefore $\sqrt{a}-\frac{1}{\sqrt{a}}$ is negative. Hence reqd. ans.$=-\sqrt{8}$ or $-2\sqrt{2}$.

23. Given Exp.$=(3+\sqrt{8})^2+\frac{1}{(3+\sqrt{8})^2}=x^2+\frac{1}{x^2}$

[where $x=3+\sqrt{8}$]

Now. $x=3+\sqrt{8}$ and we have to find the value cf

$$x^2+\frac{1}{x^2}.$$

This is exactly Q. 1.

31. $x=\frac{2+\sqrt{3}}{2-\sqrt{3}}$ and $y=\frac{2-\sqrt{3}}{2+\sqrt{3}}$

$\therefore$ $x+y=\frac{2+\sqrt{3}}{2-\sqrt{3}}+\frac{2-\sqrt{3}}{2+\sqrt{3}}=\frac{(2+\sqrt{3})^2+(2-\sqrt{3})^2}{(2-\sqrt{3})(2+\sqrt{3})}$

$=\frac{2(4+3)}{4-3}$

$[\because (a+b)^2+(a-b)^2=2(a^2+b^2)]$

$=14.$

Also $xy=\frac{2+\sqrt{3}}{2-\sqrt{3}}\times\frac{2-\sqrt{3}}{2+\sqrt{3}}=1.$

$\therefore \quad x^2+y^2=(x+y)^2-2xy$ [Formula]

$$=(14)^2-2=196-2=194.$$

33.   Find the values of $a+b$ and $a-b$, and multiply the two results

34.   Let $x=\dfrac{\sqrt{5}+2}{\sqrt{5}-2}$ and $y=\dfrac{\sqrt{5}-2}{\sqrt{5}+2}$.

Then we have to find the value of $\sqrt{x}-\sqrt{y}$.

Now, $x+y=\dfrac{\sqrt{5}+2}{\sqrt{5}-2}+\dfrac{\sqrt{5}-2}{\sqrt{5}+2}=\dfrac{(\sqrt{5}+2)^2+(\sqrt{5}-2)^2}{(\sqrt{5}-2)(\sqrt{5}+2)}$

$$=\dfrac{2(5+4)}{5-4}=18$$

and $xy=\dfrac{\sqrt{5}+2}{\sqrt{5}-2}\times\dfrac{\sqrt{5}-2}{\sqrt{5}+2}=1$ $\therefore \quad \sqrt{xy}=1.$

$\therefore \quad x+y-2\sqrt{xy}=18-2=16$

or $(\sqrt{x}-\sqrt{y})^2=16$ $[\because (\sqrt{x}-\sqrt{y})^2=x+y-2\sqrt{xy}]$

$\therefore \quad\quad\quad \sqrt{x}-\sqrt{y}=\sqrt{16}=4.$

---

149.   Irrational Equations.

Equations in which the unknown quantity occurs under the root sign are called *irrational equations*.

It should be noted that in such equations the *positive value* of the square root is meant. For example, if $x=2$, $\sqrt{4x+1}$ means 3 and not -3.

We shall take only those equations, for the solution of which knowledge of *quadratic equations* is not required.

EXERCISE 89

Solve the following equations and verify your answers :—

1. $\sqrt{3x-2}=4$.   [*Solved*]   2. $\sqrt{7x-10}=5$.

3. $\sqrt{x+1}=8\sqrt{x-7}$.   4. $\sqrt{3x-5}-2=0$.   [*Hint*]

5. $\sqrt[3]{x-1}+2=0$.   6. $\sqrt[3]{4x-7}-5=0$.

7. $\sqrt{x-1}+\sqrt{x+6}=7$. [Solved]

8. $\sqrt{x-5}+\sqrt{x+7}=6$. 　9. $\sqrt{x+7}-\sqrt{x}=1$.

10. $\sqrt{x+8}=4-\sqrt{x}$. 　11. $\sqrt{x-4}-\sqrt{x+11}+3=0$.

12. $10-\sqrt{25+9x}-3\sqrt{x}=0$.

13. $3\sqrt{x+4}-2-\sqrt{9x-8}=0$. 14. $\sqrt{4x-5}=2\sqrt{x-1}+1$.

15. $\sqrt{x^2+6x}-3+x=0$. 　16. $3x+1-\sqrt{9x^2-2x}=0$.

———

17. $\sqrt{x+2}+\sqrt{x-3}-\sqrt{4x-3}=0$. [Solved]

18. $\sqrt{x+3}+\sqrt{x+8}-\sqrt{4x+21}=0$.

19. $\sqrt{8x+17}-\sqrt{2x}-\sqrt{2x+9}=0$.

20. $\sqrt{9x+7}=\sqrt{x+2}+\sqrt{4x+1}$.

———

21. $\dfrac{2x-1}{\sqrt{2x+1}}=1+\dfrac{\sqrt{2x}-1}{2}$. [Solved]

22. $\dfrac{3x-1}{\sqrt{3x+1}}-1=\dfrac{\sqrt{3x}-1}{2}$.

23. $\dfrac{5x-4}{\sqrt{5x+2}}=3+\dfrac{\sqrt{5x}-2}{2}$.

24. $\sqrt{x}-\sqrt{x-13}=\dfrac{6}{\sqrt{x-13}}$. [Hint]

25. $\sqrt{x+\sqrt{7+x}}-\dfrac{21}{\sqrt{7+x}}=0$.

26. $\dfrac{6\sqrt{x}-11}{3\sqrt{x}}=\dfrac{2\sqrt{x}+1}{\sqrt{x+6}}$.

———

27. $\sqrt{4x+5}-\sqrt{4x-11}=2$. [Solved]

28. $\sqrt{2x+21}-\sqrt{2x+12}=1$. 29. $\sqrt{3x+7}+\sqrt{3x-5}=6$.

30. $\sqrt{5x+16}+\sqrt{5x-4}=10$.

———

31. $\sqrt{x^2+11}-\sqrt{x^2-11}=6-\sqrt{14}$. [Solved]

32. $\sqrt{2x^2-1}+\sqrt{2x^2-2}=7+4\sqrt{3}$.

33. $\sqrt{3x^2+1} - \sqrt{3x^2-8} = 7 - 2\sqrt{10}.$

34. $\sqrt{9x-2} + \sqrt{9x-19} = 5 + 2\sqrt{2}.$

35. $\dfrac{4\sqrt{x} + \sqrt{3x-1}}{4\sqrt{x} - \sqrt{3x-1}} = \dfrac{5}{3}.$ [*Solved*]

36. $\dfrac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} = \dfrac{1}{2}.$ **37.** $\dfrac{\sqrt{17+x} - \sqrt{1-x}}{\sqrt{17+x} + \sqrt{17-x}} = \dfrac{1}{4}.$

38. $\dfrac{3x - \sqrt{4x-x^2}}{3x + \sqrt{4x-x^2}} = \dfrac{1}{2}.$

SOLUTIONS & HINTS—EXERCISE 89

1. $\sqrt{3x-2} = 4$

∴ $3x - 2 = 16$ [Squaring both sides]

or $3x = 16 + 2 = 18$

or $x = \dfrac{18}{3} = 6.$

4. Transpose -2 to the right hand side. *Remember that whenever there is only one radical in an equation, it should be kept all alone on one side while squaring.*

7 $\sqrt{x-1} + \sqrt{x+6} = 7$

∴ $\sqrt{x-1} = 7 - \sqrt{x+6}$ [Transposing $\sqrt{x+6}$ to R. H. S.]

∴ $x - 1 = 49 + (x+6) - 14\sqrt{x+6}$ [Squaring both sides]

∴ $14\sqrt{x+6} = 49 + x + 6 - x + 1 = 56$

or $\sqrt{x+6} = 4$ [Dividing both sides by 14]

∴ $x + 6 = 16$ [Squaring both sides]

or $x = 16 - 6 = 10.$

Important Note. *When there are two radicals in an equation, both should not be on the same side while squaring, otherwise second power of x will enter into the equation.*

17. $\sqrt{x+2} + \sqrt{x-3} - \sqrt{4x-3} = 0$

or $\sqrt{x+2} + \sqrt{x-3} = \sqrt{4x-3}$

∴ $x+2 + x-3 + 2\sqrt{(x+2)(x-3)} = 4x - 3$

                            [Squaring both sides]

or $2\sqrt{x^2-x-6} = 4x - 3 - x - 2 - x + 3 = 2x - 2$

or $\sqrt{x^2-x-6} = x - 1$ [Dividing both sides by 2]

$$\therefore \quad x^3 - x - 6 = x^2 - 2x + 1 \qquad \text{[Squaring both sides]}$$
$$\therefore \quad x^2 - x - x^2 + 2x = 1 + 6$$
$$\text{or} \qquad\qquad x = 7.$$

21. L. H. S. $= \dfrac{2x-1}{\sqrt{2x}+1} = \dfrac{2x-1}{\sqrt{2x}+1} \times \dfrac{\sqrt{2x}-1}{\sqrt{2x}-1}$

[Rationalising the denominator]

$$= \frac{(2x-1)(\sqrt{2x}-1)}{(2x-1)} = \sqrt{2x}-1$$

$\therefore$ The given equation is $\sqrt{2x}-1 = 1 + \dfrac{\sqrt{2x}-1}{2}$.

or $2\sqrt{2x}-2 = 2 + \sqrt{2x} - 1$ [Multiplying by 2]

or $2\sqrt{2x} - \sqrt{2x} = 2 - 1 + 2 = 3$

or $\sqrt{2x} = 3$

$\therefore \qquad 2x = 9 \qquad$ [Squaring]

or $\qquad x = \frac{9}{2}$.

24. $\sqrt{x} - \sqrt{x-13} = \dfrac{6}{\sqrt{x-13}}$

or $\sqrt{x^2-13x} - (x-13) = 6$

[Multiplying both sides by $\sqrt{x-13}$]

or $\sqrt{x^2-13x} = 6 + x - 13 = x - 7$

Square both sides ; etc.

27. $\sqrt{4x+5} - \sqrt{4x-11} = 2 \qquad \qquad \ldots(i)$

Inverting both sides of the equation we get :—

$$\frac{1}{\sqrt{4x+5}-\sqrt{4x-11}} = \tfrac{1}{2}$$

On rationalising the denominator of the left hand-side, this becomes :—

$$\frac{\sqrt{4x+5}+\sqrt{4x-11}}{(4x+5)-(4x-11)} = \tfrac{1}{2} \quad \text{or} \quad \frac{\sqrt{4x+5}+\sqrt{4x-11}}{16} = \tfrac{1}{2}$$

or $\sqrt{4x+5} + \sqrt{4x-11} = 8 \qquad \ldots(ii)$

Adding (i) and (ii) we have :—

$$2\sqrt{4x+5} = 10$$

or $\sqrt{4x+5}=5$ [Dividing by 2]

∴ $4x+5=25$ [Squaring]

or $4x=25-5=20$

or $x=\frac{20}{4}=5.$

31. $\sqrt{x^2+11}-\sqrt{x^2-11}=6-\sqrt{14}$ ...(i)

Inverting both sides of (i) we get :—

$$\frac{1}{\sqrt{x^2+11}-\sqrt{x^2-11}}=\frac{1}{6-\sqrt{14}}$$

Rationalising the denominators of both sides we have :—

$$\frac{\sqrt{x^2+11}+\sqrt{x^2-11}}{22}=\frac{6+\sqrt{14}}{22}$$

or $\sqrt{x^2+11}+\sqrt{x^2-11}=6+\sqrt{14}$ ...(ii)

Adding (i) and (ii) we get :—

$2\sqrt{x^2+11}=12$

∴ $\sqrt{x^2+11}=6$

∴ $x^2+11=36$ [Squaring]

or $x^2=36-11=25$

∴ $x=+5$ or $-5,$ i.e., $\pm 5.$

35. Introduction :—

The student should know that if $\dfrac{a}{b}=\dfrac{c}{d}$, then also

$\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}.$ That is, if two fractions are equal, then :—

$$\frac{\text{Numerator of I fraction} + \text{its denominator}}{\text{Numerator of I fraction} - \text{its denominator}}$$

$$=\frac{\text{Numerator of II fraction} + \text{its denominator}}{\text{Numerator of II fraction} - \text{its denominator}}$$

This is called the *Rule of Componendo and Dividendo.* It will be proved in the next chapter, but we shall use it here to solve the given equation.

$$\frac{4\sqrt{x}+\sqrt{3x-1}}{4\sqrt{x}-\sqrt{3x-1}}=\frac{5}{3}$$ [given]

$$2\frac{8\sqrt{x}}{2\sqrt{3x-1}} = \frac{5+3}{5-3}$$ [Comp. & Dividendo]

or $$\frac{4\sqrt{x}}{\sqrt{3x-1}} = \frac{8}{2} = 4$$

or $$\frac{\sqrt{x}}{\sqrt{3x-1}} = 1$$ [Dividing by 4]

or $$\frac{x}{3x-1} = 1$$ [Squaring]

or $$3x - 1 = x$$

or $$2x = 1$$

or $$x = \tfrac{1}{2}.$$

TEST PAPERS—SET 4

(CHAPTERS XIII to XXI)

Paper 1—Ex. 90

1. Factorise :— $x^8 - 10x^4 + 9.$

2. Find the H. C. F. of :—

   $15(a^3+1),\ 20(a^4+a^2+1),\ 25(a^3 - a^3 + a^2).$

3. Simplify :—

$$\frac{\dfrac{a^2}{b^2}\left(1 + \dfrac{b}{a}\right)^2 - \left(\dfrac{a}{b} + 1\right)}{\dfrac{a^2}{b^2}\left(1 + \dfrac{b}{a}\right)^2 - \dfrac{a}{b}\left(\dfrac{a}{b} + 1\right)}$$

4. Find the square root of :—

$$9\left(\frac{a^3}{b^2} + \frac{b^2}{a^3}\right) - 30\left(\frac{a}{b} + \frac{b}{a}\right) + 43.$$

5. Simplify :—

$$\frac{\sqrt[3]{81} \times \sqrt{18} \times \sqrt[3]{40}}{\sqrt{2} \times \sqrt{12} \times \sqrt{27} \times \sqrt[3]{5}}.$$

6. Which is greater $\sqrt[3]{4}$ or $\sqrt[3]{3}$? Express their ratio as a single surd.

Paper 2—Ex. 91

1. Factorise — $64x^6 - a^6$

2. Find the L. C. M of $12x^2y(x^3 + y^3)$, $15xy^2(x^3 - y^2)$ and $21x^2y^2(x^4 + x^2y^2 + y^4)$.

3. Simplify :—

$$\frac{2}{x+y} - \frac{1}{x-y} - \frac{3y}{y^2 - x^2} + \frac{xy}{x^3 + y^3}$$

4. Find the square root of :—

$$(x^2 + 8x + 7)(2x^2 - x - 3)(2x^2 + 11x - 21).$$

5. Simplify :—

$$\frac{2^{m+2} \cdot 3^{2m-n+1} \cdot 5^{m+n+1} \cdot 6^n}{6^m \cdot 10^{n+2} \cdot 15^m}$$

6. Evaluate :—

$$\frac{(\sqrt{3} + \sqrt{5})^2(8 - 2\sqrt{15})}{(1 + \sqrt{3})(\sqrt{2} + \sqrt{6})(2 - \sqrt{3})}.$$

Paper 3—Ex. 92

1. Find the factors of $x^3 - 18xy - 27 - 8y^3$.

2. Find the H. C. F. of :—

$$(a^3 - b^3)(a^2 + 2ab + b^2),\ a^4 - b^4,\ 4a^4 + a^3b - 5a^2b^2.$$

3. Simplify :—

$$\frac{x^6 + y^6}{x^6 - y^6} \times \frac{x - y}{x + y} \div \frac{a^4 - x^2y^2 + y^4}{x^4 + x^2y^2 + y^4}.$$

4. Find the square root of $(2x - 3)(18x^3 - 51x^2 + 44x - 12)$.

5. (i) If m and n are positive integers, show that—

$$(a^m)^n = a^{mn}.$$

(ii) Find the value of $(3^2)^3 - 3^{2^3}$.

6. Simplify and express with rational denominator—

$$\frac{2 + \sqrt{8} - \sqrt{2}}{2 + \sqrt{2} - \sqrt{8}}$$

Paper—4 Ex. 93

1. Factorise :— $(x-2y)^3+(1-x)^3-(1-2y)^3$

2. Find the L. C. M. of :—

   $15(x^4y-x^3y^2)$, $25(x^4y-x^2y^3)$ and $35(x^4y-xy^4)$.

3. Add together :—

   $$\frac{1}{x-2y}, \frac{1}{x+2y}, \frac{2x}{x^2+4y^2} \text{ and } \frac{4x^3}{x^4+16y^4}.$$

4. Find the square root of :—

   $$\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{2x}{y}-\frac{2y}{x}+3.$$

5. (i) Prove that $a^0=1$.

   (ii) Simplify :— $\left(\frac{x^a}{x^b}\right)^{a+b-c} \times \left(\frac{x^b}{x^c}\right)^{b+c-a} \times \left(\frac{x^c}{x^a}\right)^{c+a-b}$

6. Find the square root of $1+\frac{\sqrt{5}}{3}$.

Paper 5—Ex. 94

1. Factorise :— x^4-2x^3+2x-1.

2. Find the H. C. F. of :—

   $8x^4+4x^3-4x^2-8x$ and $6x^4-4x^3+2x^2-4x$.

3. Simplify :—

   $$\frac{x^3}{(x-y)(x-z)}+\frac{y^3}{(y-z)(y-x)}+\frac{z^3}{(z-x)(z-y)}.$$

4. Find the square root of :—

   $$x^4+\frac{1}{x^4}+6\left(x^2-\frac{1}{x^2}\right)+7.$$

5. (i) If m and n be positive integers, show that

   $a^m \times a^n=a^{m+n}$ and $a^m \div a^n=a^{m-n}$

   (ii) Simplify $\dfrac{(2187)^{\frac{1}{4}} \times 96^{\frac{1}{5}} \times \sqrt{75}}{9^{\frac{3}{4}} \times 3^{\frac{15}{20}}}$.

6. Find the square root of $2\sqrt{3}-2$.

Paper 6—Ex. 95

1. Factorise :— $(a^2-4a)(a^2-4a-1)-20.$
2. Find the L.C.M. of a^3+a-2 and $a^4+a^3-2a-4.$
3. Simplify :—

$$\left(\frac{a+b}{a-b}-\frac{a-b}{a+b}\right)-\left(\frac{1}{\dfrac{a}{b}-\dfrac{b}{a}}-\frac{1}{\dfrac{a}{b}+\dfrac{b}{a}}\right).$$

4. Show that the product of any four consecutive odd integers increased by 16 is a perfect square.

5. Divide $x^{-1}+y-z^{-2}+3x^{-\frac{1}{3}}y^{\frac{1}{3}}z^{-\frac{2}{3}}$ by

$$x^{-\frac{1}{3}}+y^{\frac{1}{3}}-z^{-\frac{2}{3}}.$$

6. If $a=3-\sqrt{10}$, evaluate $a^3-a^{-3}.$

Paper 7—Ex. 96

1. Factorise :— $4x^3-3x+1.$
2. Find the H. C. F of :—

$$9a^4+2a^2b^2+b^4 \text{ and } 3a^4-8a^3b+5a^2b^2-2ab^3.$$

3. Simplify :—

$$\frac{a^2\left(\dfrac{1}{b}-\dfrac{1}{c}\right)+b^2\left(\dfrac{1}{c}-\dfrac{1}{a}\right)+c^2\left(\dfrac{1}{a}-\dfrac{1}{b}\right)}{a\left(\dfrac{1}{b}-\dfrac{1}{c}\right)+b\left(\dfrac{1}{c}-\dfrac{1}{a}\right)+c\left(\dfrac{1}{a}-\dfrac{1}{b}\right)}.$$

4. Find the numerical value of k which will make $a^4+6a^3+7a^2-6a+k$ a perfect square.

5. If $x=4^{\frac{1}{3}}+4^{-\frac{1}{3}}$, prove that $4x^3-12x-17=0.$

6. If $x=9-4\sqrt{5}$, evaluate $x^{\frac{1}{2}}-x^{-\frac{1}{2}}.$

Paper 8—Ex. 97

1. Factorise :— $(a+3)(a+4)(a-8)(a-9)-64.$
2. H. C. F. of two expressions is x^2+3x and L. C. M. $(2x^2-x)(x^2+2x-3)$. If one of the expressions be x^3+2x^2-3x, find the other.

3. Reduce to its lowest terms :— $\dfrac{x^4-5x^2+4}{x^5-11x+10}$.

4. Find the square root of :—

$$\left(a^2+\frac{1}{a^2}\right)^2-4\left(a-\frac{1}{a}\right)^2-4.$$

5. Solve the equation :— $2^{2x-1}\times\dfrac{4}{8^x}=\dfrac{16^{x+1}}{32}$

6. Solve the equation :— $\sqrt{9x-2}+\sqrt{x+1}-\sqrt{4x+27}=0.$

CHAPTER XXII

RATIO AND PROPORTION

150. Definition. The *ratio* of one quantity to another *of the same kind* is the quotient obtained when the first is divided by the second.

The ratio of *a* to *b* is written *a : b* and is equal to $\dfrac{a}{b}$.

The sign : is really only $\div$ with the bar omitted.

Example. Ratio of 12 annas to 8 annas $=\dfrac{12}{8}=\dfrac{3}{2}$.

151. In the ratio *a : b*, *a* is the *first term* or *antecedent*, and *b* is the *second term* or *consequent*.

Ratios are *compounded* by being multiplied together. Thus the ratio compounded of *a   b* and *c : d* is $\dfrac{a}{b}\times\dfrac{c}{d}=\dfrac{ac}{bd}$, which may be written *ac   bd.*

A ratio compounded with itself is called the *duplicate ratio.* The duplicate ratio of *a : b* is $a^2 : b^2$.

Similarly, the *triplicate ratio* of *a : b* is $a^3 : b^3$

EXERCISE 98

Find the ratio of :—

1. 4 annas to Rs. 3.   [*Solved*]
2. 8 ft. to 2 yds.   3.   100 yds. to half a mile.
4. 4 annas to 6 yards.

Find the value of x, if :—

5. The ratio of $2x+5$ to $5x+2$ be equal to $7:8$. [*Solved*]
6. The ratio of $4+x$ to $8x-4$ be equal to $3:5$.
7. The duplicate ratio of $2x+3$ to $x-1$ be equal to 4.

<div align="right">[*Hint*]</div>

8. The duplicate ratio of $3x$ to $2x-5$ be $9:4$.

Find the ratio $x:y$ from the following equations :—

9. $4x-6y=0$. [*Solved*]   10. $8x-20y=0$.
11. $4x^2-20xy+25y^2=0$. [*Hint*]
12. $9x^2-24xy+16y^2=0$.
13. $6x^2-19xy+15y^2=0$. [*Solved*]
14. $8x^2-26xy+15y^2=0$.   15. $12x^2-23xy+10y^2=0$.
16. $15(x^2+y^2)=34xy$.

---

17. If the ratio $7x-4y:8x+y$ be equal to $5:18$, find the ratio $x:y$. [*Solved*]
18. If the ratio $8x+y:10x-y$ be equal to $5:4$, find the ratio $x:y$.
19. If $2x^2-3y^2:x^2+y^2=2:41$, find the ratio $x:y$.
20. If $6x^2-xy:2xy-y^2=6:1$, find the ratio $x:y$.

---

21. If $x:y=2:3$, evaluate $6x+5y:8x+4y$. [*Solved*]
22. If $x:y=3:4$, evaluate $7x-4y:2x+y$.
23. If $\dfrac{a}{b}=\dfrac{2}{3}$, evaluate $5a^2+b^2:2a^2+b^2$. [*Hint*]
24. If $\dfrac{a}{b}=\dfrac{3}{2}$, evaluate $2a^2+ab-b^2:a^2-ab+8b^2$.

---

25. If $3a-b:3a-2b=5:4$, find $5a+b:6a+7b$. [*Hint*]
26. If $a-4b:b-3a=3:2$, evaluate $a^2-ab+b^2:a^2+ab+b^2$.

27. What number must be added to each term of the ratio 17 · 21 to make at equal to 6 : 7. [Hint]

28. What number must be subtracted from each term of of the ratio 37 : 49 to make it equal to 5 : 8.

_____

29. Two numbers are in the ratio of 5 : 6 and their difference is equal to 136 ; find them. [Hint].

30. Two numbers are in the ratio of 4 : 3 and the difference of their squares is 63 ; find them.

31. Two numbers are in the ratio of 5 : 6 and if 5 be added to each of then, the sums are in the ratio of 6 : 7 ; find the numbers.

_____

32. If a is greater than b, prove that the ratio $\dfrac{a}{b}$ is greater than the ratio $\dfrac{a+x}{b+x}$, where a, b and x are positive quantities. [Solved]

Note. Another statement of this question is · " A ratio of greater inequality is diminished by adding the same positive qual.tity to both its terms."

33. If a is less than b, prove that the ratio $\dfrac{a}{b}$ is less than the ratio $\dfrac{a+x}{b+x}$ (a, b, x are all positive).

34. Show that a ratio of greater inequality is increased by taking the same positive quantity from both its terms.
[Hint]

35. Show that a ratio of less inequality is diminished by taking the same positive quantity from both its terms.

_____

SOLUTIONS & HINTS—EXERCISE 98

1. 1st Quantity=4 annas
   2nd „ =Rs. 3=48 annas
   ∴ ·Reqd. Ratio=$\frac{4}{48}=\frac{1}{12}$ or 1 : 12

5. $\dfrac{2x+5}{5x+2}=\dfrac{7}{8}$ [given]

By cross multiplication　　　$35x+14=16x+40$

　　　　　　　　　or　$35x-16x=40-14$

　　　　　　　　　or　　　　$19x=26$

　　　　　　　　　or　　　　$x=\frac{26}{19}$.

7. The duplicate ratio of $2x+3$ to $x-1$

$$=\frac{(2x+3)^2}{(x-1)^2}=\frac{4x^2+12x+9}{x^2-2x+1}.$$

9. $4x-6y=0$

$\therefore$　　　$4x=6y$

$\therefore$　　　$\dfrac{x}{y}=\frac{6}{4}=\frac{3}{2}$

or　　$x:y=3:2$.

11. L. H. S. of the equation is $(2x-5y)^2$.

13. $6x^2-19xy+15y^2=0$　　[given]

Factorising L. H. S. we have :—

　　　$(2x-3y)(3x-5y)=0$.

Now, the product of two quantities will be zero if either of them is zero. Hence we have :—

　　　$2x-3y=0$　　or　　$3x-5y=0$.

The first equation gives $2x=3y$ $\therefore$ $\dfrac{x}{y}=\frac{3}{2}$ or $x:y=3:2$

The second equation gives $3x=5y$ $\therefore$ $\dfrac{x}{y}=\frac{5}{3}$ or $x:y=5:3$.

17. $\dfrac{7x-4y}{3x+y}=\dfrac{5}{13}$　　[given]

$\therefore$　$91x-52y=15x+5y$　　[By cross multiplication]

$\therefore$　$91x-15x=5y+52y$

or　　　$76x=57y$

or　　　$\dfrac{x}{y}=\frac{57}{76}=\frac{3}{4}$

i.e.,　　$x:y=3:4$.

21. Reqd. Ratio $= \dfrac{6x+5y}{3x+4y} = \dfrac{6\left(\dfrac{x}{y}\right)+5}{3\left(\dfrac{x}{y}\right)+4}$

[Dividing numerator and denominator by y]

$= \dfrac{6 \times \frac{2}{3}+5}{3 \times \frac{2}{3}+4} = \dfrac{4+5}{2+4} = \dfrac{9}{6} = \dfrac{3}{2}$ or $3:2$.

Or thus :—

$\dfrac{x}{y} = \dfrac{2}{3}$ [given]

$\therefore \quad \dfrac{x}{2} = \dfrac{y}{3}$

Let each of the ratios $\dfrac{x}{2}$ and $\dfrac{y}{3}$ be equal to k

$\therefore \quad x=2k$ and $y=3k$.

$\therefore \quad \dfrac{6x+5y}{3x+4y} = \dfrac{6 \times 2k+5 \times 3k}{3 \times 2k+4 \times 3k} = \dfrac{12k+15k}{6k+12k} = \dfrac{27k}{18k} = \dfrac{3}{2}$ or $3:2$.

23. Divide the numerator and denominator of the required ratio by b^2.

25. First find the value of $\dfrac{a}{b}$, then proceed as in Q. 21.

27. Suppose the reqd. number is x.

Then, according to the condition of the question we have :—

$\dfrac{17+x}{21+x} = \dfrac{6}{7}$, etc.

29. Let the numbers be x and y

Then $\dfrac{x}{y} = \dfrac{5}{9}$ or $\dfrac{x}{5} = \dfrac{y}{9} = k$ (suppose)

$\therefore \quad x=5k$ and $y=9k$.

$\therefore$ According to the condition of the question $9k-5k$
$=136$, etc.

32. *Given :* a, b, x are positive and $a > b$.

To prove : $\dfrac{a}{b} > \dfrac{a+x}{b+x}$ or $\dfrac{a}{b} - \dfrac{a+x}{b+x}$ is positive.

Proof : $\dfrac{a}{b} - \dfrac{a+x}{b+x} = \dfrac{ab+ax-ab-bx}{b(b+x)} = \dfrac{x(a-b)}{b(b+x)}.$

In this result the factors x, b and $b+x$ are clearly positive ; also $a-b$ is positive ($\because$ $a>b$). Therefore the result is positive. [Q. E. D.]

34. Let the ratio be $\dfrac{a}{b}$ $(a>b)$.

Then, if x be a positive quantity, it is required to show that $\dfrac{a}{b}$ is less than $\dfrac{a-x}{b-x}.$

152. Equality of Ratios.

Suppose $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c} = \ldots\ldots\ldots\ldots$

If we put each of these ratios equal to a single letter, say ' k ', we get $x=ak$ $y=bk$, $z=ck\ldots\ldots$ Many useful propositions can be proved by making use of these values of the numerators. [See Ex. 99]

Note. The above equality of ratios can also be expressed as follows :—

$$x : y : z : \ldots\ldots\ldots = a : b : c : \ldots\ldots\ldots$$

153. Proportion.

Four quantities are said to be in proportion when the ratio of the first to the second is equal to the ratio of the third to the fourth. Thus the four numbers a, b, c, d are in proportion if $a : b = c : d$. This equality may also be written thus :—

$$a : b :: c : d \quad \text{or} \quad \dfrac{a}{b} = \dfrac{c}{d}.$$

The proportion $a : b :: c : d$ is read " a is to b as c is to d."

The quantities a, b, c, d are called *terms* of the proportion, a being the *first term*, b the *second*. c the *third*, and d the *fourth*.

The first and the fourth terms (*viz.*, *a* and *d*) are called the *extremes* (end terms), and the second and the third terms (*b* and *c*) the *means* (middle terms).

The fourth term (*d*) is called *the fourth proportional to* *a, b, c.*

154. Theorem. *If four quantities be in proportion, the product of the extremes is equal to the product of the means.*

Let *a, b, c, d* be the four quantities. Since they are in proportion, we have :—

$$\frac{a}{b} = \frac{c}{d}$$

By cross multiplication, we get $ad = bc$, which proves the theorem.

155. Converse of the above theorem.—*If the product of two quantities be equal to the product of two other quantities, the four quantities (taken in proper order) are in proportion.*

Let $ad = bc$.

Dividing both sides by *bd* we get :—

$$\frac{a}{b} = \frac{c}{d}.$$

∴ *a, c, b, d* are in proportion.

Note. These quantities are in proportion in the following orders as well :—

a, b, c, d ; *b, a, d, c* ; *b, d, a, c* ; *d, c, b, a* ; etc. etc.

156. Continued Proportion.

Quantities $a, b, c, d, e,$........................are said to be in *continued proportion* when $a : b = b : c = c : d = d : e = $............

Thus 162, 54, 18, 6, 2 are in continued proportion, for $\frac{162}{54} = \frac{54}{18} = \frac{18}{6} = \frac{6}{2}$ [each $= 3$]

157. When **three quantities** are in **continued proportion,** the second is called the *mean proportional* between the first and the third ; also the third is called the *third proportional* to the first and the second.

For example, the numbers 2, 6, 18 are in continued proportion, because $\frac{2}{6} = \frac{6}{18}$ or $2 : 6 : : 6 : 18$.

Here 6 is the mean proportional between 2 and 18, and 18 is the third proportional to 2 and 6.

158. **Theorem.**—*Mean proportional between two given quantities is equal to the square root of their product.*

Let x be the mean proportional between a and b, then a, x, b are in continued proportion.

$$\therefore \quad \frac{a}{x} = \frac{x}{b}$$

$$x^2 = ab \quad \therefore \quad x = \sqrt{ab}.$$

159. **If** $\frac{a}{b} = \frac{c}{d}$, we have the following important results :—

(i) $\dfrac{b}{a} = \dfrac{d}{c}$ called *Invertando.*

(ii) $\dfrac{a}{c} = \dfrac{b}{d}$,, *Alternando.*

(iii) $\dfrac{a+b}{b} = \dfrac{c+d}{d}$,, *Componendo.*

(iv) $\dfrac{a-b}{b} = \dfrac{c-d}{d}$,, *Dividendo.*

(v) $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$,, *Componendo & Dividendo.*

Proofs :—

(i) $\dfrac{a}{b} = \dfrac{c}{d}$ [given]

$$\therefore \quad 1 \div \frac{a}{b} = 1 \div \frac{c}{d}$$

$$\therefore \quad \frac{b}{a} = \frac{d}{c}.$$

(ii) $\dfrac{a}{b} = \dfrac{c}{d}$,   [given]

$$\therefore \quad \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c} \quad \left[\text{Multiplying both sides by } \frac{b}{c} \right]$$

or $\dfrac{a}{c} = \dfrac{b}{d}$.

(iii) $\dfrac{a}{b} = \dfrac{c}{d}$ [given]

$\because$ $\dfrac{a}{b} + 1 = \dfrac{c}{d} + 1$ [Adding 1 to both sides]

$\therefore$ $\dfrac{a+b}{b} = \dfrac{c+d}{d}$.

(iv) $\dfrac{a}{b} = \dfrac{c}{d}$. [given]

$\therefore$ $\dfrac{a}{b} - 1 = \dfrac{c}{d} - 1$. [Subtracting 1 from both sides]

$\therefore$ $\dfrac{a-b}{b} = \dfrac{c-d}{d}$.

(v) $\dfrac{a}{b} = \dfrac{c}{d}$ [given]

$\therefore$ $\dfrac{a+b}{b} = \dfrac{c+d}{d}$ [Componendo] ...(1)

and $\dfrac{a-b}{b} = \dfrac{c-d}{d}$. [Dividendo] ...(2)

Dividing (1) by (2) we get $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$.

160. If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c} \ldots\ldots\ldots\ldots$, then each of these

tios is equal to $\left(\dfrac{px^n + qy^n + rz^n + \ldots\ldots\ldots}{pa^n + qb^n + rc^n + \ldots\ldots\ldots}\right)^{\frac{1}{n}}$

This is almost Q. 46 of the next exercise. The student ill be able to do it after attempting Questions 38 to 45.

If we take $p = q = r = \ldots\ldots = 1$ and also $n = 1$, we get the llowing simple but important theorem :—

" If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c} = \ldots\ldots\ldots\ldots$, then each of these ratios

equal to $\dfrac{x + y + z + \ldots\ldots\ldots\ldots}{a + b + c + \ldots\ldots\ldots}$. "

EXERCISE 99 (a)

Find a fourth proportional to :—

1. 2, 3, 4. [*Solved*]   2. 6, 8, 9.

3. $2ab$, a^2, b^2.   4. a^2-b^2, a^3+b^3, a^3-b^3.

_____

Find the third proportional to :—

5. 4, 6. [*Solved*]   6. 5, 10.

7. $\dfrac{a}{b}+\dfrac{b}{a}$, $\sqrt{a^2+b^2}$.   8. $\sqrt{3}+1$, 2.

_____

Find a mean proportional between :—

9. 2 and 32. [*Solved*]   10. ab^3 and a^3b.

11. $4-\sqrt{7}$ and $4+\sqrt{7}$.   12. $\dfrac{a^2+ab}{c}$ and $\dfrac{ac+bc}{a}$

13. $a+b$ and a^3+b^3.   14. $a^2-\dfrac{1}{b^3}$ and $b^2-\dfrac{1}{a^3}$.

_____

15. What number must be added to each of the numbers 1, 5, 7, 17 to give four numbers in proportion ? [*Hint*]

16. What number must be subtracted from each of the numbers 12, 16, 20, 28 that the remainders may be in proportion ?

17. What number must be added to each of the numbers 14, 6, 2 so that the sums may be in continued proportion ? [*Hint*]

18. What must be subtracted from each of the numbers 13, 19. 2 so that the remainders may be in continued proportion ?

_____

SOLUTIONS & HINTS—EXERCISE 99 (a)

1. Let the fourth proportional to 2, 3 and 4 be x. Then we have :— $\dfrac{2}{3}=\dfrac{4}{x}$

∴ By cross multiplication $2x=12$ or $x=6$.

5. Let the third proportional to 4 and 6 be x. Then we have :—

$$\frac{4}{6} = \frac{6}{x}$$

∴ By cross multiplication $4x = 36$ or $x = 9$.

9. Let the mean proportional between 2 and 32 be x. Then we have :—

$$\frac{2}{x} = \frac{x}{32}$$

∴ By cross multiplication, $x^2 = 64$ or $x = \sqrt{64} = 8$.

15. Let the reqd. Number be x. Then $1+x, 5+x, 7+x$ $17+x$ are in proportion. Hence we have :—

$$\frac{1+x}{5+x} = \frac{7+x}{17+x}$$

Solve this equation for x.

17. Let the reqd. number be x, then $14+x, 6+x, 2+x$ are in continued proportion. Hence we have :—

$$\frac{14+x}{6+x} = \frac{6+x}{2+x}$$

Solve for x.

— — —

EXERCISE 99 (b)

If $a : b :: c : d$, prove that :—

1. $a(c+d) = c(a+b)$. [Solved]
2. $b(c-d) = d(a-b)$.
3. $(a+b)(c-d) = (c+d)(a-b)$.
4. $a\sqrt{c^2+d^2} = c\sqrt{a^2+b^2}$.
5. $\dfrac{2a+3b}{4a+5b} = \dfrac{2c+3d}{4c+5d}$ [Solved]
6. $\dfrac{3a-2c}{3b-2d} = \dfrac{5a-4c}{5b-4d}$.
7. $\dfrac{ma-nb}{a+b} = \dfrac{mc-nd}{3c+d}$
8. $\dfrac{ma-nc}{pa-qc} = \dfrac{mb-nd}{pb-qd}$
9. $\dfrac{a^2+b^2}{a^2-b^2} = \dfrac{c^2+d^2}{c^2-d^2}$ [Solved]
10. $\dfrac{a^2+ab+b^2}{a^2-ab+b^2} = \dfrac{c^2+cd+d^2}{c^2-cd+d^2}$
11. $\dfrac{a^3+c^3}{b^3+d^3} = \dfrac{ac^2}{bd^2}$
12. $\dfrac{pa^2+qc^2}{pb^2+qd^2} = \dfrac{ma^2-nc^2}{mb^2-nd^2}$

13. $\dfrac{(a-c)^2}{(b-d)^2} = \dfrac{a^2}{b^2}$. [Solved]
  14. $\dfrac{(a-c)^3}{(b-d)^3} = \dfrac{a^3}{b^3}$.

15. $(a+c)^3 : (b+d)^3 :: a^3+c^3 : b^3+d^3$. [Hint]

16. $a^2b^2 : c^2d^2 :: a^4-b^4 : c^4-d^4$.

17. $(a+c)^4 : (b+d)^4 = ac^3 : bd^3$.

18. $\dfrac{a^2}{b^2} + \dfrac{c^2}{d^2} = \dfrac{2ac}{bd}$.
         19. $\dfrac{a^2c+ac^2}{b^2d+bd^2} = \dfrac{(a+c)^3}{(b+d)^3}$

20. $\dfrac{(a+c)^3}{(b+d)^3} = \dfrac{a(a+c)^2}{b(b+d)^2}$.
         21. $\dfrac{a^2d-b^2c+b^2d}{a^2c+b^2d} = \dfrac{d}{c+d}$

22. $\dfrac{pa^2+qab+rb^2}{la^2+mab+nb^2} = \dfrac{pc^2+qcd+rd^2}{lc^2+mcd+nd^2}$.

If a, b, c, d be in proportion prove that :—

23. $abcd\left(\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2} + \dfrac{1}{d^2}\right) = a^2+b^2+c^2+d^2$.   [Solved]

24. $4(a+b)(c+d) = bd\left(\dfrac{a+b}{b} + \dfrac{c+d}{d}\right)^2$.

25. $a^2+b^2 : \dfrac{a^3}{a+b} = c^2+d^2 : \dfrac{c^3}{c+d}$.   [Hint]

26. $\dfrac{a}{l} + \dfrac{b}{m} : a = \dfrac{c}{l} + \dfrac{d}{m} : c$.

27. $\dfrac{a}{b} + \dfrac{b}{a} : \dfrac{ab}{a^2+b^2} = \dfrac{c}{d} + \dfrac{d}{c} : \dfrac{cd}{c^2+d^2}$

If $a : b = c : d$, show that :—

28. $a+b : c+d = \sqrt{a^2+b^2} : \sqrt{c^2+d^2}$ [Solved]

29. $\sqrt{a^2+c^2} : \sqrt{b^2+d^2} = c : d$.

30. $\sqrt{a^2+ab+b^2} : \sqrt{c^2+cd+d^2} = a^2+b^2 : ac+bd$.

31. $\sqrt{a^4+c^4} : \sqrt{b^4+d^4} = la^2+mc^2 : lb^2+md^2$.

32. $\sqrt{a^2-b^2} + \sqrt{a^2+b^2} : \sqrt{c^2-d^2} + \sqrt{c^2+d^2} = b : d$.

If $a : b = c : d = e : f$, show that :—

33. $\dfrac{ma+nb}{mc+nd} = \dfrac{b^2c}{d^2a}$. [Solved]   34. $\dfrac{a^2+c^2+e^2}{b^2+d^2+f^2} = \dfrac{ac}{bd}$

35. $a^3+c^3+e^3 : b^3+d^3+f^3 = ace : bdf$.

36. $pa+qc+re : pb+qd+rf = \sqrt[3]{ace} : \sqrt[3]{bdf}$.

37. $27(a+b)(c+d)(e+f) = bdf\left(\dfrac{a+b}{b} + \dfrac{c+d}{d} + \dfrac{e+f}{f}\right)^3$

If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$ show that each of these ratios is equal to :—

38. $\dfrac{2a+3c+4e}{2b+3d+4f}$. [Solved.]   39. $\dfrac{a-c+e}{b-d+f}$.

40. $\dfrac{5a-7c-9e}{5b-7d-9f}$.   41. $\dfrac{ka+lc+me}{kb+ld+mf}$.

If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$, show that each of these ratio is equal to :—

42. $\dfrac{(lx^2-my^2+nz^2)^{\frac{1}{2}}}{(la^2-mb^2+nc^2)^{\frac{1}{2}}}$. [Solved.

43. $\sqrt{\dfrac{pxy+qyz-rzx}{pab+qbc-rca}}$.

44. $\left(\dfrac{x^{-2}+y^{-2}+z^{-2}}{a^{-2}+b^{-2}+c^{-2}}\right)^{-\frac{1}{2}}$ 45. $\sqrt[3]{\left(\dfrac{px^{-3}+qy^{-3}+rz^{-3}}{pa^{-3}+qb^{-3}+rc^{-3}}\right)^{-1}}$

46. $\left(\dfrac{px^n+qy^n+rz^n}{pa^n+qb^n+rc^n}\right)^{\frac{1}{n}}$.

If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$, show that :—

47. $ax+by+cz$ is a mean proportional between $a^2+b^2+c^2$ and $x^2+y^2+z^2$. [Hint]

48. $\dfrac{x^3+y^3+z^3}{a^3+b^3+c^3} = \dfrac{xyz}{abc}$.

49. $\dfrac{(ax+by+cz)^3}{xyz} = \dfrac{(a^2+b^2+c^2)^3}{abc}$

50. $\left(\dfrac{a^2x^2+b^2y^2+c^2z^2}{a^3x+b^3y+c^3z}\right)^{\frac{3}{2}} = \sqrt{\dfrac{xyz}{abc}}$.

If $ax=by=cz$, show that :

51. $\dfrac{x^2}{yz}+\dfrac{y^2}{zx}+\dfrac{z^2}{xy} = \dfrac{bc}{a^2}+\dfrac{ca}{b^2}+\dfrac{ab}{c^2}$.   [*Solved.*]

52. $\dfrac{a^2+b^2+c^2}{x^2y^2+y^2z^2+z^2x^2} = \dfrac{ab+bc+ca}{xyz(x+y+z)}$.

53. $\dfrac{a^3+b^3+c^3}{x^{-3}+y^{-3}+z^{-3}} = abcxyz$.   [*Hint*]

54. $\left(\dfrac{a^2x^{-2}+b^2y^{-2}+c^2z^{-2}}{a^3x^{-1}+b^3y^{-1}+c^3z^{-1}}\right)^{\frac{3}{2}} = \dfrac{1}{\sqrt{abcxyz}}$.

SOLUTIONS & HINTS—Ex. 99 (b)

1.
I Method

$$\dfrac{a}{b} = \dfrac{c}{d} \quad \text{[given]}$$

$$\therefore \dfrac{b}{a} = \dfrac{d}{c} \quad \text{[Invertando]}$$

$$\therefore \dfrac{b+a}{a} = \dfrac{d+c}{c} \quad \text{[Componendo]}$$

$$\therefore c(b+a) = a(d+c). \quad \text{[Cross multiplication]}$$
$$\text{[Q. E. D.]}$$

II Method

To prove that $a(c+d)=c(a+b)$
or that $ac+ad = ca+cb$
or that $ad = cb$ [cancelling ac from both sides]

or that $\dfrac{ad}{bd} = \dfrac{cb}{bd}$ [dividing both sides by bd]

or that $\dfrac{a}{b} = \dfrac{c}{d}$, which is given.

Hence the required result.

III Method

$$\dfrac{a}{b} = \dfrac{c}{d} \text{ (given)} = k(\text{suppose})$$

$$\therefore \quad \begin{matrix} a = bk \\ c = dk \end{matrix} \Big\} \qquad \qquad \dots(i)$$

Now L. H. S. $= a(c+d)$

$\qquad\qquad\qquad = bk\,(dk+d)$ [Substituting the values of a and c from (i)]

$\qquad\qquad\qquad = bdk^2 + bdk.$

and, R. H. S. $= c(a+b)$

$\qquad\qquad\qquad = dk(bk+b)$ [substituting from (i)

$\qquad\qquad\qquad = bdk^2 + bdk.$

Since the two results are equal,

$\therefore$ L. H. S.=R. H. S.

5. $\dfrac{a}{b} = \dfrac{c}{d}$ (given)$= k$ (suppose)

$\therefore$ $a = bk$ and $c = dk$

$\therefore$ L. H. S.$= \dfrac{2a+3b}{4a+5b} = \dfrac{2bk+3b}{4bk+5b}$

$\qquad\qquad\qquad = \dfrac{b(2k+3)}{b(4k+5)} = \dfrac{2k+3}{4k+5}.$

and R. H. S. $= \dfrac{2c+3d}{4c+5d} = \dfrac{2dk+3d}{4dk+5d}$

$\qquad\qquad\qquad = \dfrac{d(2k+3)}{d(4k+5)} = \dfrac{2k+3}{4k+5}$

$\therefore$ the two results are equal

$\therefore$ L. H. S =R. H. S.

8. $a^2(b^3-c^3)+b^2(c^2-a^3)+c^2(a^3-b^3)=(a-b)(b-c)(c-a)\times$
$$(ab+bc+ca)$$

9. $ab(a^3-b^3)-bc(b^3-c^3)+ca(c^3-a^3)=-(a-b)(b-c)(c-a)$
$$\times(a^2+b^2+c^2+ab+bc+ca)$$

Besides the above results, the following may also be noted : they can be easily verified by opening brackets :—

10. $(a-b)+(b-c)+(c-a)=0$

11. $a(b-c)+b(c-a)+c(a-b)=0$

12. $a^2(b^2-c^2)+b^2(c^2-a^2)+c^2(a^2-b^2)=0$

13. $a^3(b^3-c^3)+b^3(c^3-a^3)+c^3(a^3-b^3)=0,$

      etc.

EXERCISE 77

Find the value of :—

1. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-a)(b-c)}+\dfrac{1}{(c-a)(c-b)}.$ [Solved]

2. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-a)(b-c)}+\dfrac{c}{(c-a)(c-b)}.$

3. $\dfrac{b}{(a-b)(a-c)}+\dfrac{c}{(b-c)(b-a)}+\dfrac{a}{(c-a)(c-b)}.$

4. $\dfrac{y+z}{(x-y)(x-z)}+\dfrac{z+x}{(y-z)(y-x)}+\dfrac{x+y}{(z-x)(z-y)}.$ [Hint]

5. $\dfrac{y-z}{(x-y)(x-z)}+\dfrac{z-x}{(y-z)(y-x)}+\dfrac{x-y}{(z-x)(z-y)}.$

6. $\dfrac{1+a}{(a-b)(a-c)}+\dfrac{1+b}{(b-c)(b-a)}+\dfrac{1+c}{(c-a)(c-b)}.$ [Hint]

7. $\dfrac{a-x}{(a-b)(a-c)}+\dfrac{b-x}{(b-a)(b-c)}+\dfrac{c-x}{(c-a)(c-b)}.$

8. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-a)(b-c)}+\dfrac{c^2}{(c-a)(c-b)}.$ [Solved]

9. $\dfrac{bc}{(a-b)(a-c)}+\dfrac{ca}{(b-c)(b-a)}+\dfrac{ab}{(c-a)(c-b)}.$

$$= \frac{bd^2k}{bk} + \frac{bd^2k^2}{b} + \frac{db^2k}{dk} + \frac{b^2dk^2}{d} \text{ [substituting from (i)}$$

$$= d^2 + d^2k^2 + b^2 + b^2k^2$$

and R. H. S. $= a^2 + b^2 + c^2 + d^2 = b^2k^2 + b^2 + d^2k^2 + d^2$

[substituting from (*i*)]

∵ The two results are equal,

∴ L. H. S. = R. H. S.

25. L. H. S. $\dfrac{\dfrac{a^2+b^2}{a^3}}{a+b} = \dfrac{(a^2+b^2)(a+b)}{a^3}$ etc.

28. Values of a and c as in Q. 5.

∴ L. H. S. $\dfrac{a+b}{c+d} = \dfrac{bk+b}{dk+d} = \dfrac{b(k+1)}{d(k+1)} = \dfrac{b}{d}$.

and R. H. S. $\dfrac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} \sqrt{\dfrac{a^2+b^2}{c^2+d^2}} = \sqrt{\dfrac{b^2k^2+b^2}{d^2k^2+d^2}}$

$$= \sqrt{\dfrac{b^2(k^2+1)}{d^2(k^2+1)}} = \sqrt{\dfrac{b^2}{d^2}} = \dfrac{b}{d}$$

∵ The two results are equal,

∴ L. H. S. = R. H. S.

33. $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$ (given) $= k$ (suppose)

∴ $a = bk, c = dk, e = fk$

Now, L. H. S. $= \dfrac{ma+nb}{mc+nd} = \dfrac{mbk+nb}{mdk+nd} = \dfrac{b(mk+n)}{d(mk+n)} = \dfrac{b}{d}$

and, R. H. S. $= \dfrac{b^2c}{d^2a} = \dfrac{b^2 \, dk}{d^2 \, bk} = \dfrac{b}{d}$

∵ The two results are equal

∴ L. H. S. = R. H. S.

38. $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$ (given) $= k$ (suppose)

∴ $a = bk, c = dk, e = fk.$

$\therefore$ The given Exp. $= \dfrac{2a+3c+4e}{2d+3d+4f} = \dfrac{2bk+3dk+4fk}{2b+3d+4f}$

$$= \dfrac{k(2b+3d+4f)}{(2b+3d+4f)} = k.$$

But each of the given ratios is also $= k$

$\therefore$ The given exp. = each of the given ratios.

42. $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$ (given) $= k$ (suppose)

$\therefore \quad x = ak, \; y = bk, \; z = k$

$\therefore$ The given Exp. $= \left(\dfrac{lx^2 - my^2 + nz^2}{la^2 - mb^2 + nc^2} \right)^{\frac{1}{2}}$

$$= \left(\dfrac{la^2 k^2 - mb^2 k^2 + nc^2 k^2}{la^2 - mb^2 + nc^2} \right)^{\frac{1}{2}}$$

$$= \left\{ \dfrac{k^2(la^2 - mb^2 + nc^2)}{(la^2 - mb^2 + nc^2)} \right\}^{\frac{1}{2}} = (k^2)^{\frac{1}{2}} = k$$

$$= \text{each of the given ratios.}$$

47. It is required to prove that

$$\dfrac{a^2 + b^2 + c^2}{ax + by + cz} = \dfrac{ax + by + cz}{x^2 + y^2 + z^2} \qquad \dots (i)$$

[See definition of 'mean proportional' from Art. 157]
Or that $(ax+by+cz)^2 = (a^2+b^2+c^2)(x^2+y^2+z^2)$ $\qquad \dots(ii)$
[Form (i) is better than Form (ii).]

51. $ax = by = cz = k$ (suppose)

$\therefore \quad x = \dfrac{k}{a}, \; y = \dfrac{k}{b}, \; z = \dfrac{k}{c}$

Substituting these values of x, y and z, we get

$$\text{L. H. S.} = \dfrac{\dfrac{k^2}{a^2}}{\dfrac{k}{b} \cdot \dfrac{k}{c}} + \dfrac{\dfrac{k^2}{b^2}}{\dfrac{k}{c} \cdot \dfrac{k}{a}} + \dfrac{\dfrac{k^2}{c^2}}{\dfrac{k}{a} \cdot \dfrac{k}{b}}$$

$$= \dfrac{k^2}{a^2} \times \dfrac{bc}{k^2} + \dfrac{k^2}{b^2} \times \dfrac{ca}{k^2} + \dfrac{k^2}{c^2} \times \dfrac{ab}{k^2}$$

$$= \dfrac{bc}{a^2} + \dfrac{ca}{b^2} + \dfrac{ab}{c^2} = \text{R. H. S}$$

53. Values of x, y, z as in Q. 3?

$$L.\,H.\,S. = \frac{a^3 + b^2 + c^3}{\left(\dfrac{k}{a}\right)^{-3} + \left(\dfrac{k}{b}\right)^{-3} + \left(\dfrac{k}{c}\right)^{-3}}$$

$$= \frac{a^3 + b^3 + c^3}{\left(\dfrac{a}{k}\right)^3 + \left(\dfrac{b}{k}\right)^3 \left(\dfrac{c}{k}\right)^3} \quad \text{etc.}$$

EXERCISE 99 (c)

If a, b, c be in continued proportion, prove that :—

1. $a^2 + b^2 : a^2 - b^2 :: a + c : a - c$. [*Solved*]
2. $a^2 + b \quad b^2 + c^2 :: a : c$.
3. $a^2 + ab \mid b^2 : b^2 + bc + c^2 = a : c$.
4. $(a+b)^2 \cdot (b+c)^2 = a : c$.
5. $\dfrac{1}{a+b} + \dfrac{1}{b+c} = \dfrac{1}{b}$.
6. $a^2 + b^2 + c^2 = (a+b+c)(a-b+c)$. [*Solved*]
7. $(a^2 - ab + b^2)(b^2 + bc + c^2) = b^2(a^2 + ac + c^2)$.
8. $\dfrac{a+b+c}{a-b+c} = \dfrac{(a+b+c)^2}{a^2+b^2+c^2}$.
9. $a^2 b^2 c^2 (a^{-3} + b^{-3} + c^{-3}) = a^3 + \quad + c^3$.
10. $\dfrac{a^2 - b^2 + c^2}{a^{-2} - b^{-2} + c^{-2}} = b^4$.

---

11. If b is a proportional between a and c, prove that $ab + bc$ is a mean proportional between $a^2 + c^2$ and $b^2 + c^2$.

[*Hint*]

12. If a, b, c be in continued proportion, show that a has to c the duplicate ratio of $a+b$ to $b+c$. [*Hint*]

13. If $x : y = y : z$, prove that :—

$$\frac{x^2 + y^2}{x+y} - \frac{y^2 + z^2}{y+z} = \frac{(x-y)^2 + (y-z)^2}{x-z}$$

[*Hint*]

14. If y is a mean proportional between x and z, show that

$$\frac{xyz(x+y+z)^3}{(xy+yz+zx)^3}=1. \quad [Hint]$$

_____

If a, b, c, d be in continued proportion, prove that :—

15. $a : d :: a^3 : b^3.$ [Solved]

16. $a-c : c :: b-d : d.$

17. $b+c$ is a mean proportional between $a+b$ and $c+d$.

18. $(b+c)(b+d)=(c+a)(c+d).$

19. $a^3+b^3+c^3 : b^3+c^3+d^3=a : d.$

20. $la^3+mb^3+nc^3 : lb^3+mc^3+nd^3=a : d.$

21. $(a+c)^4 : ac=(b+d)^4 : d^2.$

22. b^2+c^2 is a mean proportional between a^2+b^2 and c^2+d^2.

23. $a^2+b^2+c^2, ab+bc+cd$ and $b^2+c^2+d^2$ are in continued proportion.

24. $(a-d)^2=(b-c)^2+(c-a)^2+(b-d)^2.$

25. $\sqrt{(a+b+c)(b+c+d)}=\sqrt{ab}+\sqrt{bc}+\sqrt{cd}.$

26. If a, b, c be in continued proportion and if $a(b-c)=2b$,

prove that $a-c=\dfrac{2(a+b)}{a}.$ [Solved]

27. If a, b, c be in continued proportion and $a+c=2b$, prove that $a^2(b+c)=a^3+b^3.$

_____

SOLUTIONS & HINTS—EXERCISE 99 (c)

1. $\because$ a, b, c are in continued proportion,

$\therefore \qquad \dfrac{a}{b}=\dfrac{b}{c}=\dfrac{1}{k}$ (suppose)

$\therefore \qquad b=ak$ ...(i)

and $\qquad c=bk=ak.k=ak^2$ ...(ii)

It is reqd. to prove that $\dfrac{a^2+b^2}{a^2-b^2}=\dfrac{a+c}{a-c}$

Now, $\dfrac{a^2+b^2}{a^2-b^2}=\dfrac{a^2+a^2k^2}{a^2-a^2k^2}$ [using result (i)]

$$=\frac{a^2(1+k^2)}{a^2(1-k^2)}=\frac{1+k^2}{1-k^2} \qquad \ldots(iii)$$

and $\dfrac{a+c}{a-c}=\dfrac{a+ak^2}{a-ak^2}$ [using result (ii)]

$$=\frac{a(1+k^2)}{a(1-k^2)}=\frac{1+k^2}{1-k^2} \qquad \ldots(iv)$$

From (iii) and (iv) we have $\dfrac{a^2+b^2}{a^2-b^2}=\dfrac{a+c}{a-c}$ [Q. E. D.]

6.   Values of b and c as in Q. 1.

Now, L. H. S. $=a^2+b^2+c^2$

$\qquad\qquad =a^2+(ak)^2+(ak^2)^2$ $[\because\quad b=ak,\ c=ak^2]$

$\qquad\qquad =a^2+a^2k^2+a^2k^4=a^2(1+k^2+k^4)$

and   R. H. S. $=(a+b+c)(a-b+c)$

$\qquad\qquad =(a+ak+ak^2)(a-ak+ak^2)$

$\qquad\qquad =a^2(1+k+k^2)(1-k+k^2)=a^2(1+k^2+k^4)$

$\therefore$ The two results are equal.

$\therefore$ L. H. S. = R. H. S.

11.   Given : $\dfrac{a}{b}=\dfrac{b}{c}$.

   To prove : $\dfrac{a^2+b^2}{ab+bc}=\dfrac{ab+bc}{b^2+c^2}$.

12.   To prove : $\dfrac{a}{c}=\dfrac{(a+b)^2}{(b+c)^2}$.

13. $\dfrac{x}{y}=\dfrac{y}{z}=\dfrac{1}{k}$ (suppose)

$\therefore$ $y=xk,\ z=xk^2$.   etc.

14.   *Either* substitute the values of y and z obtained in the last question.

                   Or thus :—

   Given : $\dfrac{x}{y}=\dfrac{y}{z}$ or $y^2=zx$.

The given exp. $= \dfrac{xyz(x+y+z)^3}{(xy+yz+zx)^3} = \dfrac{y \cdot y^2(x+y+z)^3}{(xy+yz+y^2)^3}$

[Replacing zx by y^2]

$= \dfrac{y^3(x+y+z)^3}{\{y(x+z+y)\}^3} = \dfrac{y^3(x+y+z)^3}{y^3(x+y+z)^3} = 1.$

15. $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d}$ (given) $= \dfrac{1}{k}$ (suppose)

$\therefore \quad b = ak$

$c = bk = ak \cdot k = ak^2$

$d = ck = ak^2 \cdot k = ak^3.$

Now, $\dfrac{a}{d} = \dfrac{a}{ak^3} = \dfrac{1}{k^3}$ ...(i)

and $\dfrac{a^3}{b^3} = \dfrac{a^3}{(ak)^3} = \dfrac{a^3}{a^3 k^3} = \dfrac{1}{k^3}$ ...(ii)

From (i) and (ii) we get $\dfrac{a}{d} = \dfrac{a^3}{b^3}$ or $a : d :: a^3 : b^3.$

26. Here we are given that :—

(i) $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{1}{k}$ (say), which gives $b = ak$, $c = ak^2$

(ii) $a(b-c) = 2b$, which gives

$a(ak - ak^2) = 2ak$ [substituting the values of b and c from (i)]

or $a^2 k(1-k) = 2ak$

or $a(1-k) = 2$ [Dividing both sides by ak].

Now, $a - c$ will be equal to $\dfrac{2(a+b)}{a}$

if $a - ak^2 = \dfrac{2(a+ak)}{a}$

[Substituting the values of b and c from (i).]

or if $a(1+k)(1-k) = 2(1+k)$ [Factorising L. H. S. and simplifying R. H. S.]

or if $a(1-k) = 2$ [Dividing both sides by $1+k$]

But this is result (ii) which is given to us.

$\therefore \quad a - c = \dfrac{2(a+b)}{a}.$

EXERCISE 99 (d)

1. If $6a+7b : 6c+7d : : 6a-7b : 6c-7d$, show that $a : b : : c : d$. [Solved]

2. If $2a-3b : 2a+3b = 2c-3d : 2c+3d$, show that $a : b : : c : d$.

3. If $(3a+8b)(3c-8d)=(3a-8b)(3c+8d)$, then will $a : b = c : d$. [Hint]

4. If $(a+b+2c+2d)(a-b-2c+2d)=(a-b+2c-2d) \times (a+b-2c-2d)$, show that a, b, c, d are proportionals. [Solved]

5. If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$, then a, b, c, d are in proportion.

6. If $(a+2b+3c+4d)(a-2b-3c+4d)=(a-2b+3c-4d) \times (a+2b-3c-4d)$, show that $a : b : : 3c : 2d$.

7. If $(2a+3b+4c+6d)(2a-3b-4c+6d)$
   $=(2a+3b-4c-6d)(2a-3b+4c-6d)$ prove that a, b, c, d are proportionals.

----

8. If $x=\dfrac{4ab}{a+b}$, find the value of $\dfrac{x+2a}{x-2a}+\dfrac{x+2b}{x-2b}$. [Solved]

9. If $x=\dfrac{2mn}{m+n}$, find the value of $\dfrac{x+m}{x-m}+\dfrac{x+n}{x-n}$.

10. If $x=\dfrac{2\sqrt{6}}{\sqrt{3}+\sqrt{2}}$, evaluate $\dfrac{x+\sqrt{3}}{x-\sqrt{3}}+\dfrac{x+\sqrt{2}}{x-\sqrt{2}}$. [Hint]

11. If $x=\dfrac{2\sqrt{35}}{\sqrt{5}+\sqrt{7}}$, evaluate $\dfrac{x+\sqrt{5}}{x-\sqrt{5}}+\dfrac{x+\sqrt{7}}{x-\sqrt{7}}$.

12. If $x=\dfrac{12mn}{m+n}$, evaluate $\dfrac{x+4n}{x-4n}-\dfrac{x+4m}{x-4m}$.

13. If $x=\dfrac{2ab}{b^2+1}$, evaluate $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$. [Hint]

14. If $x=\sqrt{\dfrac{m^4+1}{2}}$, evaluate $\dfrac{\sqrt{x^2+m^2}+\sqrt{x^2-m^2}}{\sqrt{x^2+m^2}-\sqrt{x^2-m^2}}$.

----

15. If $x = \dfrac{\sqrt{a+3b} + \sqrt{a-3b}}{\sqrt{a+3b} - \sqrt{a-3b}}$, show that $3bx^2 - 2ax + 3b = 0$.

[*Hint*].

16. If $x = \dfrac{\sqrt[3]{a+1} + \sqrt[3]{a-1}}{\sqrt[3]{a+1} - \sqrt[3]{a-1}}$, show that $x^3 - 3ax^2 + 3x - a = 0$.

———

Solve the following equations :—

17. $\dfrac{\sqrt{3} + \sqrt{3-x}}{\sqrt{3} - \sqrt{3-x}} = 3$. [*Solved*]

18. $\dfrac{\sqrt{7x} + \sqrt{4x-3}}{\sqrt{7x} - \sqrt{4x-3}} = 6$.

19. $\dfrac{2 - 2\sqrt{1-x}}{1 + \sqrt{1-x}} = 1$. *Hint*.

20. $\dfrac{6x + 2\sqrt{x^2-16}}{3x - \sqrt{x^2-16}} = 3$.

21. $\dfrac{6(\sqrt{4x+9} - \sqrt{2x+5})}{\sqrt{4x+9} + \sqrt{2x+5}} = 1$,

22. $\dfrac{1+x+x^2}{1-x+x^2} = \dfrac{62(1+x)}{63(1-x)}$

[*Hint*]

30. $\dfrac{14(1+x+x^2)}{13(1-x+x^2)} = \dfrac{1+x}{1-x}$.

24. $16\left(\dfrac{a-x}{a+x}\right)^3 = \dfrac{a+x}{a-x}$.

[*Hint*]

———

25. If $\dfrac{x}{b-c} = \dfrac{y}{c-a} = \dfrac{z}{a-b}$, show that $x+y+z=0$.   [*Solved*]

26. If $\dfrac{x}{b+c-2a} = \dfrac{y}{c+a-2b} = \dfrac{z}{a+b-2c}$ show that $x+y+z=0$.

27. If $\dfrac{a}{q+r-p} = \dfrac{b}{r+p-q} = \dfrac{c}{p+q-r}$, show that $a(q-r) + b(r-p) + c(p-q) = 0$.

28. If $\dfrac{a}{y+z} = \dfrac{b}{z+x} = \dfrac{c}{x+y}$, show that

$$\dfrac{a(b-c)}{y^2-z^2} = \dfrac{b(c-a)}{z^2-x^2} = \dfrac{c(a-b)}{x^2-y^2}.$$

———

29. If $\dfrac{a}{b+c} = \dfrac{b}{c+a} = \dfrac{c}{a+b}$, prove that each ratio is either equal to $\frac{1}{2}$ or -1. [*Solved*]

30. If $\dfrac{x}{y+z-x}=\dfrac{y}{z+x-y}=\dfrac{z}{x+y-z}$ prove that each ratio $=1$ or $-\frac{1}{2}$,

31. If $\dfrac{y+z}{ay+bz}=\dfrac{z+x}{az+bx}=\dfrac{x+y}{ax+by}$ and $x+y+z$ is not zero,

    then each ratio $=\dfrac{2}{a+b}$

32. If $\dfrac{x}{ax+by+cz}=\dfrac{y}{bx+cy+az}=\dfrac{z}{cx+ay+bx}$ and $x+y+z$

    is not zero, show that each ratio $=\dfrac{1}{a+b+c}$.

33. If $a+b:b+c=c+d:d+a$, show that either $a=c$ or $a+b+c+d=0$.

———

34. If $\dfrac{y+z-x}{a}=\dfrac{z+x-y}{b}=\dfrac{x+y-z}{c}$, prove that

    $\dfrac{x}{b+c}=\dfrac{y}{c+a}=\dfrac{z}{a+b}$. [Solved]

35. If $\dfrac{2y+2z-x}{a}=\dfrac{2z+2x-y}{b}=\dfrac{2x+2y-z}{c}$, show that

    $\dfrac{x}{2b+2c-a}=\dfrac{y}{2c+2a-b}=\dfrac{z}{2a+2b-c}$.

36. If $\dfrac{b+c-a}{x}=\dfrac{c+a-b}{y}=\dfrac{a+b-c}{z}$, show that

    $\dfrac{y+z+2x}{b+c}=\dfrac{z+x+2y}{c+a}=\dfrac{x+y+2z}{a+b}$.

37. If $(x+y+z)a=(y+z-x)b=(z+x-y)c=(x+y-z)d$,

    prove that $\dfrac{1}{a}=\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}$ [Hint]

38. If $(a^2+b^2)(x^2+y^2)=(ax+by)^2$, prove that x, a, y, b are proportional. [Solved]

39. If $a^2+c^2:b^2+d^2::ac:bd$, show that a, b, c, d are proportional.

40. If $2a+3c : 3a-4c = 2b+3d : 3b-4d$, show that a, b, c, d are in proportion.

41. If $ax+by+cz$ is a mean proportional between $x^2+y^2+z^2$ and $a^2+b^2+c^2$, show that $x : a = y : b = z : c$.

[Hint]

SOLUTIONS AND HINTS EX. 99 (d)

1. $\dfrac{6a+7b}{6c+7d} = \dfrac{6a-7b}{6c-7d}$.   [given]

$\therefore \dfrac{6a+7b}{6a-7b} = \dfrac{6c+7d}{6c-7d}$.   [alternando]

$\therefore \dfrac{12a}{14b} = \dfrac{12c}{14d}$.   [Comp. & Div.]

$\therefore \dfrac{a}{b} = \dfrac{c}{d}$ or $a : b :: c : d$.

3. Transfer $(3a-8b)$ to L. H. S. and $8c-8d$ to R. H. S.,

then : $\dfrac{3a+8b}{3a-8b} = \dfrac{3c+8d}{3c-8d}$, etc., etc.

4. The given relation may be written as :—

$$\dfrac{a+b+c+d}{a-b+c-d} = \dfrac{a+b-c-d}{a-b-c+d}$$

$\therefore \dfrac{2a+2c}{2b+2d} = \dfrac{2a-2c}{2b-2d}$ [Comp. & Div.]

or $\dfrac{a+c}{b+d} = \dfrac{a-c}{b-d}$

$\therefore \dfrac{a+c}{a-c} = \dfrac{b+d}{b-d}$ [Alternando]

$\therefore \dfrac{2a}{2c} = \dfrac{2b}{2d}$ [Comp. & Div.]

or $\dfrac{a}{c} = \dfrac{b}{d}$

$\therefore \dfrac{a}{b} = \dfrac{c}{d}$ [Alternando]

$\therefore$ a, b, c, d are proportionals.

8. $x = \dfrac{4ab}{a+b}$ [given]   ...(i)

∴ $\dfrac{x}{2a} = \dfrac{2b}{a+b}$ [Dividing both sides by $2a$]

∴ $\dfrac{x+2a}{x-2a} = \dfrac{3b+a}{b-a}$ [Comp. & Div.]   .(ii)

Again. dividing both sides of (i) by $2b$, we get :—

$$\dfrac{x}{2b} = \dfrac{2a}{a+b}$$

∴ $\dfrac{x+2b}{x-2b} = \dfrac{3a+b}{a-b}$ [Comp. & Div.]   ...(iii)

Adding (ii) and (iii) we have :—

$$\dfrac{x+2a}{x-2a} + \dfrac{x+2b}{x-2b} = \dfrac{3b+a}{b-a} + \dfrac{3a+b}{a-b} = \dfrac{3b+a}{b-a} - \dfrac{3a+b}{b-a}$$

$$= \dfrac{3b+a-3a-b}{b-a} = \dfrac{2b-2a}{b-a} = \dfrac{2(b-a)}{(b-a)} = 2. \quad \textbf{Ans.}$$

10. $x = \dfrac{2\sqrt{6}}{\sqrt{3}+\sqrt{2}} = \dfrac{2\sqrt{3}\cdot\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \dfrac{2ab}{a+b}$ [where $a = \sqrt{3}$, $b = \sqrt{2}$]

Dividing both sides by b we get $\dfrac{x}{b} = \dfrac{2a}{a+b}$.   From

this get the value of $\dfrac{x+b}{x-b}$ by Comp. and Div.)

Similarly find the value of $\dfrac{x+a}{x-a}$.   Add the two

values.

13. $x = \dfrac{2ab}{b^2+1}$ [given]

∴ $\dfrac{x}{a} = \dfrac{2b}{b^2+1}$ [Dividing both sides by a]

∴ $\dfrac{a}{x} = \dfrac{b^2+1}{2b}$ [Invertando]

Now apply Componendo and Dividendo, take sq. root of
both sides, and apply Componendo and Dividendo
once again.

15.　Apply Componendo and Dividendo ; then square both sides and multiply.

17　$\dfrac{\sqrt{3}+\sqrt{3-x}}{\sqrt{3}-\sqrt{3-x}}=\dfrac{3}{1}$　[Given]

∴　$\dfrac{2\sqrt{3}}{2\sqrt{3-x}}=\dfrac{3+1}{3-1}$　[Comp. & Div.]

i.e.　$\dfrac{\sqrt{3}}{\sqrt{3-x}}=2$

∴　$\dfrac{3}{3-x}=4$　[Squaring both sides]

or　$3=12-4x$

or　$4x=12-3=9$

or　$x=\dfrac{9}{4}$　Ans.

19.　Dividing both sides of the given equation by 2, we get

$$\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}=\tfrac{1}{2}$$

Now apply Componendo and Dividendo, etc.

Note.　The student should note carefully why both sides of the given equation have been divided by 2

22.　Transferring $1+x$ and $1-x$ from R. H. S. to L. H. S. we get

$$\frac{(1+x+x^2)(1-x)}{(1-x+x^2)(1+x)}=\frac{62}{63}\quad\text{or}\quad\frac{1-x^3}{1+x^3}=\frac{62}{63}$$

Now apply Componendo and Dividendo ; etc.

24.　Transferring $a+x$ and $a-x$ to L. H. S. and 16 to R. H. S., we get :—

$$\left(\frac{a-x}{a+x}\right)^3\times\frac{a-x}{a+x}=\tfrac{1}{16}\quad\text{or}\quad\left(\frac{a-x}{a+x}\right)^4=\tfrac{1}{16}.$$

∴　$\dfrac{a-x}{a+x}=\tfrac{1}{2}$, etc.

25. $\dfrac{x}{b-c}=\dfrac{y}{c-a}=\dfrac{z}{a-b}=k$ (Suppose)

$\therefore$ $x=k(b-c)$

$y=k(c-a)$

$z=k(a-b)$

$\therefore$ $x+y+z=k(b-c)+k(c-a)+k(a-b)$

$\qquad\qquad =k(b+c+c-a+a-b)$

$\qquad\qquad =k\times o=0.$

29. $\dfrac{a}{b+c}=\dfrac{b}{c+a}=\dfrac{c}{a+b}$ [given]

$\therefore$ Each of these ratios$=\dfrac{a+b+c}{(b+c)+(c+a)+(a+b)}$ [Art. 160]

$\qquad\qquad\qquad =\dfrac{(a+b+c)}{2(a+b+c)}$

In this result we can cancel the factor $(a+b+c)$ only if $a+b+c$ is not equal to zero ($\because$ division by zero of the numerator and denominators is not admissible)

If $a+b+c$ is not$=0$, we have each ratio$=\frac{1}{2}$

But if $a+b+c=0$, we get $b+c=-a$

$\therefore$ First ratio$=\dfrac{a}{b+c}=\dfrac{a}{-a}=-1$

$\therefore$ Each ratio$=-1$ [$\because$ The three ratios are equal]

Hence each ratio$=-\frac{1}{2}$ if $a+b+c=0$

$\qquad\qquad$ and $=1$ if $a+b+c=0$ $\Big\}$ Q. E. D.

34. $\dfrac{y+z-x}{a}=\dfrac{z+x-y}{b}=\dfrac{x+y-z}{c}=\dfrac{1}{k}$ (suppose)

$\therefore$ $a=k(y+z-x)$

$b=k(z+x-y)$

$c=k(x+y-z)$

$\therefore$ $\dfrac{x}{b+c}=\dfrac{x}{k(z+x-y)+k(x+y-z)}=\dfrac{x}{k\times 2x}=\dfrac{1}{2k}$

Similarly $\dfrac{y}{c+a}=\dfrac{1}{2k}$ and $\dfrac{z}{a+b}=\dfrac{1}{2k}$

$\therefore \quad \dfrac{x}{b+c}=\dfrac{y}{c+a}=\dfrac{z}{a+b}.$

37. $(x+y+z)a=(y+z-x)b=(z+x-y)c=(x+y-z)d$

$$=\frac{1}{k} \quad \text{(Suppose)}$$

$$\frac{1}{a}=k(x+y+z)$$

$$\frac{1}{b}=k(y+z-x)$$

$$\frac{1}{c}=k(z+x-y)$$

$$\frac{1}{d}+k(x+y-z) \text{ etc.}$$

38. $(a^2+b^2)(x^2+y^2)=(ax+by)^2$ [Given],

$\therefore \quad a^2x^2+a^2y^2+b^2x^3+b^2y^2=a^2x^2+b^2y^2+2abxy$

$\therefore \quad a^2y^2+b^2y^2=2abxy$

or $a^2y^2+b^2x^2-2abxy=0$

or $(ay-bx)^2-0$

$\therefore \quad ay-bx=0$

or $bx=ay$

$\therefore \quad \dfrac{x}{a}=\dfrac{y}{b}$ [Dividing both sides by ab]

i.c., x, a, y, b are proportionals.

41. $(x^2+y^2+z^2)(a^2+b^2+c^2)=(ax+by+cz)^2$ [given]

Open the brackets, transfer all terms to L. H. S. and simplify, we get :—

$$(bx-ay)^2+(az-cx)^2+(cy-bz)^2=0$$

L. H. S. is the sum of three terms, none of which is negative [a squared quantity can never be negative].

Hence their sum can be equal to zero only if each of them is zero. This gives :—

$$bx - ay = 0, \quad az - cx = 0, \quad cy - bz = 0$$

$$\therefore \quad \frac{x}{a} = \frac{y}{b}, \quad \frac{x}{a} = \frac{z}{c}, \quad \frac{y}{b} = \frac{z}{c}$$

$$\therefore \quad \frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$$

CHAPTER XXIII

ELIMINATION

161. Literally **elimination** means *removing, throwing off, or getting rid of,* but algebraically the term is used for a process by means of which we find out the condition, under which a certain set or system of equations may be *simultaneously true or consistent.*

For example, $ax + b = 0$ is true *only* if $x = -\dfrac{b}{a}$

and $cx + d = 0$ is true *only* if $x = -\dfrac{d}{c}$

Hence both equations will be *simultaneously true* if

$$-\frac{b}{a} = -\frac{d}{c} \quad \text{or} \quad \frac{b}{a} = \frac{d}{c} \quad \text{or} \quad ad = bc.$$

Here x has been *thrown out* or *eliminated* and the result derived is that *the two equations will be consistent if $ad = bc$.*

This condition (*viz.,* $ad = bc$ or $ad - bc = 0$) is called the **eliminant** of the given equations, $ax + b = 0$ and $cx + d = 0$.

An exhaustive treatment of elimination is beyond the scope of this book. We shall take typical cases only, providing, as usual, with a sufficient number of solutions for the guidance of the student.

EXERCISE 100 (a)

Eliminate x from :—

1. $\left. \begin{array}{l} ax + b = 0 \\ cx + d = 0 \end{array} \right\}$ *[Solved]*

2. $\left. \begin{array}{l} lx - m = 0 \\ px - q = 0 \end{array} \right\}$

3. $lx+m+n=0$ $\}$
   $a+b+cx=0$ $\}$

4. $px=qx+r$ $\}$ [Hint]
   $mx-nx=k$ $\}$

5. $ax+b=0$ $\}$ [Solved]
   $px^2+q=0$ $\}$

6. $x+a=0$ $\}$
   $bx^2+c=0$ $\}$

7. $px+q=0$ $\}$
   $ax^2+b=0$ $\}$

8. $a^2x^2-b^2=0$ $\}$
   $cx-d=0$ $\}$

9. $ax^2-bx+c=0$ $\}$
   $mx+l=0$ $\}$

10. $px^2+qx+r=0$ $\}$
    $qx-1=0$ $\}$

11. $x+y=a$ $\}$
    $x^2+y^2=b^2$ $\}$

12. $ax^3+bx^2+cx+d=0$ $\}$
    $lx+m=0$ $\}$

---

13. Eliminate t from the equations $x=at^2$ $\}$ (P. U. 1914)
    $y=2at$ $\}$

14. Eliminate r from the equations $x=ar^2+br+c$ $\}$
    $y=lr+m$ $\}$
    (P. U. 1918)

15. Eliminate r from the equations :
    $l(r-a)=m(r-b)$ $\}$
    $l(r-c)=m(r-d)$ $\}$ [Hint]
    (P. U. 1914)

16. Eliminate r from the equations :
    $l(r-a)=m(r-b)$ $\}$
    $p(r-c)=q(r-d)$ $\}$ Hint

17. $a-b=r+1$ $\}$ Eliminate r.
    $(1+r)^2=a-b$ $\}$

18. $\dfrac{b}{t}=d-\dfrac{t}{a}$ $\}$
    $\dfrac{t}{a}=b$ $\}$ Eliminate t

19. If $v=u+ft$ and $s=ut+\frac{1}{2}ft^2$, show that $v^2=u^2+2fs$
    [Solved]

20. If $v=u+ft$ and $v^2=u^2+2fs$, show that $s=ut+\frac{1}{2}ft^2$.

21. If $s=ut+\frac{1}{2}ft^2$ and $v^2=u^2+2fs$, show that $v=u+ft$.

22. If $ax+by+cz=0$ and $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=0$ show that
    $(ax+by)\left(\dfrac{a}{x}+\dfrac{b}{y}\right)=c^2.$ [Hint]

23. If $x+\dfrac{1}{y}=1$ and $y+\dfrac{1}{z}=1$, prove that $z+\dfrac{1}{x}=1$

[P. U. 1938]

Eliminate x from :—

24. $\left.\begin{array}{l} ax^4-b=0 \\ px^6-q=0 \end{array}\right\}$ [Solved]

25. $\left.\begin{array}{l} x^6-a=0 \\ x^3-b=0 \end{array}\right\}$ [Hint]

26. $\left.\begin{array}{l} lx^8+m=0 \\ px^{13}+q=0 \end{array}\right\}$

27. $\left.\begin{array}{l} ax^n-b=0 \\ cx^m-d=0 \end{array}\right\}$ [Hint]

---

28. $\left.\begin{array}{l} x+\dfrac{1}{x}=m \\[2mm] x-\dfrac{1}{x}=n \end{array}\right\}$ Eliminate x. [Solved]

29. $\left.\begin{array}{l} t+\dfrac{1}{t}=a \\[2mm] t-\dfrac{1}{t}=b \end{array}\right\}$ Eliminate t

30. $\left.\begin{array}{l} p+\dfrac{1}{p}=a+b \\[2mm] p-\dfrac{1}{p}=a-b \end{array}\right\}$ Eliminate t

31. $\left.\begin{array}{l} 5t+\dfrac{8}{t}=8c+4d \\[2mm] 5t-\dfrac{8}{t}=3c-4d \end{array}\right\}$ Eliminate t

32. $\left.\begin{array}{l} p+\dfrac{1}{p}=\dfrac{a}{m}+\dfrac{m}{a} \\[2mm] q+\dfrac{1}{q}=\dfrac{a}{m}-\dfrac{m}{a} \end{array}\right\}$ Eliminate m.

33. $\left.\begin{array}{l} x+\dfrac{1}{x}=2b \\[2mm] x^3-\dfrac{1}{x^3}=4ab \end{array}\right\}$ Eliminate x. [Hint]

34. $\left.\begin{array}{l} t-\dfrac{1}{t}=a+b \\[2mm] t^2-\dfrac{1}{t^2}=a^2-b^2 \end{array}\right\}$ Eliminate t

35. $\left.\begin{array}{l} a\left(t-\dfrac{1}{t}\right)=b \\[3mm] c\left(t+\dfrac{1}{t}\right)=d \end{array}\right\}$ Eliminate t.   [*Hint*]

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36. $\left.\begin{array}{l} x+\dfrac{1}{x}=a \\[2mm] x^2+\dfrac{1}{x^2}=b^2 \end{array}\right\}$ Eliminate x.   [*Solved*]

36-A. $\left.\begin{array}{l} x=t+\dfrac{1}{t} \\[3mm] y^2=t^2+\dfrac{1}{t^2} \end{array}\right\}$ Eliminate t.

37. $\left.\begin{array}{l} x+y=2a+\dfrac{1}{2a} \\[3mm] x^2+y^2=4a^2+\dfrac{1}{4a^2} \end{array}\right\}$ Eliminate a.

38. $\left.\begin{array}{l} xy+\dfrac{y}{x}=1 \\[3mm] zx^2+\dfrac{z}{x^2}=1 \end{array}\right\}$ Eliminate x.   [*Hint*]

39. $\left.\begin{array}{l} ax-\dfrac{a}{x}=b \\[3mm] bx^2+\dfrac{b}{x^2}=a \end{array}\right\}$ Eliminate x.

40. $\left.\begin{array}{l} t+\dfrac{1}{t}=k \\[3mm] t^4+\dfrac{1}{t^4}=l^4 \end{array}\right\}$ Eliminate t.   [*Hint*]

41. $\quad x - \dfrac{1}{x} = a$
$\qquad x^4 + \dfrac{1}{x^4} = b$
$\qquad\qquad$ Eliminate x.

42. $\quad \left(t - \dfrac{1}{t} \right) = 1$
$\qquad m\left(t^4 + \dfrac{1}{t^4} \right) = 1$
$\qquad\qquad$ Eliminate t.

_____

43. $\quad x + \dfrac{1}{x} = p$
$\qquad x^3 + \dfrac{1}{x^3} = q^3 - 3p$
$\qquad\qquad$ Eliminate x.

44. $\quad x - \dfrac{1}{x} = a$
$\qquad a^3 - \dfrac{1}{x^3} = b^3 + 3b$
$\qquad\qquad$ Eliminate x.

45. Eliminate l from :—

$$a\left(l + \dfrac{1}{l} \right) = b\left(l^3 + \dfrac{1}{l^3} \right) = 1. \quad [Hint)$$

46. Eliminate t from $\dfrac{t - \dfrac{1}{t}}{p} = \dfrac{t^3 - \dfrac{1}{t^3}}{q} = \dfrac{1}{pq}$

_____

47   Eliminate x from $\quad \begin{matrix} a_1 x^2 + b_1 x + c_1 = 0 \\ a_2 x^2 + b_2 x + c_2 = 0 \end{matrix} \Big\}$ [Solved]

48.   Eliminate z from $\quad \begin{matrix} z^2 + fz + x = 0 \\ z^3 + gz + y = 0 \end{matrix} \Big\}$

49.   Eliminate t from $\quad \begin{matrix} xt^2 - yt + z = 0 \\ x't^2 + y't - z' = 0 \end{matrix} \Big\}$

50.   Eliminate x from $\quad \begin{matrix} ax^2 + bx + c = 0 \\ x^2 + d = 0 \end{matrix} \Big\}$ [Hint]

_____

Eliminate x from :—

51. $\left.\begin{array}{l} a_1x^3+b_1x+c_1=0 \\ a_2x^3+b_2x+c_2=0 \end{array}\right\}$ [Hint]

52. $\left.\begin{array}{l} px^3-qx=r \\ lx^3=mx-n \end{array}\right\}$

53. $\left.\begin{array}{l} ax^3+bx+c=0 \\ cx^3+bx+a=0 \end{array}\right\}$ [Hint]

54. $\left.\begin{array}{l} ax^3+bx^2+c=0 \\ a'x^3+b'x^2+c'=0 \end{array}\right\}$ [Hint]

55. $\left.\begin{array}{l} a_1x^4+b_1x^3+c_1=0 \\ a_2x^4+b_2x^3+c_2=0 \end{array}\right\}$

———

Eliminate t :—

56. $\left.\begin{array}{l} x=\dfrac{a(1+t^2)}{1-t^2} \\[2mm] y=\dfrac{2bt}{1-t^2} \end{array}\right\}$ [Solved]

57. $\left.\begin{array}{l} x=\dfrac{1+t^2}{a(1-t^2)} \\[2mm] y=\dfrac{2t}{b(1-t^2)} \end{array}\right\}$

58. $\left.\begin{array}{l} (1-t^2)x=1+t^2 \\ (1-t^2)y=2t \end{array}\right\}$ [Hint]

59. $\left.\begin{array}{l} x=\dfrac{a(1-t^2)}{1+t^2} \\[2mm] y=\dfrac{2bt}{1+t^2} \end{array}\right\}$ [Hint]

60. $x(1+t^2)=1-t^2$
$y(1+t^2)=2t.$

———

61. Eliminate x from $\left.\begin{array}{l} x^3+1=2ax \\ x^3-1=2bx \end{array}\right\}$ [Hint]

62. Eliminate t from $\left.\begin{array}{l} t^3+\dfrac{3}{t}=4(a^3+b^3) \\[2mm] 3t+\dfrac{1}{t^3}=4(a^3-b^3) \end{array}\right\}$ [Hint].

———

SOLUTIONS & HINTS—EX. 100 (a)

1. $ax+b=0$ ...(i)
   $cx+d=0$ ...(ii)

From (i) $ax=-b$ $\therefore$ $x=-\dfrac{b}{a}$ ..(iii)

From (ii) $cx=-d$ $\therefore$ $x=-\dfrac{d}{c}$ ...(iv)

From (iii) and (iv) we have $-\dfrac{b}{a}=-\dfrac{d}{c}$ or $\dfrac{b}{a}=\dfrac{d}{c}$

or $ad=bc$, *which is the required eliminant.*

Second Method

Multiply   (i) by c we get $acx+bc=0$

  ,,   (ii) by a we get $acx+ad=0$

By subtraction $bc-ad=0$, *the reqd. eliminant.*

Third Method

From (i) $ax=-b$, $\therefore$ $x=-\dfrac{b}{a}$

Substituting this value in (ii) we get $c\left(-\dfrac{b}{a}\right)+d=0$

or $\dfrac{-bc}{a}+d=0$ or $-bc+ad=0$, *the reqd. eliminant.*

Fourth Method

From (i) $ax=-b$ ...(v)
From (ii) $cx=-d$ ...(vi)

Dividing (v) by (vi) we get $\dfrac{ax}{cx}=\dfrac{-b}{-d}$ or $\dfrac{a}{c}=\dfrac{b}{d}$

or $ad=bc$, *the reqd. eliminant.*

Fifth Method

From (i) $ax=-b$ ...(vii)
From (ii) $d=-cx$ ...(viii)

Multiplying (vii) and (viii), we get $adx=bcx$

$\therefore$ $ad=bc$ [Dividing both sides by x]

the reqd. eliminant.

Note 1. The object of giving so many methods for this simple question is to bring home to the student that we have to get rid of x *by any means whatever.* The student should carefully remember that :—

(i) *the result should be an equation, not an expression.*

(ii) *it should not contain the letter required to be eliminated.*

It should also be borne in mind that the result should be put in as simple a form as possible. If simplification is possible, it must be carried out. The indices of powers are generally required to be positive integers and not fractional or negative.

Note 2. The results obtained by different methods in the above example are all the same. There is absolutely no difference between the equations $ad=bc$, $bc-ad=0$, $-bc+ad=0$, etc.

4. The first equation gives $px-qx=r$ or $x(p-q)=r$

or $x=\dfrac{r}{p-q}$.

Similarly get the value of x from the second equation and equal the two values.

5. $ax+b=0$ ...(i)

$px^2+q=0$ ...(ii)

From (i) $ax=-b$ $\therefore$ $x=-\dfrac{b}{a}$

Substituting this value of x in (ii), we get :—

$$p\left(-\frac{b}{a}\right)^2+q=0$$

$\therefore$ $\dfrac{pb^2}{a^2}+q=0$

$\therefore$ $pb^2+qa^2=0$, *the reqd. eliminant.*

Or thus :—

From (i) $x=-\dfrac{b}{a}$ (as before), $\therefore$ $x^2=\dfrac{b^2}{a^2}$ [squaring]

From (ii) $px^2=-q$ $\therefore$ $x^2=-\dfrac{q}{p}$

Equating the two values of x^2 we get $\dfrac{b^2}{a^2} = -\dfrac{q}{p}$

$\therefore \quad b^2p = -a^2q \quad$ or $\quad b^2p + a^2q = 0.$

15. Open the brackets and subtract one equation from the other ; the terms containing r cancel out.

16. From the first equation, $lr - la = mr - mb$

$\therefore \quad lr - mr = la - mb \qquad \therefore \quad r(l - m) = la - mb.$

$$\therefore \quad r = \frac{la - mb}{l - m}.$$

Similarly find the value of r from the second equation and equate the two values.

1>. [In the required result there is no "t" Hence we have to eliminate t between the given equations]

$$v = u + ft \qquad \qquad \ldots(i)$$
$$s = ut + \tfrac{1}{2}ft^2 \qquad \qquad \ldots(ii)$$

From (i) $ft = v - u \qquad \therefore \quad t = \dfrac{v - u}{f}$

Substituting this value of t from (ii) we get :—

$$s = \frac{u(v-u)}{f} + \frac{f}{2}\left(\frac{v-u}{f}\right)^2 = \frac{uv - u^2}{f} + \frac{f(v^2 + u^2 - 2uv)}{2f^2}$$

or $\quad s = \dfrac{uv - u^2}{f} + \dfrac{v^2 + u^2 - 2uv}{2f}$

$\therefore \quad 2fs = 2uv - 2u^2 + v^2 + u^2 - 2uv$

[Multiplying both sides by $2f$]

or $\quad 2fs = v^2 - u^2 \quad$ or $\quad v^2 = u^2 + 2fs.$ \qquad [Q. E. D.]

Or thus :—[*Artifice*]

Squaring (i) we get $v^2 = u^2 + f^2t^2 + 2uft \qquad \ldots(iii)$

Also $(ii) \times 2f$ gives $2fs = 2fut + f^2t^2 \qquad \ldots(iv)$

From (iii) and (iv) by subtraction $v^2 - 2fs = u^2$

or $\quad v^2 = u^2 + 2fs.$ \quad [Q. E. D.]

Note. The student should note that the two results obtained above are the same.

22.   From the first equation $ax + by = -cz$

,,   second   ,, $\dfrac{a}{x} + \dfrac{b}{y} = -\dfrac{c}{z}$

Multiply these two equations.

24. $ax^4 - b = 0$ ...(i)

$px^6 - q = 0$ ...(ii)

From (i) $ax^4 = b$ $\therefore$ $x^4 = \dfrac{b}{a}$ $\therefore$ $(x^4)^3 = \left(\dfrac{b}{a}\right)^3$

or $x^{12} = \dfrac{b^3}{a^3}$

From (ii) $px^6 = q$ $\therefore$ $x^6 = \dfrac{q}{p}$ $\therefore$ $(x^6)^2 = \left(\dfrac{q}{p}\right)^2$

or $x^{12} = \dfrac{q^2}{p^2}$

Equating the two values of x^{12} we get

$\dfrac{b^3}{a^3} = \dfrac{q^2}{p^2}$ $\therefore$ $b^3 p^2 = a^3 q^2$; *the reqd. eliminant*

Note. We have obtained the value of x^{12} from each equation. Note that 12 is the L. C. M. of the two indices 4 and 6.

27   Find the value of x^{mn} from each equation and equate the two values.

28. $x + \dfrac{1}{x} = m$ ...(i)

$x - \dfrac{1}{x} = n$ ...(ii)

(i)+(ii) gives $2x = m + n$ ...(iii)

(i)−(ii) gives $\dfrac{2}{x} = m - n$ ...(iv)

(iii)×(iv) gives $2x \times \dfrac{2}{x} = (m+n)(m-n)$

or $4 = m^2 - n^2$, *the reqd. eliminant.*
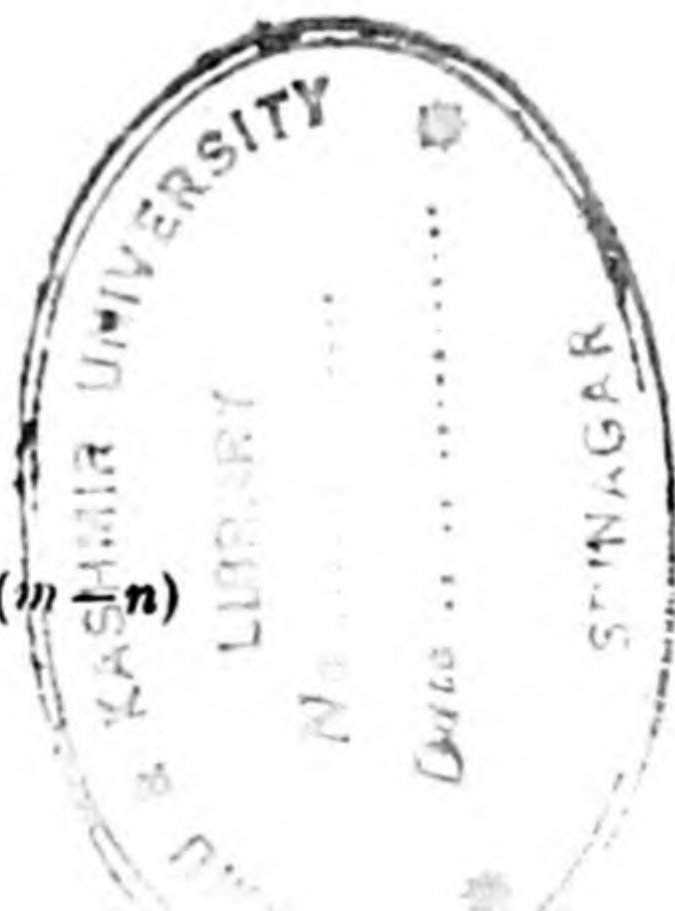
Or thus :—

Squaring (i) $x^2 + \dfrac{1}{x^2} + 2 = m^2$

Squaring (ii) $x^2 + \dfrac{1}{x^2} - 2 = n^2$

By subtraction $4 = m^2 - n^2$, *the reqd. eliminant.*

33. Dividing the second equation by the first we get

$$\dfrac{x^2 - \dfrac{1}{x^2}}{x + \dfrac{1}{x}} = \dfrac{4ab}{2a} \quad \text{or} \quad x - \dfrac{1}{x} = 2b$$

Take this equation and the first and proceed as in Q. 28.

35. From (i) $t - \dfrac{1}{t} = \dfrac{b}{a}$ [Dividing both sides by a]

Similarly from (2) $t + \dfrac{1}{t} = \dfrac{d}{c}$ Now proceed as before.

36. $x + \dfrac{1}{x} = a$ ...(1)

$x^2 + \dfrac{1}{x^2} = b^2$ ...(2)

Squaring (1), $x^2 + \dfrac{1}{x^2} + 2 = a^2$

$\therefore b^2 + 2 = a^2$, [Substituting the value of $x^2 + \dfrac{1}{x^2}$

from (2)]

the reqd. eliminant.

38. From the first equation $x + \dfrac{1}{x} = \dfrac{1}{y}$

[Dividing both sides by y]

,, ,, second ,, $x^2 + \dfrac{1}{x^2} = \dfrac{1}{z}$

[Dividing both sides by z]

etc.

40. Get the valu of $t^4 + \frac{1}{t^4}$ from the first equation; another value of $t^4 + \frac{1}{t^4}$ is given in the second equation. Equate the two values.

45. The given equation are $a\left(l + \frac{1}{l}\right) = 1$ and

$$b\left(l^3 + \frac{1}{l^3}\right) = 1$$

or $l + \frac{1}{l} = \frac{1}{a}$ and $l^3 + \frac{1}{l^3} = \frac{1}{b}$ etc.

47. $a_1 x^2 + b_1 x + c_1 = 0$ ...(i)
$a_2 x^2 + b_2 x + c_2 = 0$ ...(ii

By cross multiplication,

$$\frac{x^2}{b_1 c_2 - c_1 a_2} = \frac{x}{ -a_1 c_2} = \frac{1}{a_1 b_2 - b_1 a_2}$$

$$x^2 = \frac{b_1 c_2 - c_1 a_2}{a_1 b_2 - b_1 a_1} \qquad ...(iii)$$

and $x = \frac{c_1 a_2 - a_1 c_2}{a_1 b_2 - b_1 a_2}$

∴ Squaring $x^2 = \frac{(c_1 a_2 - a_1 c_2)^2}{(a_1 b_2 - b_1 a_2)^2}$ ...(iv)

Equating the values of x^2 from (ii) and (iv) we get :—

$$\frac{b_1 c_2 - c_1 b_2}{a_1 b_2 - b_1 a_2} = \frac{(c_1 a_2 - a_1 c_2)^2}{(a_1 b_2 - b_1 a_2)^2}$$

Multiplying both sides by $(a_1 b_2 - b_1 a_2)^2$ we get :—

$(b_1 c_2 - c_1 b_2)(a_1 b_2 - b_1 a_2) = (c_1 a_2 - a_1 c_2)^2$, *the reqd. eliminant.*

50 Write the second equation as $x^2 + 0x + d = 0$ and cross multiply.

51. Get the values of x^3 and x by cross multiplication Cube the second value to get another value of x^3 Equate the two values of x^3.

53. Either proceed as in Q. 51 or thus :—

[*Artifice*]

From the given equations, by subtraction,

$(a-c)x^3+(c-a)=0$ or $(a-c)x^3=(a-c)$ or $x^3=1$

[Dividing both sides by $(a-c)$] $\therefore$ $x=1$

Substituting this value of x in the first equation we get

$$a+b+c=0.$$

54. By cross multiplication, get the values of x^3 and x^2 Square the first value and cube the second, thus getting two values of x^6. Equate these values.

56. $x=\dfrac{a(1+t^2)}{1-t^2}$ ...(i)

$y=\dfrac{2bt}{1-t^2}$ ...(ii)

From (i) $\dfrac{x}{a}=\dfrac{1+t^2}{1-t^2}$ $\therefore$ $\dfrac{x^2}{a^2}=\dfrac{(1+t^2)^2}{(1-t^2)^2}$ [Squaring]

From (ii) $\dfrac{y}{b}=\dfrac{2t}{1-t^2}$ $\therefore$ $\dfrac{y^2}{b^2}=\dfrac{4t^2}{(1-t^2)^2}$ [Squaring]

By subtraction we get

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{(1+t^2)^2-4t^2}{(1-t^2)^2}=\frac{(1-t^2)^2}{(1-t^2)^2}=1$$

or $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, *the reqd. eliminant.*

58. The given equations may be written $x=\dfrac{1+t^2}{1-t^2}$ and

$y=\dfrac{2t}{1-t^2}$ Square and subtract.

59. The given equations are $\dfrac{x}{a}=\dfrac{1-t^2}{1+t^2}$ and $\dfrac{y}{b}=\dfrac{2t}{1+t^2}$. Square and add.

66. Dividing each equation by x we get $x+\dfrac{1}{x}=2a$

$x-\dfrac{1}{x}=2b$. etc.

62. By addition $\quad t^3 - 3t + \dfrac{3}{t} + \dfrac{1}{t^3} = 8a^3$

or $\quad \left(t + \dfrac{1}{t} \right)^3 = (2a)^3 \quad$ or $\quad t + \dfrac{1}{t} = 2a$

Similarly, by subtraction, $\quad t - \dfrac{1}{t} = 2b$. etc.

EXERCISE 100 (b)

Eliminate x and y from the equations :—

1. $a_1x + b_1y + c_1 = 0, \; a_2x + b_2y + c_2 = 0, \; a_3x + b_3y + c_3 = 0.$
$\hfill$ [*Solved*]

2. $ax + hy + g = 0, \; hx + by + f = 0, \; gx + fy + c = 0.$

3. $ax + by + c = 0, \; px + qy + r = 0, \; x + y + 1 = 0.$

Eliminate x, y, z from the equations :—

4. $ax + by + cz = 0, \; bx + cy + az = 0, \; cx + ay + bz = 0.$ [*Hint*]

5. $a(y + z) = x, \; b(z + x) = y, \; c(x + y) = z.$

6. $ax + by = z, \; by + cz = x, \; cz + ax = y.$

7. $\dfrac{x}{y - z} = l, \; \dfrac{y}{z - x} = m, \; \dfrac{z}{x - y} = n$. [*Hint*]

8. $\dfrac{ax + by}{c} = \dfrac{bx + cy}{a} = \dfrac{cx + ay}{b} = -z.$

Eliminate x and y from the equations :—

9. $\left. \begin{array}{l} px + qy = 0 \\ ax^2 + bxy + cy^2 = 0 \end{array} \right\}$ [*Hint*]
$\quad$ **10.** $\left. \begin{array}{l} lx^2 - mxy + ny^2 = 0, \\ bx + ay = 0 \end{array} \right\}$

11. $\left. \begin{array}{l} a_1x^2 + b_1xy + c_1y^2 = 0 \\ a_2x^2 + b_2xy + c_2y^2 = 0 \end{array} \right\}$ [*Hint*

12. $\left. \begin{array}{l} ax^2 + bxy + cy^2 = 0 \\ x^2 + dy^2 = 0 \end{array} \right\}$
$\quad$ **13.** $\left. \begin{array}{l} a_1x^3 + b_1xy^2 + c_1y^3 = 0 \\ a_2x^3 + b_2xy^2 + c_2y^3 = 0 \end{array} \right\}$

14. $\left. \begin{array}{l} ax^3 + bx^2y + cy^3 = 0 \\ a'x^3 + b'x^2y + c'y^3 = 0 \end{array} \right\}$

Eliminate x and y from :—

15. $x+y=a$, $x^2+y^2=b^2$, $x^3+y^3=c^3$. [Solved]
16. $x-y=m$, $x^2+y^2=n^2$, $x^3-y^3=p^3$.
17. $x+y=a$, $x^2+y^2=b^2$, $x^4+y^4=c^4$.
18. $x-y=a$, $x^2-y^2=b^2$, $x^3-y^3=c^3$. [Hint]
19. $x+y=l$, $x^2-y^2=m^2$, $x^3-y^3=n^3$.
20. $x+y=a$, $x^3+y^3=b^3$, $x^4+y^4=c^4$.

———————

[Miscellaneous types]

21. Eliminate x, y, z from $x=ay$, $y=bz$, $z=cx$. [Hint]
22. Eliminate x, y, z from $bx=ay$, $dy=cz$, $fz=ex$.

23. Eliminate x, y, z from $\dfrac{x+y}{x-y}=a$, $\dfrac{y+z}{y-z}=b$, $\dfrac{z+x}{z-x}=c$.

[Hint]

24. Eliminate x, y, z from $\dfrac{y}{z}+\dfrac{z}{y}=a$, $\dfrac{z}{x}+\dfrac{x}{z}=b$,

$\dfrac{x}{y}+\dfrac{y}{x}=c$. [Hint]

25. Eliminate x and y from $x+\dfrac{1}{x}=a$, $y+\dfrac{1}{y}=b$,

$xy+\dfrac{1}{xy}=c$.

26. Eliminate p and q from $qx+py=a$, $qy-px=0$, $p^2+q^2=1$. [Hint]

27. Eliminate m_1 and m_2 from $y=m_1x+\dfrac{a}{m_1}$,

$y=m_2x+\dfrac{a}{m_2}$, $m_1m_2+1=0$. [Hint]

28. Eliminate a, b, c from $a+b+c=x$, $a^2+b^2+c^2=y^2$, $ab+bc+ca=z^2$. [Hint]

29. Eliminate x and y from $x+y=a$, $x^2-xy+y^2=b^2$, $x^3+y^3=c^3$.

30. Eliminate x, y, z from the equations :—
$x^2(y-z)=a^3$, $y^2(z-x)=b^2$, $z^2(x-y)=c^2$, $xyz=abc$.
[Hint]

SOLUTIONS & HINTS—Ex. 100 (b)

1. $a_1x+b_1y+c_1=0$ ...(i)

$a_2x+b_2y+c_2=0$ ...(ii)

$a_3x+b_3y+c_3=0$ ...(iii)

From (i) and (ii) by cross multiplication we have :—

$$\frac{x}{b_1c_2-c_1b_2}=\frac{y}{c_1a_2-a_1c_2}=\frac{1}{a_1b_2-b_1a_2}$$

$\therefore\quad x=\dfrac{b_1c_2-c_1b_2}{a_1b_2-b_1a_2},\qquad y=\dfrac{c_1a_2-a_1c_2}{a_1b_2-b_1a_2}$

Substituting these values of x and y in (iii) we get :—

$$a_3\times\frac{b_1c_2-c_1b_2}{a_1b_2-b_1a_2}+b_3\times\frac{c_1a_2-a_1c_2}{a_1b_2-b_1a_2}+c_3=0.$$

or $a_3(b_1c_2-c_1b_2)+b_3(c_1a_2-a_1c_2)+c_3(a_1b_2-b_1a_2)=0$,
the reqd. eliminant.

4. Divide each equation by z ; then we get three equations from which we have to eliminate $\dfrac{x}{z}$ and $\dfrac{y}{z}$. Proceed as in Q. 1.

7. The first equation may be written $x=ly-lz$ or $x-ly+lz=0$.

Similarly modify the other two equations and proceed as in Q. 4.

9. Divide the first equation by y and the second by y^2 ; we get two equations in which the only unknown quantity is $\dfrac{x}{y}$.

Find the value of $\dfrac{x}{y}$ from the first equation and substitute it in the second.

11. Divide both the equations by y^2 and cross multiply.

                            [See Ex. 100 (a) Q. 47]

15. $x+y=a$ ...(i)

$x^2+y^2=b^2$ ...(ii)

$x^3+y^3=c^3$ ..(iii)

Squaring (i) we get $x^2 + y^2 + 2xy = a^2$

or $\quad b^2 + 2xy = a^2 \qquad [\because \quad x^2 + y^2 = b^2$ by (ii)

or $\qquad 2xy = a^2 - b^2$

or $\qquad xy = \dfrac{a^2 - b^2}{2} \qquad .. (iv)$.

Cubing (i) we get $\qquad x^3 + y^3 + 3xy(x+y) = a^3$

or $\qquad c^3 + 3xy(a) = a^3 \quad$ [by substitution from (iii) and (i)]

or $\qquad 3xya = a^3 - c^3$

or $\qquad xy = \dfrac{a^3 - c^3}{3a} \qquad ...(v)$

Erom (iv) and (v) we have $\dfrac{a^2}{2} - \dfrac{b^2}{2} = \dfrac{a^3 - c^3}{3a}$

or $\quad 3a^3 - 3ab^2 = 2a^3 - 2c^3$

or $\quad a^3 + 2c^3 - 3ab^2 = 0$, the reqd. eliminant.

18. Dividing (ii) by (i) we get the value of $x+y$.

Then applying the formula $(x+y)^2 - (x-y)^2 = 4xy$, we get the value of xy.

Then writing the third equation as $(x-y)^3 + 3xy(x-y) = c^3$, and substituting the values of $x-y$ and xy we get the reqd. eliminant.

21. From the first equation $\dfrac{x}{y} = a$. Similarly get the values of $\dfrac{y}{z}$ and $\dfrac{z}{x}$ and multiply the three results

Or

Multiply the three equations as they stand and divide both sides by xyz.

23. We can easily get the values of $\dfrac{x}{y}$, $\dfrac{y}{z}$ and $\dfrac{z}{x}$ from the three equations respectively by componendo and dividendo. Multiply the three results.

24.   It can be easily verified that

$$\left(\frac{y}{z}+\frac{z}{y}\right)^2+\left(\frac{z}{x}+\frac{x}{z}\right)^2+\left(\frac{x}{y}+\frac{y}{x}\right)^2$$

$$=4+\left(\frac{y}{z}+\frac{z}{y}\right)\left(\frac{z}{x}+\frac{x}{z}\right)\left(\frac{x}{y}+\frac{y}{x}\right)$$

Substitute the values of $\frac{y}{z}+\frac{z}{y}$, etc., from the given

equations.

26.   Square and add the first two equations : left-hand-side
will be found equal to $(x^2+y^2)(p^2+q^2)$.   Substitute
the value of x^2+y^2 from the third equation.

27.   Subtract the second equation from the first ; substitute
for $m_1 m_2$ from the third equation and divide both
sides by (m_1-m_2).

28.   Square the first equation and substitute the values of
$a^2+b^2+c^2$ and $ab+bc+ca$ from the other two
equations.

30.   Multiply the first three equations and substitute for
xyz from the fourth.   This gives the value of

$$(x-y)(y-z)(z-x)$$

Again by adding the first three equations we get the value
of the same expression.   Equate the two values.

———

LINEAR GRAPHS

162. On a piece of squared paper take two straight lines XOX' and YOY' intersecting at right angles in O [Fig. 1]

Let the lengths measured to the *right* along or parallel to OX be *positive*, and consequently tho·e measured to the left along or parallel to OX' *negative.*

Also let lengths measured *upwards* along or parallel to OY be *positive* and consequently those measured *downwards* along or parallel to OY' *negative.*

unit. length = 1 small division

Fig. 1.

Take any point P and draw PM, PN perpendiculars to OX and OY respectively.

It is evident that if the point P be given in position, we can find the lengths PN and PM. [In the above figure PN = +8 units and PM = +4 units] and if the lengths PN and PM be given, with proper signs, we can mark the point P.

Definitions :—

The lines of reference, XOX' and YOY' from which distances of a point are measured, are called the **axes.**

XOX' is called the **axis of x.**

YOY'   „   „   **axis of y.**

The point O is called the **origin.** [It is so called because it is the point from which measurements start].
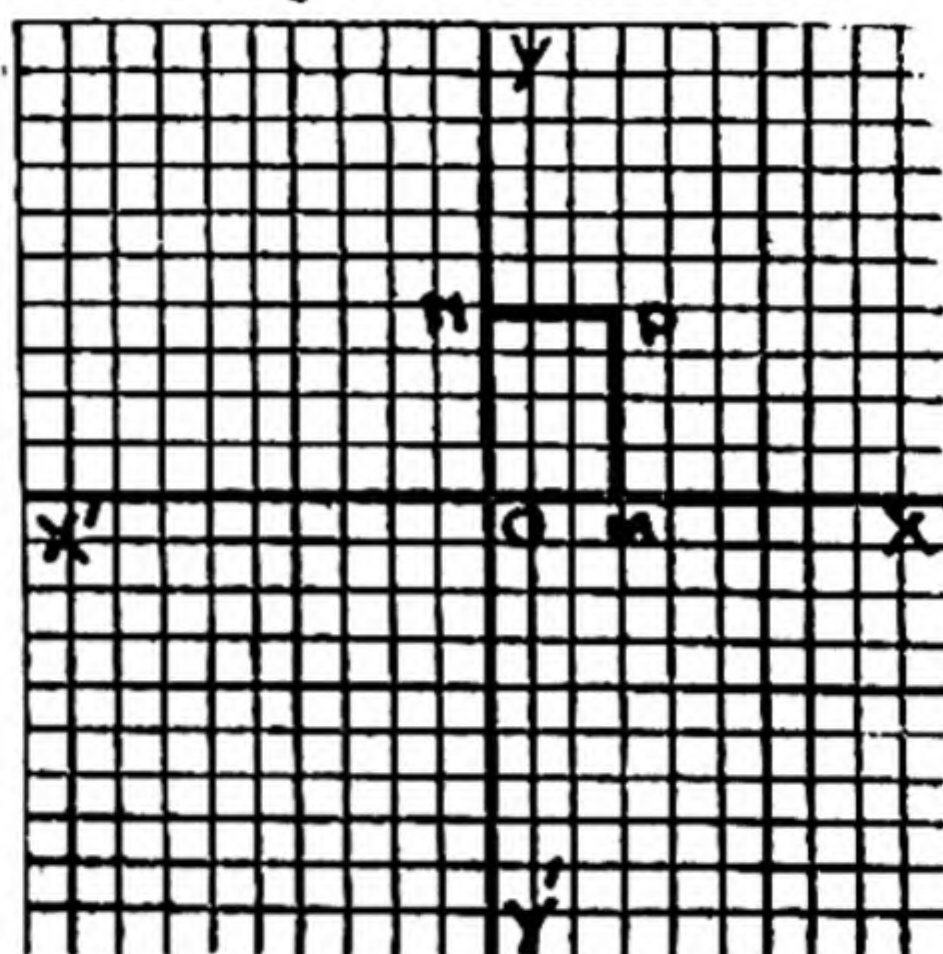
OM and MP are called the **co-ordinates** of the point P. To distinguish them, OM is called the **abscissa** or **x-co-ordinate of P** and MP is called the **ordinate** or **y-co-ordinate of P**.

Marking the position of a point whose co-ordinates are given is called the **plotting the point**.

The point whose co-ordinates are a and b is briefly written as **the point (a, b)**. Thus, the point P [Fig. 1], whose abscissa is 8 and ordinate 4, is described as the point (3, 4). Note that in writing thus the abscissa is always put first.

The four divisions into which the plane of the paper is divided by the axes are spoken of as **quadrants**.

XOY is the **first quadrant**. It will be noticed that any point situated in this quadrant has both its abscissa and ordinata positive. Thus, for instance, P [Fig. 2] is the point (5, 8).

unit length = 1 small division

YOX' is known as the **second quadrant**. Any point situated in this quadrant has its abscissa negative and ordinate positive. Thus, for instance, Q is the point (−2, 5).

X'OY' is known as the **third quadrant.** Any point situated in this quadrant has both its abscissa and ordinate negative. Thus, for instance, R is the point (−6, −7).
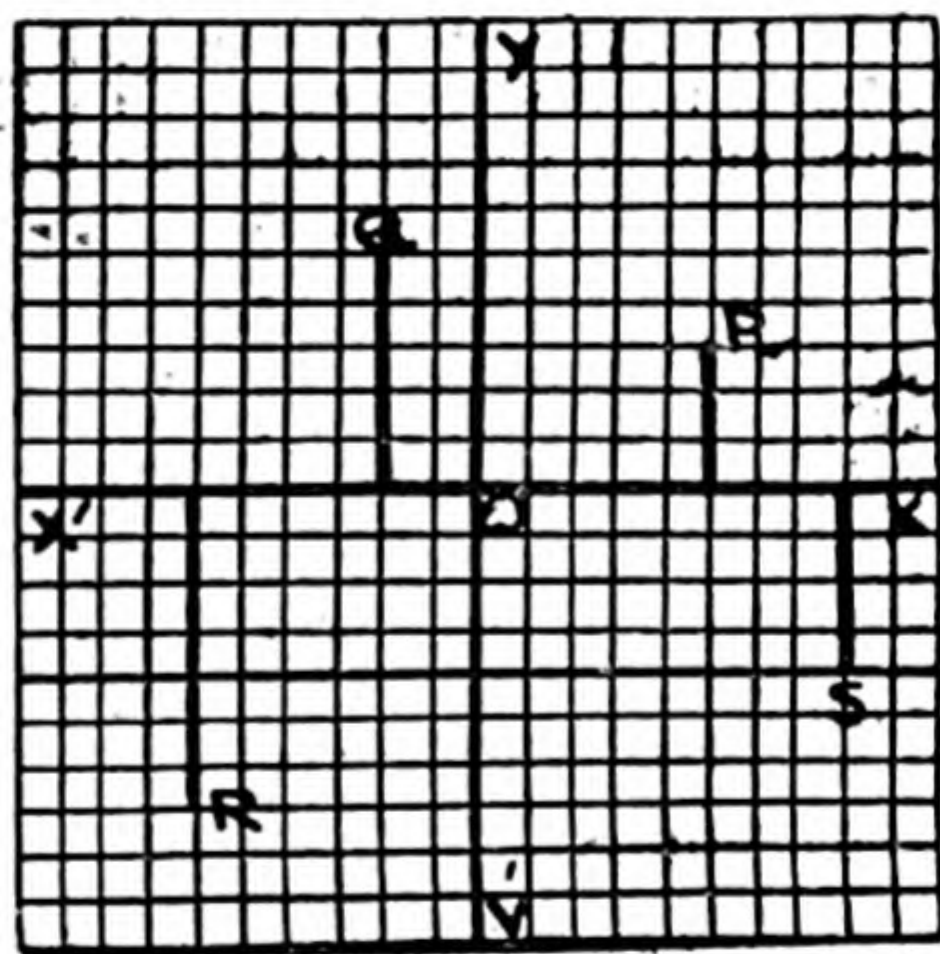
Y'OX is known as the **fourth quadrant.** Any point situated in this quadrant has its abscissa positive and ordinate negative. Thus, for instance, S is the point (8, −4).

Fig. 2.

Example. Plot the point (6, 5), (−5, 8), (−4, −4,) (6, −8), (7, 0), (0, −6), (−5, 0), (0, 5), (0, 0).

Solution

On a piece of squared paper take two st. lines XOX' and YOY' inters·cting in rt. angles at O [Fig. 3].

(i) *To plot the point* (6. 5)

Since the absci·sa is 6 and ordirate 5; from O we go 6 divisions to the *right* along OX, and then 5 divisions *upwards* parallel t OY and reach the required point marked A.

(ii) *To plot the point* (—5, 3).

We go 5 divisions from O to the *left* along OX' and then divisions *upwards* parallel to OY, getting to the reqd; point B.

unit length = 1 small division

Fig. 3.

(iii) *To plot the point* (—4. —4) We go 4 divisions from O to the *left* along OX' and then 4 div sions *downwards* parallel to OY', gettn g to the reqd. pomt C.

(iv) *To plot the point* (6 —8). We go 6 divisions from O to the *right* along OX a d then 8 divisions *downwards* parallel to OY' get ing to the roqd pomt D)

(v) *To plot th point* (7. 0) we go 7 divisions from O to the *right* along OX gotting to the reqd point E. Note that in t is case we do not move parallel to the *y*-axis at all, because the oidmate of The point is O.

(vi) *To plot the point* (0 6) we do not move along the axis of *x* at all, because the abscissa is O : moving 6 divisions from O *downwards* along OY' we get to the reqd. point F.

(vii) *To plot the point* (—5. 0) we have to go 5 divisions from O to the *left* along OX' getting to the reqd. point G.

(viii) *To plot the point* (0 5) we have not to move along the axis of *x*. becau the a seis u is O We go 5 divisions from O *upwards* along OY getting to the reqd point H
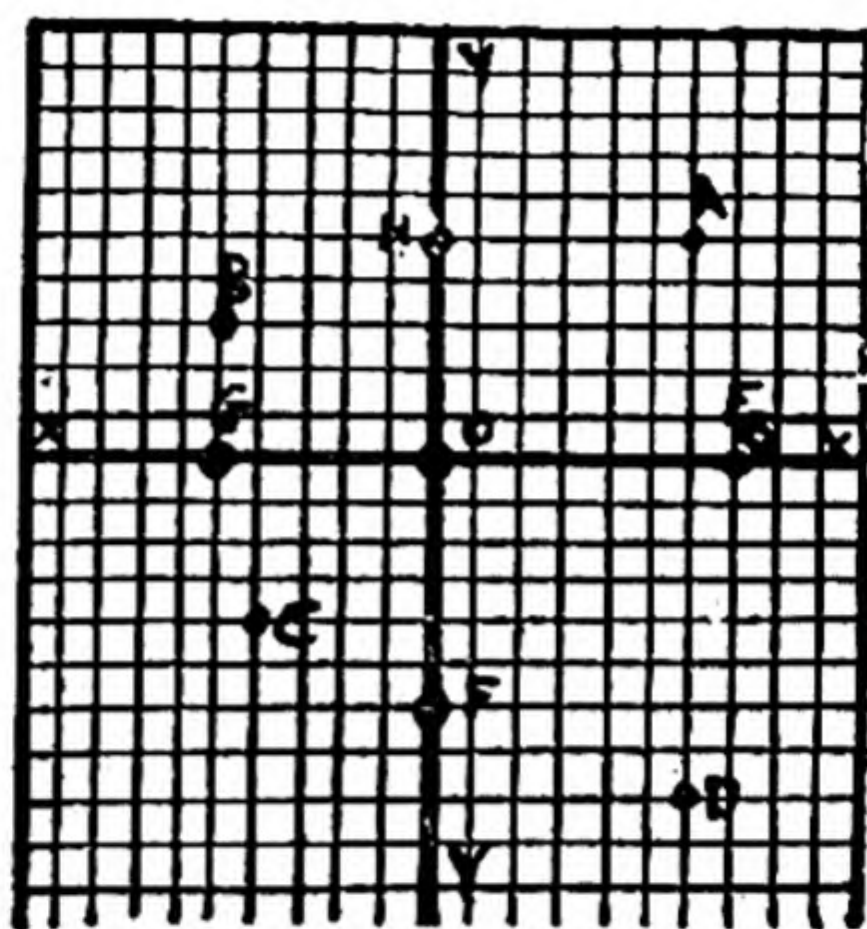
(ix) *To plot the point* $(0, 0)$ we note that we have neither to move along the x-axis and nor along (or parallel to) the y-axis. Hence the reqd. point is the starting point, *viz.*, O (the *origin*).

163. Distance between two points

(i) By measurement   unit length=1 small divisson

To find the distance between the two points A(2, 8) and B(5, 7) [Fig. 4], we open the compasses equal to AB and place the two ends of the compasses on any st. line of the graph paper and count the unit divisions between them. (The length AB will be found equal to 5 units).
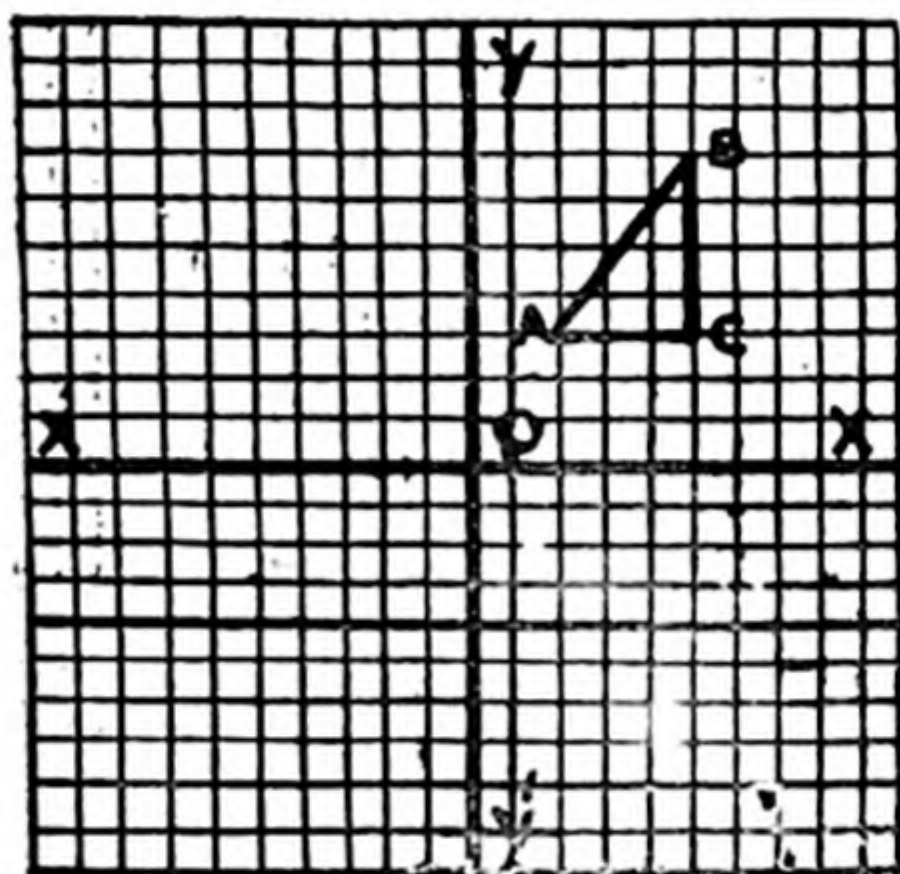
(ii) By Calculation

Draw AC and BC parallel to the x-axis and y-axis respectively, meeting in C. Clearly, AC=abscissa of B—abrcissa of A=5—2=3, and BC=ordinate of B—ordinate of A =7—3=4 [verify these facts by studying the figure very carefully]. Hence, from the right-angled triangle ABC we have $AB=\sqrt{AC^2+BC^2}=\sqrt{3^2+4^2}$

$=\sqrt{9+16}=\sqrt{25}=5$.

The student may remember the general rule :—

Distance between two points

$=\sqrt{\text{(Difference of their absscissac)}^2+\text{(difference of their ordinates)}^2}$

Fig. 4.

EXERCISE 101

1. Plot the following points on squared paper :—
A(6, 8), B(—5, 7), C(4, —11), D(—7, —9), E(0, 12),
    F(10, 0),   G(—8, —8),   H(—8, 0),   I(0, —12),
    J(—1, —11).

2. Plot the points (9, 8) and (3, 0). Measure and calculate the distance between them.

3. Plot the points (10, 4) and (−5, 12). Measure and calculate the distance between them.

4. Plot the points (−2, −7) and (10, −2). Measure and calculate the distance between them.

5. Show that the triangle formed by joining the points (−3, −3), (−2, 3), (3, −4) is isosceles. [*Hint*]

6. Show that the points (4, 2), (12, 8) and (1, 6) are the vertices of a right-angled triangle. [*Hint*]

7. Show that the points (−3, −4), (0, 5), (3, −4), (−5, 0) and (4, 3) lie on the circumference of a circle with the origin as its centre. [*Hint*]

8. Read off the co-ordinates of the middle points of the sides of the triangle formed by joining the points A(−2, 0), B(8, −10), C(−4, −6)

HINTS ON EXERCISE 101

5. Calculate all the sides. (For this we have only to find the distances between the three points taken in pairs); two of the sides will be found equal.

6. Calculate the three sides as in Q. 5. It will be seen that the square of one of the sides is equal to the sum of the squares of the other two.

7. Find the distance between the origin (0, 0) and each of the given points. All these distances will be found equal = (=5 each).

164. LOCUS, GRAPHS OR CURVE

The *locus* of a point moving in a plane is the *path* traced by that point when it moves according to a given *condition.*

For example, the locus of a point which remains at a distance of +5 units from a fixed straight line AB [Fig 5] is a straight line PQ drawn parallel to AB at a distaoce of 5 units from AB, and on that side of is which it supposed to be positive [∵ the signs of 5 is positive].
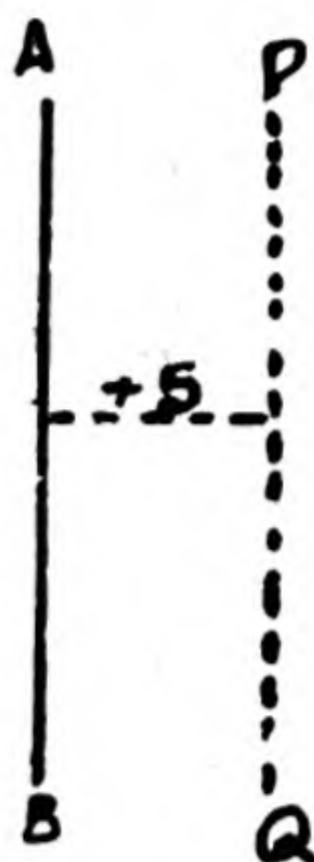
Now suppose that the fixed straight line is the axis of y [Fig. 6] ; then the above condition that of remaining at a distance of +5 units from the fixed . st. line) may be expressed in symbols as "$x = +5$" or "$x - 5 = 0$". [∵ x tands for the abscissa of the moving point, which is nothing else but the distance of the point from YY'].

The st. line PQ (produced both ways) is in this case the locus, graph or curve of the equation $x - 5 = 0$.

The student may take it for granted that to *every equation* in x and y corre ponds a *graph* (or *locus* or *curve*) and the *every graph* corr. sponds an equation *in* x and y, where x and y stand for the abscissa and ordinate of the moving point respectively.

Further, it may be assumed that a *linear* equation in x and y (*i e* an equation containing no power of x or y higher than the first and no product xy) always represents a straight line. It is only such equations we propose to deal with.

FIG 5

unit length = 1 small division
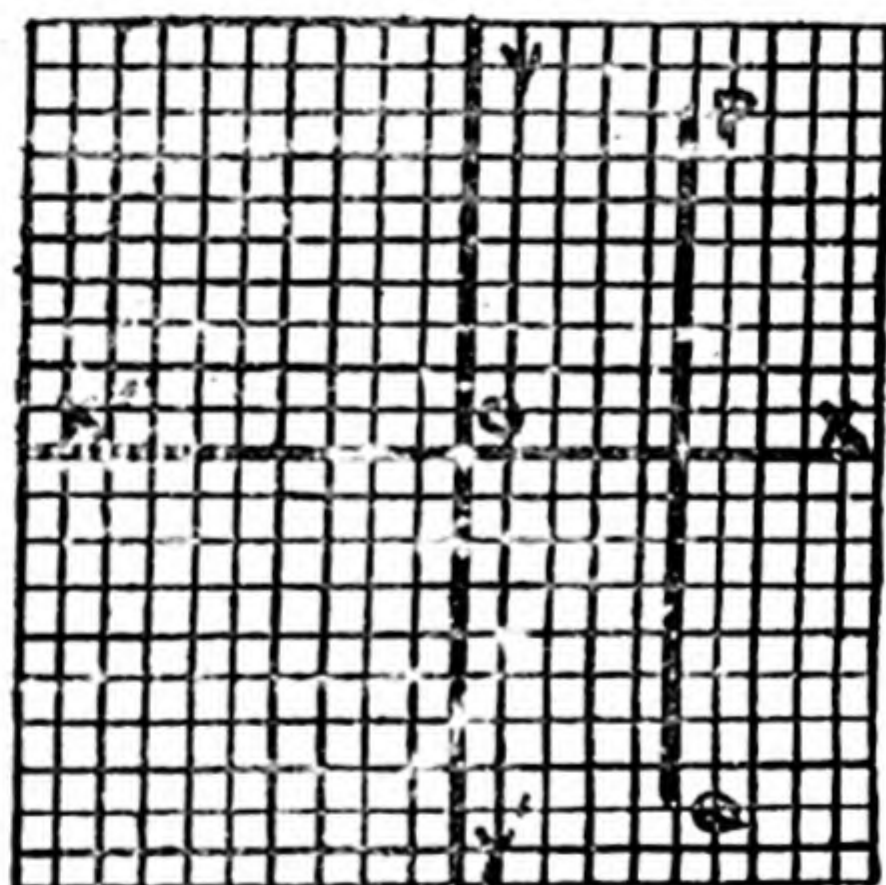
Fig. 6.

The only aspect we have to discuss is :—

165. To draw the graphs of a given linear equation.

Equations from which y is missing.

Such equations can always be put in the form $x=a$ (where a may be $+ve$ or $-ve$) and this clearly states that the moving point remains at a distance of 'a' units from YY'($\because$ x means abscissa, *i.e.*, distance from YY']. Therefore the required graph is the straight line drawn parallel to YY' at a distance 'a' units from it.

Example. Draw the graph of the equation $x+7=0$.

Solution :—

The given equation may be written as $x=-7$.

This shows that the moving point is always at a distance of -7 from YY' [$\because$ x means absscissa or distance from YY']

unit length $=1$ small division.

Fig. 7

Hence the reqd. locus (or graph or curve) is the st. lme AB (produced both ways) drawn parallel to YY' at a distance -7 from it (*i.e.* at a distance of 7 units to the *left* of YY'). [Fig. 7]

(b) Equations from which x is missing

Such equations can always be put in the form $y=b$ (where b may be positive or negative) and since y stands for "ordinate" or "distance from XX'", the required graph (or locus or curve) is the straight line drawn parallel to XX' at a distance 'b' from it.

Example. Draw the graphs of the equations $y-6=0$ and $y+5=0$.
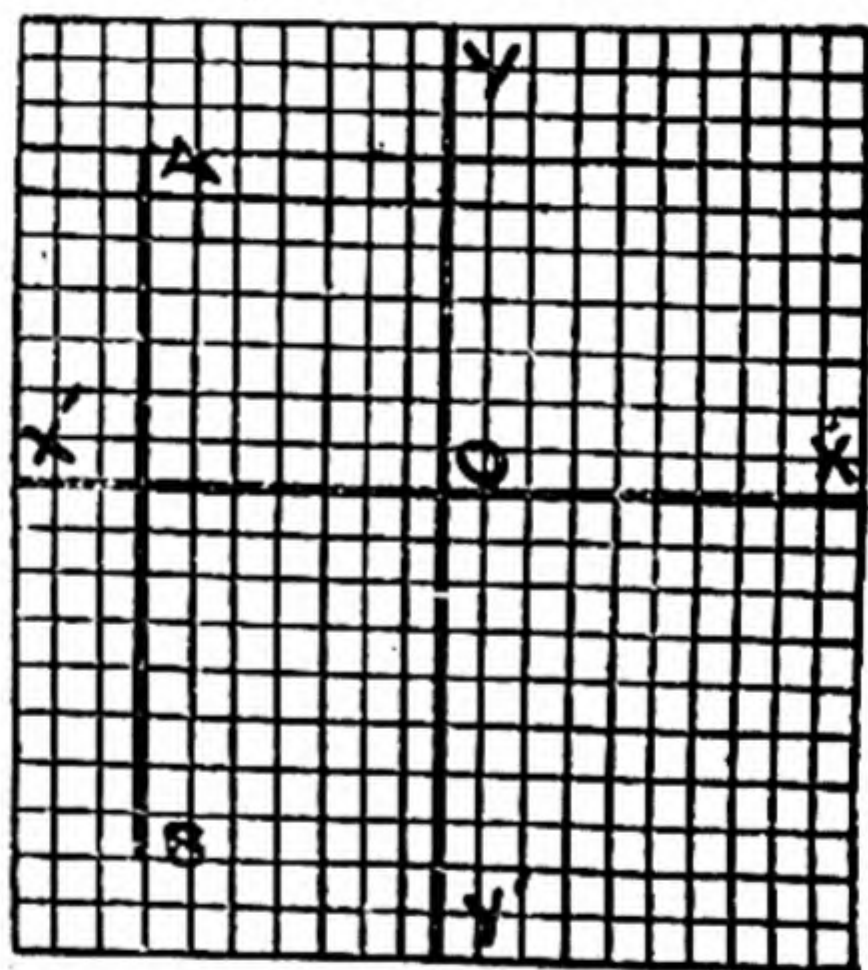
The equation $y-6=0$ may be written as $y=6$ which means that the ordinate of the moving point should be always 6. Now, ordinate is the distance from XX'. Hence the reqd. locus (or curve or graph) is the straight line PQ drawn ‖ XX' at a distance of 6 units from it. [Fig. 8]

Similarly, the graph of the equation $y+5=0$ or $y=-5$ is the st. line RS drawn ‖ XX' at a distance -5 from it. [Fig 8]

unit length=1 small division
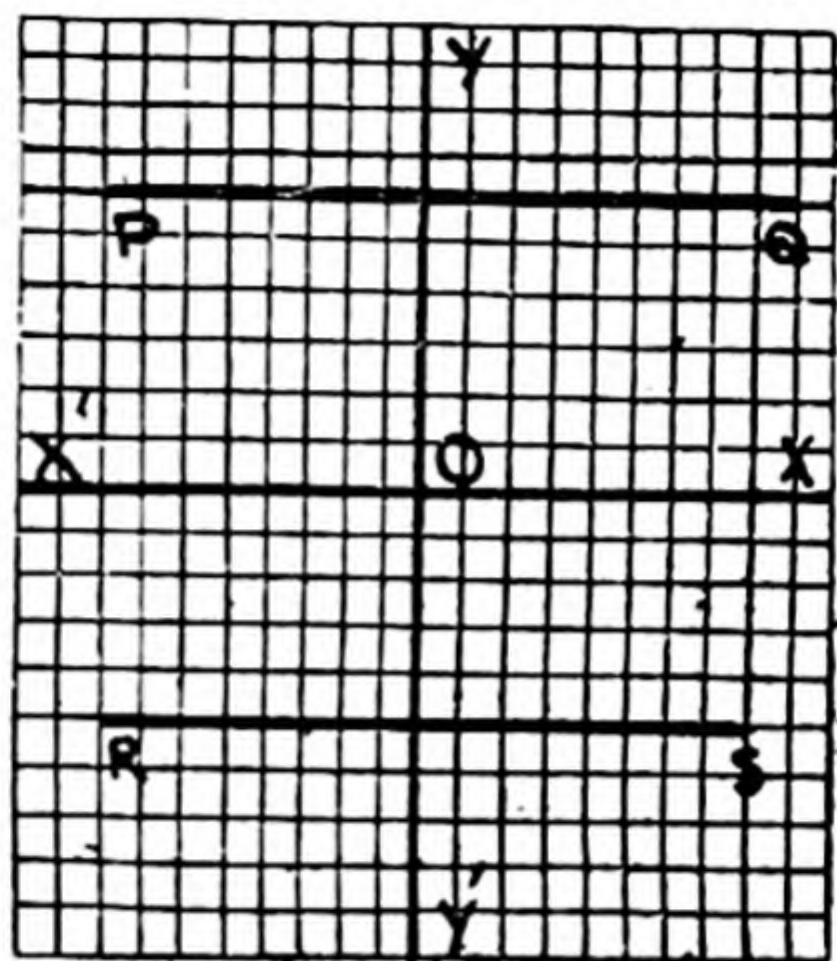
Fig. 8

(c) Equations in which x and y are both present

We shall discuss a particular case :— $2x-3y=6$.

The equation clearly states that *twice the abscissa of the moving point minus thrice its ordinate should always be equal to 6*. If abscissa be equal to 6 and ordinate equal to 2 (*i.e.* $x=6$ and $y=2$) this condition is satisfied, for $2\times6-3\times2=6$. Hence (6,2) is one of the positions of the moving point. Similarly $(0,-2)$, $(3, 0)$, $(9, 4)$ are three more positions of the moving point.

If we plot the four points $(6, 2)$, $(0,-2)$, $(3, 0)$, $(9, 4)$ we find that they are in a straight line. This straight line, produced both ways, is the required locus. [Fig. 9].

Note 1. The given equation being *linear* (*i e.* of the first degree), it necessarily represents a straight line Hence it is enough to find only two points on it. But we find three or four points to be on the safe side.

Note 2. If the given equation be put into the form

*$y = \dfrac{2x-6}{3}$, or $y = \dfrac{2(x-3)}{3}$, the pairs of values of x and y satisfying the equation can be easily obtained. For, we have only to give a value to x and evaluate R. H S. which gives the corresponding value of y. For example, if $x = 6$, R. H. S. $= \dfrac{2(6-3)}{3} = \dfrac{2 \times 3}{3} = 2$, so that $y = 2$. This gives the point (6, 2) already found above.

We now present the solution of the above question, viz " *Draw the graph of the equation $2x - 3y = 6$* " in a precise form avoiding unnecessary explanations), as expected from the student :—

Equation :—
$2x - 3y = 6$

or $\quad y = \dfrac{2(x-3)}{3}$

Points on the graph :—

	A	B	C	D
$x =$	6	0	3	9
$y =$	2	−2	0	4

Plot the points A (6, 2), B(0, −2), C(3, 0), D(9, 4). St line BCAD, produced both ways is the reqd. graph [Fig 9].

unit 1 ngth = 1 small d vision
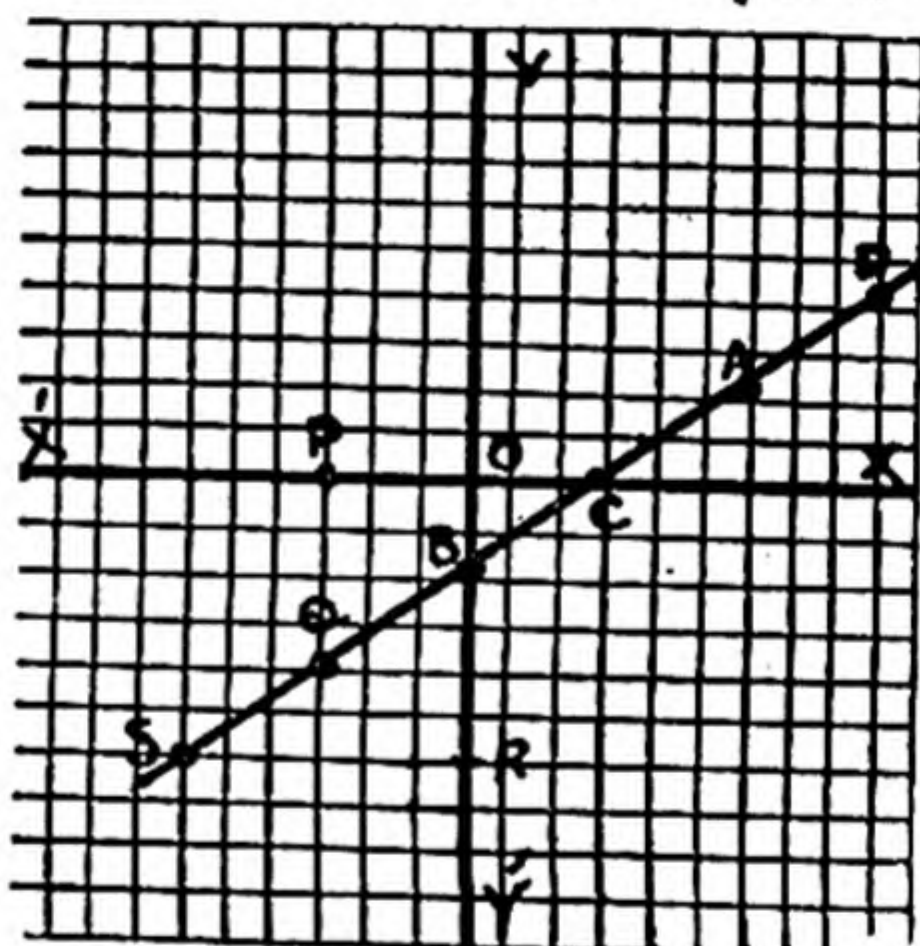
Fig. 9

166. *Given the abscissa of a point on a graph ; to read off its ordinate, and vice versa.*

*The form is obtained thus :—

$\quad -3y = 6 - 2x$ [transposing $2x$ to R. H. S.].

$\quad$ or $\quad 3y = 2x - 6$

$\quad$ or $\quad y = \dfrac{2x-6}{3}$ or $\dfrac{2(x-3)}{3}$

(i) In the last example (graph of the equation $2x-3y=6$), suppose it is required to read off the ordinate of the point whose abscissa is -3.

We go 3 units from O to the left along OX' [∵ abscissa $=-3$] reaching the point P. From P we move parallel to YY' in order to reach the graph (viz the st. line BCAD). For this we have to go 4 units *downwards* parallel to OY', getting to the point Q on the graph. Hence the reqd. ordinate is -4

Algebraically this result can be obtained by putting $x=-3$ in the given equation and getting the value of y from the resulting equation.

(ii) Now suppose it is required to find (on the same graph) the abscissa of the point whose ordinate is -6.

We go 6 units from O *downwards* along OY' [∵ ordinate $=-6$] reaching the point R; from this position we move parallel to XX' in order to reach the graph. For this we have to go 6 units to the *left* along OX', getting to the point S Hence the required abscissa is -6.

Algebraically this result can be obtained by putting $y=-6$ in the given equation and getting the value of x from the resulting equation

167   Intercepts on the axes

If a straight line cuts the axis of x in A, the length OA (with proper sign) is called the *intercept of the line on the axis of x*. Similarly, if it cuts the axis of y in B, the length OB (with proper sign) is termed the *intercept of the line on the axis of y*

Thus for example, the intercept of the st line $2x-3y=6$ [See Fig. 9] on the axis of x is $+3$ and on the axis of y is -2.

Algebraically the intercept on the axis of x can be found by putting $y=0$ in the given equation and getting the value of x from the resulting equation, for this intercept is nothing but the abscissa of a point on the graph, whose ordinate is 0. [ordinate of every point or the axis of x is 0].

Similarly the intercept on the axis of y can be found by putting $x=0$ in the given equation and getting the value of

y from the resulting equation, for this intercept is nothing but the ordinate of a point on the graph, whose abscissa is 0. [abscissa of every point on the axis of y is 0].

168. Function of x and its graph

Any expression which contains a variable quantity x, so that its value depends upon the value of x, is called a *function of x*. Thus the expression $\dfrac{2(x-3)}{3}$ is a function of x, for its value changes as we change the value of x. If we put $x=6$. 0, 8, 9 in turn, we get the corresponding values of the function as 2, -2, 0, 4 respectively. And if we agree to call the function ' y ', we have the following pairs of the values of x and y :—

$x=$	6	0	3	9
$y=$	2	-2	0	4

Further, if the values of x be represented by lengths measured from the origin along or parallel to the axis of x and the values of y along or parallel to the axis of y, we get the same graph as in Fig 9.

Hence we conclude that the graph of a function of x is obtained by equating the function to y and proceeding exactly as in Art 165.

EXERCISE 102

Draw on squared paper the graphs of :—

1. $x=8$
2. $x+2=0$
3. $x-7=0$
4. $x=-8$

5. $y-5=0$
6. $y+6=0$
7. $y=-4$
8. $y=8$

9. $y=x$
10. $y=2x$
11. $y+3x=0$
12. $y-4x=0$

13. $2x+5y=0$

14. $3x-4y=0$

15. $5x-2y=0$

16. $\dfrac{x}{2}+\dfrac{y}{8}=0$

———

Draw the graphs of the following equations and find their intercepts on the axis of y :—

17. $3x-4y=12$

18. $5x-2y=10$

19. $\dfrac{x}{3}+\dfrac{y}{4}=1$

20. $\dfrac{y}{4}-\dfrac{x}{6}=\frac{3}{2}.$

———

Draw the graphs of the following equations and find their intercepts on the axis of x :—

21. $2x+3y+12=0$

22. $4y-3x-12=0.$

23. $\frac{1}{3}x+\frac{1}{4}y=2.$

24. $\dfrac{x-2}{9}=\dfrac{y}{7}.$

———

25. Draw the graph of the equation $5y+4x-20=0$. What is the ordinate of that point on the graph, whose abscissa is 5 ?

26. Draw the graph of the equation $3x-2y-18=0$. What is the abscissa of the point on the graph, whose ordinate is -3 ?

27. Draw the graph of the function $\dfrac{6-x}{2}$.   Read off its value when $x=-8$. For what value of x does the function become 5 ?

28. Draw the graph of the equation $2x-3y=12$.   What are the co-ordinates of the point where it is cut by the st. line $x-6=0$?

29. Draw the graph of the equation $3x+4y=8$.   In this graph read the value of x when $y=-10$. (*P. U. 1918*)

30   Plot the graph of the function $\dfrac{3x+18}{2}$ and from it

find out the value of the function when $x=3$. Find also the intercepts of the graph on the axes. Show that the points $(-1, 5)$ and $(1, 8)$ lie on the graph.

<div align="right">(P. U. 1923)</div>

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169. Graphical Solution of two linear equations in x and y.

Since the equations are linear, therefore their graphs are two straight lines. Draw these straight lines and read off their point of intersection. The co-ordinates of this point constitute the required solution.

[Proof :—The co-ordinates of any point on one of the st. lines satisfy the corresponding equation ; Similarly the co-ordinates of any point on the other line satisfy the other equation. Hence the co-ordinates of their common point must satisfy both the given equations].

Example. Solve the following equations graphically and also find the intercept of the graph of the first equation on the y-axis. $4x+8y=12$, $2x+y=2$.

Solution :—

The first equation is

$$4x+8y=12$$

or $$y=\frac{12-4x}{8}$$

Some of the points on its graph are :—

	A	B	C	D
$x=$	0	8	6	9
$y=$	4	0	-4	-8

The second equation is $2x+y=2$ or $y=2-2x$

Some of the points on its graph are :—
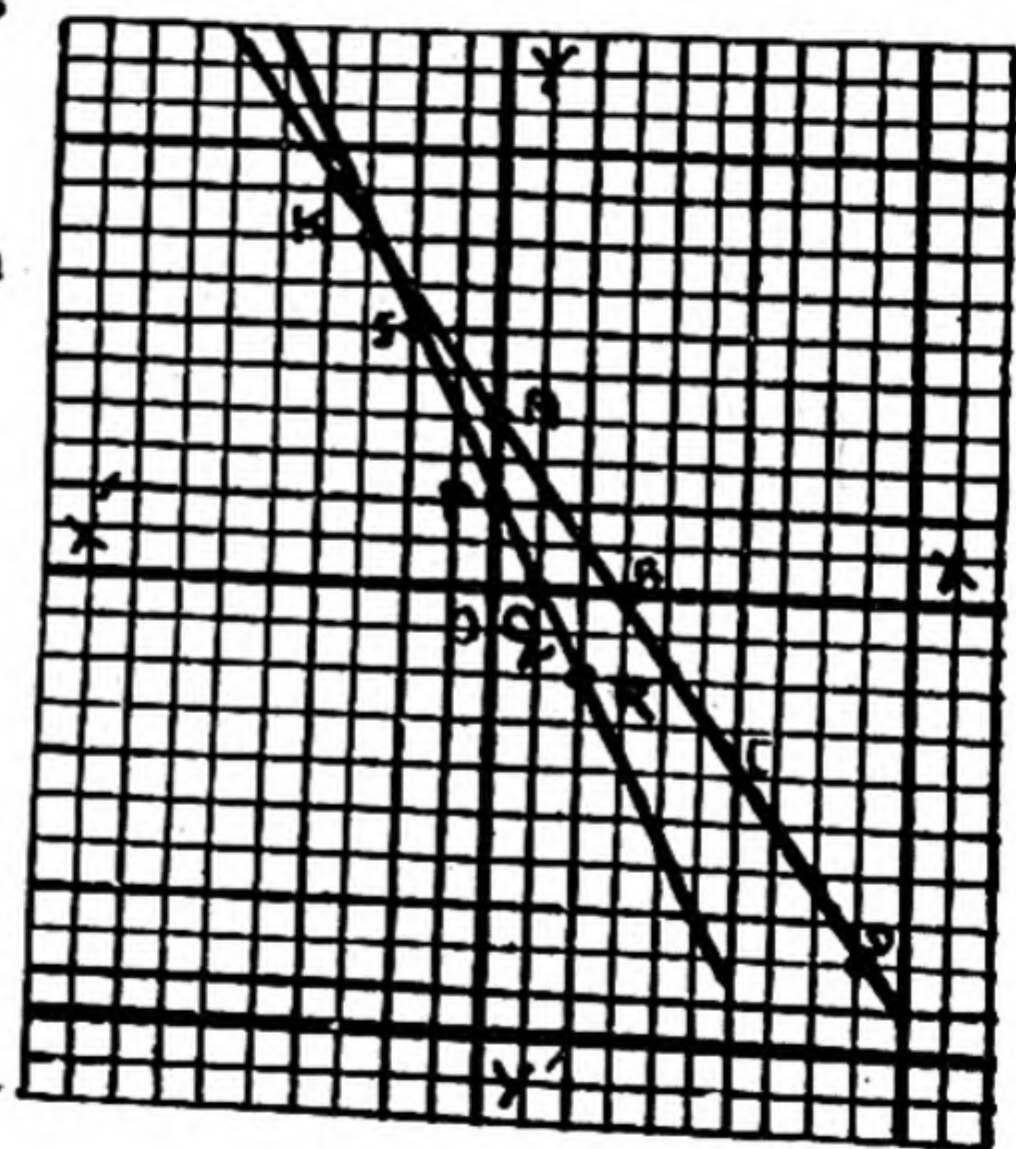
Fig. 10

	P	Q	R	S
$x=$	0	1	2	−2
$y=$	2	0	−2	6

The two graphs (*i.e.* st. lines AD and SR produced both ways) intersect in the point K, whose co-ordinates are found to be (−3, 8)

Hence the solution is $x=-3$ and $y=8$. **Ans I.**

Again, the graph of the first equation, *viz.*, st. line AD cuts the y-axis is the point A. Therefore its intercepts on y-axis =OA=4. **Ans II.**

EXERCISE 103

Solve the following equations graphically :—

1. $x+y=8$ }
   $x-y=2$ }

2. $2x+y=9$ }
   $3x+y=15$ }

3. $3x+y-14=x-2y=0$.   (*P. U. 1914*)

4. $3x+5y=13$, $5x-3y+1=0$.   (*P. U. 1915*)

5. $3x+2y=5$ }
   $5x-2y=3$ } (*Patna, 1932*)

6. $2x+3y=6$ }
   $6x-5y=4$ } (*Dacca, 1935*)

7. Draw the graphs of the three equations $3x-4y=12$, $4x+3y=41$ and $x+y=11$ and find the values of x and y which satisfy all of them.   (*P. U 1917*)

8. Draw the graphs of $4x+3y=12$ and $2x+y=2$ and read the values of x and y at the point of intersection.
                                       (*P. U. 1919*)

9. Solve graphically the equations $4x+3y-12=0$ and $2x-y-16=0$. Find the intercepts of the graph of the latter on the axis of x and y.   (*P U. 1925*)

10. (a) Solve graphically the equations $2x-3y=5$, $3x-4y=6$.   (*P. U 1916*)

    (b) Find the abscissa of that point on the graph of the former which has ordinate$=-9$.

11. Solve graphically the equations $x-2y+11=0$, $2x-3y+18=0$, and verify your solution

Find the intercepts of the graph of the latter on the axes of x and y. (P. U. 1924)

12. Draw a graph of $x=1+y$ and of $2x+4y=17$ in the same diagram. Hence solve the equations. (P. U. 1926)

13. Solve the equations : $5x-8y=9$ and $3x+5y=19$. Verify the solutions by graphs, and measure the angle between the lines represented by the equations.
(P. U. 1929)

14. Solve the equations $2x+y=18$ and $3y=33+x$; verify the result by means of graph. (P. U. 1933)

15. Draw the graphs of $2x-8y=6$ and $8x+2y=9$, and read the co-ordinates of their point of intersection.
(P. U. 1935)

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170. It has already been remarked that an equation of the first degree in x and y always represents a straight line. The most general equation of the first degree in x and y appears to be $ax+by+c=0$, but this can be easily reduced to the form $Ax+By+1=0$. For, dividing both sides of the

equation $ax+by+c=0$ by c we get $\dfrac{a}{c}x+\dfrac{b}{c}y+1=0$, and

since $\dfrac{a}{c}$ and $\dfrac{b}{c}$ are some constants, they may be replaced

by single letters A and B.

Hence. $Ax+By+1=0$ may be taken to represent any straight line whatever.

171. Given the co-ordinates of two points on a straight line ; to find its equation.

The method will be clear from the following :—

Example. Find the equation of the straight line passing through the points $(8, -4)$ and $(-5, -12)$.

Solution. Let the equation of the st. line be
$$Ax+By+1=0 \qquad ...(i) \qquad [\text{Art. 170}]$$
Our object is now to find the values of A and B from the conditions of the question.

∵   the point (3, −4) lies on the line, therefore its co-ordinates ($x=3$, $y=-4$) must satisfy equation (i). Hence we have

$$3A - 4B + 1 = 0 \qquad ...(ii)$$

Similarly, substituting the co-ordinates of the point (−5, −12) in (i) we get :—

$$-5A - 12B + 1 = 0 \qquad ...(iii)$$

Solving equations (ii) and (iii) we easily get $A = -\frac{1}{7}$, $B = \frac{1}{7}$.

Substituting these values of A and B in (i) we get the reqd. equation to be $-\frac{1}{7}x + \frac{1}{7}y + 1 = 0$ or $-x + y + 7 = 0$
or $x - y - 7 = 0$. Ans.

EXERCISE 104

Find the equations of the straight lines passing through the following pairs of points :—

1. (1, 2), (2, 4).   2. (1, 8), (2, 10).
3. (1, −4), (2, −2).   4. (0, −2), (3, 0)
5. (0, 17), (10, −3).   6. (0, −4), (8, 8).
7. (0, 6), (−3, 4).   8. $(-2, -22)$, $(\frac{1}{2}, \frac{2}{2})$

9.   The vertices of a triangle are (1, 2), (−3, 4) and (0, 0). Find the equations of its sides.

10. Prove that the points (0, 3), (1, 5) and (4, 11) lie on a st. line. Also find its equation. [Hint]

11. Plot the points (3, −3) and (−3, 3) and write down the equation of the straight line joining them.
                                             (P. U. 1918)

12. Plot the points (12, 4), (−3, −5) and find if the st. line joining them passes through the point (8, 2).
                                             (P. U. 1919)

13. Plot the points A(5, 7), B(7, 10), C(2, 2) and D(−1, −4) and find the equations of the straight lines AB and CD   Find the point of their intersection (unit=$\frac{1}{2}$ inch)   (Bombay, 1912)

14. Plot the points (4, 2) and (7, 3) ; join them and find the equation of the straight line passing through them. Write down from the figure ordinates of the points on this line whose abscissa are respectively 1 and −5. (unit=$\frac{1}{3}$ inch) (*Bombay, 1915*)

HINT—Ex. 104

10. Find the equation of the straight line joining any two of the given points and show that the co-ordinates of the third point satisfy that equation.

CHAPTER XXV

HARDER EQUATIONS

172. No introductory remarks or explanations [3] in general terms are necessary for the examples of this chapter. The student is to be guided by the solutions provided, which contain explanations wherever necessary.

EXERCISE 105

Solve the following equations :—

1. $ax+b=cx+d$. [*Solved.*]   2. $x+b=a-x$.

3. $ax-2c=5cx-3a$.   4. $b^2+cx=bx+c^2$.

5. $a(x-a)+b(x-b)+2ab=0$.

6. $c(x-b)+d(x-a)=ca+bd$.

7. $a^2(x-a)+b^2(x-b)=abx$. 8. $a^2(a-x)-abx=b^2(b+x)$.

9. $(l+x)(m+x)=x(x-n)$.

10. $(p-q)(x-p)=(p-r)(x-q)$.

11. $(x+a)(x+b)=(x+c)(x+d)$.

12. $x(x-m)+x(x-n)=2(x-m)(x-n)$.

13. $(x-a)(x-b)=(x-a-b)^2$. [*Hint.*]

14. $(ax-b)(bx+a)=a(bx^2-a)$.

15. $(x+a)(x+a+b)=(x+b)(x+3a)$.

16. $(x-ac)^2+(x-bd)^2=(x-bc)^2+(x-ad)^2$.

17. $\dfrac{b}{a} - \dfrac{dx}{c} = \dfrac{ax}{b} - \dfrac{c}{d}$. [Solved.]

18. $\dfrac{x}{b} = c - x$.

19. $\dfrac{1}{a} - \dfrac{1}{x} = \dfrac{1}{x} - \dfrac{1}{b}$?

20. $p + \dfrac{x}{q} = q + \dfrac{x}{p}$.

21. $\dfrac{a-b}{x-c} = \dfrac{a+b}{x+c}$.

22. $\dfrac{p}{x} = \dfrac{q}{x-p+q}$.

23. $\dfrac{m'(m-x)}{n} - \dfrac{n'(n+x)}{m} = x$.

24. $k \cdot \dfrac{x-k}{l} + l \cdot \dfrac{x-l}{k} = x$.

25. $\dfrac{7x+16}{21} - \dfrac{x+8}{4x-11} = \dfrac{x}{3}$. [Solved.]

26. $\dfrac{4x-3}{3} - \dfrac{x-2}{5x-14} = \dfrac{20x-6}{15}$.

27. $\dfrac{10x-3}{30} - \dfrac{7x-24}{8x-2} = \dfrac{2x-3}{6}$.

28. $\dfrac{1}{x-2} - \dfrac{3}{3x-1} = \dfrac{5}{3x-6}$. [Hint.]

29. $\dfrac{3}{x-2} - \dfrac{1}{2x-1} = \dfrac{12}{5x-10}$.

30. $\dfrac{3}{3x-5} - \dfrac{6}{28x-49} = \dfrac{7}{27x-45} + \dfrac{2}{4x-7}$.

31. $\dfrac{5}{x-1} + \dfrac{3}{x-2} = \dfrac{8}{x-4}$. [Solved.]

32. $\dfrac{2}{x-5} + \dfrac{3}{x-3} = \dfrac{5}{x-4}$.

33. $\dfrac{9}{x-7} + \dfrac{15}{x-13} = \dfrac{24}{x-12}$.

34. $\dfrac{15}{x-3} - \dfrac{4}{x-5} = \dfrac{11}{x-2}$.

35. $\dfrac{3}{4x+1} + \dfrac{4}{4x+5} = \dfrac{7}{4x+8}$. [*Hint.*]

36. $\dfrac{5}{3x-4} + \dfrac{6}{3x+1} = \dfrac{11}{3x-7}$.

37. $\dfrac{5}{3-4x} + \dfrac{9}{4x+13} - \dfrac{4}{4x+5} = 0$. [*Hint.*]

38. $\dfrac{6}{5-6x} + \dfrac{13}{6x+19} = \dfrac{7}{6x+7}$.

39. $\dfrac{9}{8x+2} + \dfrac{1}{x-8} = \dfrac{8}{2x+5}$. [*Hint.*]

40. $\dfrac{10}{2x-5} + \dfrac{1}{x+5} = \dfrac{18}{3x-5}$.

41. $\dfrac{10}{5x-9} + \dfrac{14}{2x+9} = \dfrac{9}{x+8}$.

42. $\dfrac{5}{4x+3} + \dfrac{2}{3x-6} = \dfrac{23}{12x-18}$.

43. $\dfrac{15}{4x-1} - \dfrac{2}{x-1} = \dfrac{7}{4x+5}$.

44. $\dfrac{1}{x-2} + \dfrac{3}{2(x-4)} = \dfrac{5}{2x-7}$.

45. $\dfrac{5}{x-1} + \dfrac{3}{4-3x} - \dfrac{8}{2x-1} = 0$. [*Hint.*]

46. $\dfrac{13}{6x+1} + \dfrac{2}{4-3x} = \dfrac{3}{2x-7}$.

47. $\dfrac{a}{x-a} + \dfrac{b}{x-b} = \dfrac{a+b}{x-a-b}$. *Hint*

48. $\dfrac{a}{x-a} - \dfrac{b}{x-b} = \dfrac{a-b}{x-c}$.   49. $\dfrac{a}{x+b} + \dfrac{b}{x+a} = \dfrac{a+b}{x}$.

50. $\dfrac{c}{x-a} - \dfrac{a}{x+c} = \dfrac{c-a}{x}$.

51. $\dfrac{6x+8}{2x+1} - \dfrac{2x+38}{x+12} = 1$ [Solved]

52. $\dfrac{3x-1}{x-2} = \dfrac{9x+5}{3x-5}$

53. $\dfrac{x+2}{x-3} + \dfrac{x-2}{x-6} = 2.$

54. $\dfrac{x+16}{x-4} + \dfrac{5x-3}{x-2} = 6.$

55. $\dfrac{2x}{x-4} = 9 - \dfrac{7x-3}{x+1}.$

56. $\dfrac{4x-7}{2x-5} = 5 - \dfrac{9x+7}{3x+1}.$

57. $\dfrac{x+18}{x-2} - \dfrac{27-3x}{3x-19} = 2.$

58. $\dfrac{6x+8}{2x+1} - \dfrac{2x+38}{x+12} = 1.$

59. $\dfrac{px+r}{x+s} + \dfrac{qx+s}{x+r} = p+q.$

60. $\dfrac{px-q^2}{x-p} - \dfrac{qx-p^2}{x-q} = p-q.$

61. $\dfrac{x^2+7x+12}{x+2} + \dfrac{x^2+14x+13}{x+11} = 2x+8.$ [Hint]

62. $\dfrac{x^2+x+1}{x+1} + \dfrac{x^2-x+1}{x-1} - 2x = 0.$

63. $\dfrac{2x-1}{x-1} + \dfrac{3x-4}{x-2} = \dfrac{5x-12}{x-3}.$ [Hint]

64. $\dfrac{x+4}{x-1} + \dfrac{2x-1}{x-2} = \dfrac{3x-4}{x-4}.$

65. $\dfrac{x+2}{x-7} + \dfrac{x+2}{x-13} = \dfrac{2x}{x-12}.$

66. $\dfrac{3x+6}{x-3} - \dfrac{x-1}{x-5} = \dfrac{2x+7}{x-2}.$

67. $\dfrac{8x+11}{8x+2} + \dfrac{x-2}{x-3} = \dfrac{4x+18}{2x+5}.$

68. $\dfrac{4x+14}{4x-1} - \dfrac{x+1}{x-1} = \dfrac{7}{4x+5}.$

69. $\dfrac{x-1}{x-2} + \dfrac{x-6}{x-7} = \dfrac{x-2}{x-3} + \dfrac{x-5}{x-6}.$ [Hint]

70. $\dfrac{x+5}{x+4}+\dfrac{x-15}{x-16}=\dfrac{x-4}{x-5}+\dfrac{x-6}{x-7}.$

71. $\dfrac{2x+2}{x}+\dfrac{2x-12}{x-5}-\dfrac{2x+4}{x+1}=\dfrac{2x-10}{x-4}.$

72. $\dfrac{6x-18}{x-4}+\dfrac{8x-10}{2x-3}=\dfrac{6x-23}{2x-7}+\dfrac{4x-3}{x-1}.$

SOLUTIONS & HINTS—Ex. 105

1. $ax+b=cx+d$

$\therefore ax-cx=d-b$ [By transposition]

or $x(a-c)=d-b$ or $x=\dfrac{d-b}{a-c}.$

Note. Carefully that we take *all terms containing x* to one side and the others to the other side. Then both sides are divided by the coefficient of x.

13. $(x-a)(x-b)=\{x-(a+b)\}^2$

$\therefore x^2-(a+b)x+ab=x^2-2(a+b)x+(a+b)^2$

Cancel x^2 from both sides, etc. etc.

17. $\dfrac{b}{a}-\dfrac{dx}{c}=\dfrac{ax}{b}-\dfrac{c}{d}$

or $\dfrac{b}{a}+\dfrac{c}{d}=\dfrac{ax}{b}+\dfrac{dx}{c}$ [Taking terms containing x to one side]

or $\dfrac{bd+ac}{ad}=\dfrac{acx+bdx}{bc}=\dfrac{x(ac+bd)}{bc}$

or $\dfrac{1}{ad}=\dfrac{x}{bc}$ [Cancelling the factor $(ac+bd)$ from both sides]

or $x=\dfrac{bc}{ad}.$ **Ans**

25. $\dfrac{7x+16}{21}-\dfrac{x+8}{4x-11}=\dfrac{x}{3}$

[Bring together fractions which do not contain x in the denominator]

or $\qquad \dfrac{7x+16}{21} - \dfrac{x}{3} = \dfrac{x+8}{4x-11}$

or $\qquad \dfrac{7x+16-7x}{21} = \dfrac{x+8}{4x-11}$

or $\qquad \dfrac{16}{21} = \dfrac{x+8}{4x-11}$

or $\qquad 64x-176 = 21x+168$　[By cross multiplication]

or $\qquad 64x-21x = 168+176 \qquad$ or $\qquad 43x = 844$

or $\qquad x = \dfrac{844}{43} = 8.$　**Ans.**

31.　$\dfrac{5}{x-1} + \dfrac{8}{x-2} = \dfrac{8}{x-4}$

or $\qquad \dfrac{5}{x-1} + \dfrac{8}{x-2} = \dfrac{5}{x-4} + \dfrac{8}{x-4}$

$\qquad \left[\text{Splitting } \dfrac{8}{x-4} \text{ into two parts, } \dfrac{5}{x-4} \text{ and } \dfrac{8}{x-4} \right]$

or $\qquad \dfrac{5}{x-1} - \dfrac{5}{x-4} = \dfrac{3}{x-4} - \dfrac{3}{x-2}$ 　[By transposition]

or $\qquad \dfrac{5x-20-5x+5}{(x-1)(x-4)} = \dfrac{3x-6-3x+12}{(x-4)(x-2)}$

or $\qquad \dfrac{-15}{(x-1)(x-4)} = \dfrac{6}{(x-4)(x-2)}$

or $\qquad \dfrac{-5}{x-1} = \dfrac{2}{x-2}$ 　[Cancelling the factor 3 from the

numerators and $(x-4)$ from the denominators]

or $\qquad 2x-2 = -5x+10$ 　[By cross multiplication]

or $\qquad 2x+5x = 10+2 \qquad$. or $\qquad 7x = 12$

or $\qquad x = \dfrac{12}{7} = 1\tfrac{5}{7}.$　**Ans.**

Note. Note carefully the form of the given equation. The numerator of R. H. S. (*viz* 8) is equal to the sum of the two numerators on the L. H. S. (*viz.*, 5 and 8). We write 8 as

5+8, and thus split the fractions on R. H. S. into two fractions. Then we combine fractions with equal numerators, etc., etc.

It should also be noted that the coefficient of x in each denominator is 1. The device can be used even if the coefficients of x in the denominators are *equal*, though not equal to 1 each, (See Q. 85).

35. Write R. H. S. as $\dfrac{8}{4x+3} + \dfrac{4}{4x+3}$ and proceed as in Q. 81.

37. The equation may be written as

$$-\dfrac{5}{4x-8} + \dfrac{9}{4x+18} - \dfrac{4}{4x+5} = 0.$$ Split $\dfrac{9}{4x+13}$ into two parts

$$\dfrac{5}{4x+18} + \dfrac{4}{4x+13}$$ and proceed as before.

39. $\dfrac{9}{3x+2} + \dfrac{1}{x-4} = \dfrac{8}{2x+5}$

The coefficients of x in the denominators are not equal, but they can be made equal as follows :—

The L. C. M. of the coefficients (3, 1, 2) is 6. Multiply the numerators and the denominators of the fractions by such numbers as to make the coefficient of x equal to 6 in each denominator. Evidently these numbers are :—

2 for the first fraction, 6 for the second, 3 for the third. Thus we have :—

$$\dfrac{18}{6x+4} + \dfrac{6}{6x-18} = \dfrac{24}{3x+15}$$

Now proceed as in Q. 35 or 31.

45. The given equation is $\dfrac{5}{x-1} - \dfrac{3}{3x-4} - \dfrac{8}{2x-1} = 0.$

[Taking minus sign out the second denominator]

[L. C. M. of the coefficient x (*i.e.* of 1, 3 and 2) is 6]

$$\therefore \dfrac{30}{6x-6} - \dfrac{6}{6x-8} - \dfrac{24}{6x-3} = 0, \text{ etc., etc.}$$

47   Split R. H. S. into two fractions $\dfrac{a}{x-a-b}+\dfrac{b}{x-a-b}$

and proceed as in Q. 31.

51 $\dfrac{6x+8}{2x-1}-\dfrac{2x+38}{x+12}=1$

Dividing $6x+8$ by $2x+1$ we get 3 as quotient and 5 as remainder.

$$2x+1\overline{)6x+8(}3$$
$$\underline{6x+3}$$
$$5$$

$\therefore \quad \dfrac{6x+8}{2x+1}$ may be written as $3+\dfrac{5}{2x+1}$

Similarly $\dfrac{2x+38}{x+12}$ may be written as $2+\dfrac{14}{x+12}$

Hence we have :—

$$\left(3+\dfrac{5}{2x+1}\right)-2+\left(\dfrac{14}{x+12}\right)=1$$

or $3+\dfrac{5}{2x+1}-2-\dfrac{14}{x+12}=1$

or $\dfrac{5}{2x+1}=\dfrac{14}{x+12}=1-3+2$ [By transposition]

$$=0.$$

$\therefore$ $28x+14=5x+60$ [By cross multiplication]

or $23x=46$

or $x=\dfrac{46}{23}=2$.   **Ans.**

61. $\dfrac{x^2+7x+12}{x+2}$ may be written as

$$x+5+\dfrac{2}{x+2}$$ [why ?]

etc., etc.

$$x+2\overline{)x^2+7x+12(}x+5$$
$$\underline{x^2+2x}$$
$$5x+12$$
$$\underline{5x+10}$$
$$2$$

63.   The equation may be written as :—

$$2+\dfrac{1}{x-1}+3+\dfrac{2}{x-2}=5+\dfrac{3}{x-3}$$

or $\qquad \dfrac{1}{x-1}+\dfrac{2}{x-2}=\dfrac{3}{x-3}$ [Taking away $2+3$ from L.H.S

and 5 from R. H. S.]

Now proceed as in Q. 31, by writing $\dfrac{3}{x-3}$ as $\dfrac{1}{x-3}+\dfrac{2}{x-3}$.

69. The given equation may be written as :—

$$1+\dfrac{1}{x-2}+1+\dfrac{1}{x-7}=1+\dfrac{1}{x-3}+1+\dfrac{1}{x-6}$$

or $\qquad \dfrac{1}{x-2}+\dfrac{1}{x-7}=\dfrac{1}{x-3}+\dfrac{1}{x-6}$

or $\qquad \dfrac{1}{x-2}-\dfrac{1}{x-3}=\dfrac{1}{x-6}-\dfrac{1}{x-7}$

[Note this step of transposition : it is to get rid of x from the numerators. If we do not make this transposition and add up the fractions on the two sides, x does not clear off from the numerators.]

$\therefore \qquad \dfrac{(x-3)-(x-2)}{(x-2)(x-3)}=\dfrac{(x-7)-(x-6)}{(x-6)(x-7)}$

or $\qquad \dfrac{-1}{(x-2)(x-3)}=\dfrac{-1}{(x-6)(x-7)}$

Remove negative sign from both the numerators and cross multiply.

_____

EXERCISE 106

Solve the following equations :—

1. $\dfrac{x-a}{b+c}+\dfrac{x-b}{c+a}+\dfrac{x-c}{a+b}=3.$ [Solved]. 2. $\dfrac{x-p}{q}+\dfrac{x-q}{p}=2.$

3. $\dfrac{x-a^2}{b^2+c^2}+\dfrac{x-b^2}{a^2+c^2}+\dfrac{x-c^2}{a^2+b^2}=3.$

4. $\dfrac{x-ab}{a+b}+\dfrac{x-bc}{x+c}+\dfrac{x-ca}{c+a}=a+b+c.$ [Hint]

5. $\dfrac{x+a}{b+c}+\dfrac{x+b}{c+a}+\dfrac{x+c}{a+b}+3=0.$

6. $\dfrac{x-m^3}{m^2+m}+\dfrac{x-m^2}{m^2+1}+\dfrac{x-m}{m+1}=1+m+m^2.$

7. $\dfrac{x}{x+1}+\dfrac{x+1}{x}=\dfrac{25}{12}.$ [Solved]   8. $x+\dfrac{1}{x}=\dfrac{13}{6}.$

9. $\dfrac{x-3}{x+3}+6\dfrac{6}{7}=\dfrac{x+3}{x-3}.$ 10. $\dfrac{4}{x-1}-\dfrac{5}{x+2}=\dfrac{3}{x}.$

11. $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c).$ [Solved]
12. $(x+1)^3+(x+2)^3+(x+3)^3=3(x+1)(x+2)(x+3)$
13. $x^3+(x-2)^3+(x-4)^3-3x(x^2-6x+8)=0.$ [Hint]
14. $x^3+(x+a^3)+(x-a)^3-3x(x^2-a^2)=0.$

SOLUTIONS AND HINTS—EX. 106

1. $\dfrac{x-a}{b+c}+\dfrac{x-b}{c+a}+\dfrac{x-c}{a+b}=3$

or $\dfrac{x-a}{b+c}+\dfrac{x-b}{c+a}+\dfrac{x-c}{a+b}=1+1+1$

or $\dfrac{x-a}{b+c}-1+\dfrac{x-b}{c+a}-1+\dfrac{x-c}{a+b}-1=0$

or $\dfrac{x-a-b-c}{b+c}+\dfrac{x-b-c-a}{c+a}+\dfrac{x-c-a-b}{a+b}=0$

or $(x-a-b-c)\left\{\dfrac{1}{b+c}+\dfrac{1}{c+a}+\dfrac{1}{a+b}\right\}=0$

or $x-a-b-c=0$

$\left[\text{Dividing both sides by }\left(\dfrac{1}{b+c}+\dfrac{1}{c+a}+\dfrac{1}{a+b}\right)\right]$

or $x=a+b+c.$ Ans.

4. Transpose $a+b+c$ to L. H. S., combining c with the first fraction, a with the second and b with the third.

7. $\dfrac{x}{x+1}+\dfrac{x+1}{x}=\dfrac{25}{12}$

L. C. M. of the denominators is $12x(x+1)$. Multiplying both sides by their L. C. M. we have :—

$$12x^2+12(x+1)^2=25x(x+1)$$

or $12x^3+12x^2+24x+12=25x^3+25x$

or $-x^2-x+12=0$

[Transposing all terms to L. H. S. and simplifying]

or $x^2+x-12=0$ or $(x+4)(x-3)=0$

∴ Either* $x+4=0$ or $x-3=0$

∴ Either $x=-4$ or $x=3$

∴ $x=-4$ or 3.   **Ans.**

[Test the two solutions].

11. $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c)$

∴ $(x-a)^3+(x-b)^3+(x-c)^3-3(x-a)(x-b)(x-c)=0$.

Applying the formula :—

$$x^3+y^3+z^3-xyz=\tfrac{1}{2}(x+y+z)\{(x-y)^2+(y-z)^2+(z-x)^2\}$$

to the L. H. S. we get :—

$\tfrac{1}{2}\{(x-a)+(x-b)+(x-c)\}[\{(x-a)-(x-b)\}^2$
+two similar terms]$=0$

or $(3x-a-b-c)[(b-a)^2+$two similar terms]$=0$

[Multiplying both sides by 2]

or $3x-a-b-c=0$

[Dividing both sides by $\{(b-a)^2+$two similar terms $\}$

or $3x=a+b+c$

or $x=\dfrac{a+b+c}{3}$.   **Ans**

13. Write x^2-6x+8 as $(x-2)(x-4)$ and proceed as before.

EXERCISE 107

Solve the following simultaneous equations :—

1. $\left.\begin{array}{l}a_1x+b_1y+c_1=0\\a_2x+b_2y+c_2=0\end{array}\right\}$. [Solved]

2. $\left.\begin{array}{l}ax+by=p\\bx+ay=q\end{array}\right\}$

3. $\left.\begin{array}{l}px-qy=r\\rx-py=q\end{array}\right\}$

4. $\left.\begin{array}{l}lx+my=l^2\\mx+ly=m^2.\end{array}\right\}$

---

*When product of two factors is zero, either of them can be zero.

5. $\left.\begin{array}{l} \dfrac{x}{m} + \dfrac{y}{n} = \dfrac{1}{mn} \\ \dfrac{x}{m'} - \dfrac{y}{n'} = \dfrac{1}{m'n'} \end{array}\right\}$

6. $\left.\begin{array}{l} \dfrac{x}{p} - \dfrac{y}{q} = 0 \\ qx + py - 4pq = 0. \end{array}\right\}$

7. $\left.\begin{array}{l} \dfrac{x}{a+b} + \dfrac{y}{a-b} = 2 \\ (a+b)x - (a-b)y = 4ab \end{array}\right\}$

8. $\left.\begin{array}{l} (a-b)x + (a+b)y = 2(a^2 - b^2) \\ ax - by = a^2 + b^2 \end{array}\right\}$

9. $\left.\begin{array}{l} \dfrac{a}{x} + \dfrac{b}{y} = c \\ \dfrac{b}{x} + \dfrac{a}{y} = d \end{array}\right\}$ [Hint]

10. $\left.\begin{array}{l} 2ab(x-y) = xy(a-b) \\ 2ab(x+y) + xy(a+b+2ab) = 0 \end{array}\right\}$ [Hint]

11 $\left.\begin{array}{l} x+y=8 \\ xy=15 \end{array}\right\}$ Hint

12. $\left.\begin{array}{l} x+y=10 \\ xy=24 \end{array}\right\}$

13. $\left.\begin{array}{l} x-y=2 \\ xy=48 \end{array}\right\}$ [Hint]

14. $\left.\begin{array}{l} x-y=9 \\ xy=-20 \end{array}\right\}$

15. $\left.\begin{array}{l} 6(x+y)=1 \\ 6xy+1=0 \end{array}\right\}$ [Hint]

16. $\left.\begin{array}{l} 4(x-y)=3 \\ 8xy+1-0 \end{array}\right\}$

17 $\left.\begin{array}{l} x^2 - y^2 = 24 \\ x-y=4 \end{array}\right\}$ [Hint]

18. $\left.\begin{array}{l} x^2 - y^2 = 32 \\ x+y=8. \end{array}\right\}$

19. $\left.\begin{array}{l} \dfrac{1}{x^2} - \dfrac{1}{y^2} = 33 \\ \dfrac{1}{x} - \dfrac{1}{y} = 3 \end{array}\right\}$

20. $\left.\begin{array}{l} x+y=7xy \\ y^2 - x^2 + 21xy = 0 \end{array}\right\}$

21 $\left.\begin{array}{l} xy=6 \\ yz=12 \\ zx=8 \end{array}\right\}$ [Solved]

22. $\left.\begin{array}{l} xy=20 \\ yz=30 \\ zx=24 \end{array}\right\}$

23. $\left.\begin{array}{l} 6xy+1=0 \\ 15yz-1=0 \\ 10zx+1=0 \end{array}\right\}$ [Hint]

24. $\left.\begin{array}{l} 8xy+1=0 \\ 5yz+2=0 \\ 15zx-8=0 \end{array}\right\}$

—— · ——

25. $\left.\begin{array}{l} y+z=2a \\ z+x=2b \\ x+y=2c \end{array}\right\}$ [Solved]

26. $\left.\begin{array}{l} ax+by=2c \\ by+cz=2a \\ cz+ax=2b \end{array}\right\}$

27 $\left.\begin{array}{l} cy+bz=bc \\ az+cx=ca \\ bx+ay=ab \end{array}\right\}$ [Hint]

28. $\left.\begin{array}{l} \dfrac{1}{ay}+\dfrac{1}{bx}=\dfrac{c}{xy} \\[2mm] \dfrac{1}{cz}+\dfrac{1}{ay}=\dfrac{b}{yz} \\[2mm] \dfrac{1}{cz}+\dfrac{1}{bx}=\dfrac{a}{zx} \end{array}\right\}$

39 $\left.\begin{array}{l} x(y+z)=5 \\ y(z+x)=8 \\ z(x+y)=9 \end{array}\right\}$ [Hint]

30. $\left.\begin{array}{l} x(x+y+z)=18 \\ y(x+y+z)=27 \\ z(x+y+z)=36 \end{array}\right\}$ [Hint]

—— · ——

31 $\left.\begin{array}{l} x+y+z=0 \\ ax+by+cz=0 \\ bcx+cay+abz+(b-c)(c-a)(a-b)=0 \end{array}\right\}$ [Solved]

32. $\left.\begin{array}{l} x+y+z=0 \\ bcx+cay+abz=0 \\ ax+by+cz+(b-c)(c-a)(a-b)=0 \end{array}\right\}$

33. $\left.\begin{array}{l} ax+by+cz=0 \\ a^2x+b^2y+c^2z=0 \\ x+y+z+(b-c)(c-a)(a-b)=0 \end{array}\right\}$

34. $\left.\begin{array}{l} x+y+z=0 \\ (b+c)x+(c+a)y+(a+b)z=0 \\ bcx+cay+abz=1 \end{array}\right\}$ (Calcutta 1906)

——————

SOLUTIONS & HINTS—EX. 107

1. $\begin{array}{l} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{array}$

$\begin{array}{ccc} b_1 & a_1 & a_1 & b_1 \\ \times & \times & \times \\ b_2 & c_2 & a_2 & b_2 \end{array}$

By cross multiplication, $\dfrac{x}{b_1c_2-c_1b_2}=\dfrac{y}{c_1a_2-a_1c_2}$

$$=\dfrac{1}{a_1b_2-b_1a_2}$$

$\therefore x=\dfrac{b_1c_2-c_1b_2}{a_1b_2-b_1a_2}$ and $y=\dfrac{c_1a_2-a_1c_2}{a_1b_2-b_1a_2}$

9. By cross multiplication, $\dfrac{\frac{1}{x}}{\text{what?}}=\dfrac{\frac{1}{y}}{\text{what?}}=\dfrac{1}{\text{what?}}$,

etc.

10. Dividing both sides of the first equation by xy we get

$$2ab\left(\dfrac{1}{y}-\dfrac{1}{x}\right)=a-b.$$

$\therefore \dfrac{1}{y}-\dfrac{1}{x}=\dfrac{a-b}{2ab}$. Similarly get the value of

$\dfrac{1}{y}+\dfrac{1}{x}$ from the second equation. Add & substract

the two results.

11. $x+y=8$ ...(i)
$xy=15$ ...(ii)

We have $(x-y)^2=(x+y)^2-4xy$ [Formula]
$$=(8)^2-4\times15=4$$

$\therefore$ Either $x-y=2$ ...(iii)

or $x-y=-2$ ...(iv)

Solve (i) & (iii) and also (i) & (iv).

13. Apply the formula $(x+y)^2=(x-y)^2+4xy$.

15. Given equations may be written as $x+y=\frac{1}{6}$ and $xy=-\frac{1}{6}$.

17. Dividing the first equation by the second we get
$x+y=6$, etc.

21. $xy=6$ ...(i)
$yz=12$ ...(ii)
$zx=8$ ...(iii)

Multiplying (i), (ii) & (iii) we get :—

$$x^2 y^2 z^2 = 6 \times 12 \times 8.$$

∴ Either $xyz = 24$ ...(iv) or $xyz = -24$ ...(v)

Dividing (iv) by (i) we get $\dfrac{xyz}{xy} = \dfrac{24}{6}$, i.e., $z = 4$

Similarly dividing (iv) by (ii) & (iii) we get $x = 2$ and $y = 3$.

If we use equation (v) instead of (iv) we get $x = -2$, $y = -3$, $z = -4$.

Hence $\qquad x = 2, y = 3, z = 4$ ⎫
or $\qquad x = -2, y = -3, z = -4.$ ⎭ **Ans.**

23. From the given equations we get :—

$$xy = -\frac{1}{6}, \quad yz = \frac{1}{15}, \quad zx = -\frac{1}{10}. \text{ etc.}$$

25 $\quad y + z = 2a$ $\qquad\qquad\qquad$ ...(i)
$\qquad z + x = 2b$ $\qquad\qquad\qquad$ ...(ii)
$\qquad x + y = 2c$ $\qquad\qquad\qquad$ ...(iii)

(i)+(ii)+(iii) gives $\quad 2(x + y + z) = 2(a + b + c)$

or $\qquad\qquad\qquad x + y + z = a + b + c$ $\qquad$ ...(iv)

(iv)−(i) gives $x = a + b + c - 2a = b + c - a$ ⎫
(iv)−(ii) ,, $y = a + b + c - 2b = c + a - b$ ⎬
(iv)−(iii) ,, $z = a + b + c - 2c = a + b - c$ ⎭ **Ans.**

27. The first equation may be written as $\dfrac{y}{b} + \dfrac{z}{c} = 1$

[Dividing both sides by bc]

Similarly, $\dfrac{z}{c} + \dfrac{x}{a} = 1$, $\dfrac{x}{a} + \dfrac{y}{b} = 1$. Now proceed as in Q. 25.

29. Add the three equations and get the value of $xy + yz + zx$.

From this subtract the given equations in turn, thus getting the values of yz, zx and xy. Then proceed as in Q. 21.

30.   Adding the three equations we get : —

$$x(x+y+z)+y(x+y+z)+z(x+y+z)=81$$

or $(x+y+z)(x+y+z)=81$

or $(x+y+z)^2=81$.   Either $x+y+z=9$ ...(iv)

          or $x+y+z=-9$ ...(v)

Dividing the given equations by (iv) we get the values of x, y and z.

Dividing them by (v) we get another set of values.

31. $x+y+z=0$ ...(i)
        $ax+by+cz=0$ ...(ii)
        $$\overset{1}{\underset{b}{\times}}\overset{1}{\underset{c}{\times}}\overset{1}{\underset{a}{\times}}\overset{1}{\underset{b}{}}$$

$$bcx+cay+abz+(b-c)(c-a)(a-b)=0 \qquad \text{...(iii)}$$

From (i) & (ii) by cross multiplication : —

$$\frac{x}{c-b}=\frac{y}{a-c}=\frac{z}{b-a}$$

or $$\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}=k \text{ (suppose)}$$

∴ $x=(b-c)k,\ y=(c-a)k,\ z=(a-b)k$ ...(iv)

Substituting these values of x, y and z in (iii) we have : —

    $bc(b-c)k+ca(c-a)k+ab(a-b)k+(b-c)(c-a)(a-b)=0$

                    or

  $k\{bc(b-c)+ca(c-a)+ab(a-b)\}+(b-c)(c-a)(a-b)=0$

or $-k(a-b)(b-c)(c-a))+(b-c)(c-a)(a-b)=0$

or $-k+1=0$ [Dividing both sides by $(a-b)(b-c)(c-a)$]

or $k=1$.

Substituting this value of k in results (iv) we have : —

    $x=b-c,\ y=c-a,\ z=a-b$.   **Ans.**

CHAPTER XXVI

CONDITIONAL IDENTITIES & INDETERMINATE CO-EFFICIENTS

173. A Conditional identity is one which is ·true **only** when given condition is satisfied by the letters involved.

For example, if $a+b+c=0$ then we have $a^3+b^3+c^3 \equiv 3abc$.

[See Ex. 108, Q. 9.]

Note :

Proofs for conditional identities mostly depend upon artifices. The student is therefore to be guided by the solutions and hints to the next exercise.

EXERCISE 108

1. If $\left(x+\dfrac{1}{x}\right)^2 = 3$, prove that $x^3 + \dfrac{1}{x^3} = 0$ [*Hint*]

2. If $x+y=2z$, prove that $\dfrac{x}{x-z} + \dfrac{y}{y-z} = 2$ [*Hint*]

3. If $x+\dfrac{1}{y}=1$ and $y+\dfrac{1}{z}=1$, prove that $z+\dfrac{1}{x}=1$.

    [*Hint*]

4. If $y-z=ax$, $z-x=by$ and $x-y=cz$, show that $a+b+c+abc=0$.

If $2s=a+b+c$, prove that :—

5. $s^2 + (s-a)^2 + (s-b)^2 + (s-c)^2 = a^2 + b^2 + c^2$. [*Solved*]
6. $(s-a)^3 + (s-b)^3 + (s-c)^3 + 3abc = s^3$.
7. $(s-a)^3 + (s-b)^3 + 3(s-a)(s-b)c = c^3$. [*Hint*]
8. $(s-a)^3 + (s-b)^3 + (s-c)^3 - 3(s-a)(s-b)(s-c)$
    $$= \tfrac{1}{3}(a^3 + b^3 + c^3 - 3abc).$$

If $a+b+c=0$ show that :—

9. $a^3 + b^3 + c^3 = 3abc$. [*Solved*]
10. $ab(a+b) = bc(b+c) = ca(c+a)$. [*Hint*]
11. $(a+b)(b+c)(c+a) = -abc$.

12. $a^2 - bc = b^2 - ca = c^2 - ab.$ [*Hint*]

13. $a^2 + b^2 + c^2 = -2(ab + bc + ca).$

14. $a^2 + ab + b^2 = b^2 + bc + c^2 = c^2 + ca + a^2.$

15. $\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right)^2 = \dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}.$ [*Solved*]

16. $\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right)^3 = \dfrac{1}{a^3} + \dfrac{1}{b^3} + \dfrac{1}{c^3} - \dfrac{8}{abc}.$ [*Hint*]

17. $\dfrac{1}{b^2 + c^2 - a^2} + \dfrac{1}{c^2 + a^2 - b^2} + \dfrac{1}{a^2 + b^2 - c^2} = 0.$ [*Hint*]

18. $\dfrac{a^2}{2a^2 + bc} + \dfrac{b^2}{2b^2 + ca} + \dfrac{c^2}{2c^2 + ab} = 1.$ [*Hint*]

19. $a(b-c)^2 + b(c-a)^2 + c(a-b)^2 + 9abc = 0.$

20. $2(a^4 + b^4 + c^4) = (a^2 + b^2 + c^2)^2.$ [*Solved*]

21. $4(b^2c^2 + c^2a^2 + a^2b^2) = (a^2 + b^2 + c^2)^2.$

22. Prove that $\dfrac{1}{(y-z)^2} + \dfrac{1}{(z-x)^2} + \dfrac{1}{(x-y)^2}$

$$= \left(\dfrac{1}{y-z} + \dfrac{1}{z-x} + \dfrac{1}{x-y}\right)^2.$$ [*Hint*]

23 Prove that $(y-z)^2(z-x)^2 + (z-x)^2(x-y)^2 + (x-y)^2(y-z)^2$

$$= (x^2 + y^2 + z^2 - yz - zx - xy)^2.$$

SOLUTIONS & HINTS—Ex. 108

1. $\left(x + \dfrac{1}{x}\right)^2 = 3$ [given]

$$\therefore \qquad x + \dfrac{1}{x} = \sqrt{3}$$

Now, $x^3 + \dfrac{1}{x^3} = \left(x + \dfrac{1}{x}\right)^3 - 3x \cdot \dfrac{1}{x}\left(x + \dfrac{1}{x}\right)$ etc.

2. $x + y = 2z$ [given] $\therefore$ $x + y = z + z$

$\therefore$ $x - z = z - y$ [By transposition]

$$\therefore \qquad \dfrac{x}{x-z} + \dfrac{y}{y-z} = \dfrac{x}{z-y} + \dfrac{y}{y-z}$$ etc.

Eliminate y between the first two equations and put the result thus obtained in the required form.

5. $s^2+(s-a)^2+(s-b)^2+(s-c)^2$

$=s^2+(s^2-2as+a^2)+(s^2-2bs+b^2)+(s^2-2cs+c^2)$

$=4s^2-2as-2bs-2cs+a^2+b^2+c^2$

$=2s(2s-a-b-c)+a^2+b^2+c^2$

$=2s(a+b+c-a-b-c)+a^2+b^2+c^2$ [$\because$ $2s=a+b+c$]

$=2s\times0+a^2+b^2+c^2=a^2+b^2+c^2$

$=a^2+b^2+c^2$.

7. $2s=a+b+c$ $\quad\therefore$ $2s-a-b=c$

or $(s-a)+(s-b)=c$

Cubing both sides we get the required identity.

9. $a+b+c=0$ $\quad\therefore$ $a+b=-c$ $\quad\therefore$ $(a+b)^3=(-c)^3$

or $a^3+b^3+3ab(a+b)=-c^3$

or $a^3+b^3+3ab(-c)=-c^3$ [$\because$ $a+b=-c$]

or $a^3+b^3-3abc=-c^3$ $\qquad$ or $a^3+b^3+c^3=3abc$.

10. $a+b+c=0$ $\qquad\therefore$ $a+b=-c$

$\therefore$ $ab(a+b)=ab(-c)=-abc$

Similarly prove the other expressions equal to $-abc$ each.

12. $a^2-bc=a\times a-bc=a(-b-c)-bc$ [$\because$ $a=-b-c$]

$\qquad\qquad=-ab-ac-bc$.

Similarly prove other expressions equal to $-ab-ac-bc$ each.

15. $\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)^2=\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}+\dfrac{2}{ab}+\dfrac{2}{bc}+\dfrac{2}{ca}$

$=\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}+\dfrac{2}{abc}(c+a+b)$

$=\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}+\dfrac{2}{abc}\times0$ [$\because$ $a+b+c=0$]

$=\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}$.

16. Use the formula $(x+y+z)^3$

$=x^3+y^3+z^3+3(x+y)(y+z)(z+x)$

17. $a+b+c=0$ $\therefore$ $b+c=-a$ $\therefore$ $(b+c)^2=(-a)^2$

or $b^2+c^2+2bc=a^2$ or $b^2+c^2-a^2=-2bc.$

Hence the first fraction $=\dfrac{1}{-2bc}$

Similarly get the values of other fractions and add.

18. $a+b+c=0$ $\therefore$ $a=-(b+c)$

Multiplying both sides by a we get $a^2=-a(b+c)$

$\therefore$ $2a^2+bc=a^2+a^2+bc=a^2-a(b+c)+bc$ $[\because \ a^2=-a(b+c)]$

$\qquad\qquad =(a-b)(a-c)$

$\therefore$ First fraction $=\dfrac{a^2}{(a-b)(a-c)}$ etc. etc.

20. $a+b+c=0$ $\therefore$ $(a+b+c)^2=0$

or $a^2+b^2+c^2+2(ab+bc+ca)=0$

or $a^2+b^2+c^2=-2(ab+bc+ca)$

Squaring again $a^4+b^4+c^4+2(a^2b^2+b^2c^2+c^2a^2)$

$\qquad\qquad =4(a^2b^2+b^2c^2+c^2a^2+2ab^2c+2bc^2a+2ca^2b)$

$\qquad\qquad =4\{\ a^2b^2+b^2c^2+c^2a^2+2abc(b+c+a)\ \}$

$\qquad\qquad =4(a^2b^2+b^2c^2+c^2a^2)$ $[\because \ b+c+a=0]$

$\therefore$ $a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)$

Adding $a^4+b^4+c^4$ to both sides we get,

$\qquad\qquad 2(a^4+b^4+c^4)=a^4+b^4+c^4+2(a^2b^2+b^2c^2+c^2a^2)$

$\qquad\qquad =(a^2+b^2+c^2)^2.$

22.   Put $y-z=a$, $z-x=b$, $x-y=c$

Then we have $a+b+c=y-z+z-x+x-y=0$

and we have to show that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2.$$

This is Q. 15.

174. Theorem. *If two expressions are identically equal, the coefficients of like powers of a variable in the two expressions are equal,*

Proof. Let $a+bx+cx^2+dx^3+\ldots\ldots\ldots$

$$\equiv A+Bx+Cx^2+Dx^3+\ldots\ldots\ldots$$

This relation is true for all values of x [Definition]

Put $x=0$, then we have :—

$$a+0+0+0+\ldots\ldots\ldots=A+0+0+0+\ldots\ldots\ldots$$

i.e., $a=A$. $\therefore$ Given identity gives

$$bx+cx^2+dx^3+\ldots\ldots\ldots=B+Cx+Dx^3+\ldots\ldots\ldots$$

Dividing both sides by x we get :—

$$b+cx+dx^2+\ldots\ldots\ldots=B+Cx+Dx^2+\ldots\ldots\ldots$$

Again putting $x=0$ we get $b=B$.

Similarly $c=C$, $d=D$, etc.

Hence the coefficients of like powers of x are equal.

175. The student now knows the following two important facts about an identity :—

(i) the coefficients of like powers of a variable on the two sides are equal.

(ii) the relation is satisfied by all values of the variable.

Either of these facts may be used to solve questions of the next exercise. However, the student should see which of them is more suitable to a particular case.

EXERCISE 109

1. If $2ax^3+3bx^2-4cx-d\equiv 6x^3-4x^2+3$, find the values of a, b, c and d. [*Solved*]

2. If $ax^3-4bx^2+5cx+2d\equiv 4x^2-3x-8$, find the values of a, b, c and d.

3. If $(x+p)^2\equiv x^2+3x+q$, evaluate p and q. [*Hint*]

4. If $ax^2+bx+c\equiv 6(x-2)(x-3)$, evaluate a, b and c.

5. If $3x-5\equiv A(x-2)+B(x-3)$, evaluate A and B. [*Solved*

6. If $2x-7\equiv A(x-3)+B(x-4)$, evaluate A and B.

7   If $2-11x \equiv A(1-3x)+B(1-2x)$, evaluate A and B.

                                                       [*Hint*]

8.   If $12x+5 \equiv P(3x+4)-Q(2x-1)$, evaluate P and Q.

Evaluate A, B and C, given that :—

9. $2x^2+9x+6 \equiv Ax(x+1)+Bx(x+2)+C(x+1)(x+2)$.

                                                       [*Solved*]

10. $3x^2+11x-7 \equiv (Ax+B)(x-2)+C(x^2+3x-1)$.

11. $4x^2+3x-2 \equiv A(2x^2+3)+B(x+2)+C$.

12. $3x^2-4x+5 \equiv A(x-1)^2+B(x-1)+C$.

13. $2x^2+x-14 \equiv A(x+2)^2-B(x+3)-C$.

14. $x^2-2x+8 \equiv A(x-1)(x-2)+B(x-2)(x-3)$

        $+C(x-3)(x-1)$.   [*Hint*]

15. $p^2-10p+13 \equiv A(p^2-5p+6)+B(p-1)(p-2)$

        $+C(p-3)(p-1)$.   (*P. U. 1931*)

16. $3x^3+9x^2+7x+2 \equiv A(x+1)^3+B(x+1)+C$.   (*P. U. 1932*)

17.   If $\dfrac{3x}{x^2-x-2} \equiv \dfrac{A}{x+1}+\dfrac{B}{x-2}$,   evaluate   A and B.

                                                         [*Solved*]

18.   If $\dfrac{8x}{x^2+2x-3} \equiv \dfrac{A}{x-1}+\dfrac{B}{x+3}$, evaluate A and B.

19.   If $\dfrac{2x-5}{2x^2+5x+2} \equiv \dfrac{A}{x+2}+\dfrac{B}{2x+1}$, evaluate A and B.

20.   If $\dfrac{2x^2-x-8}{(x-1)^2(x-2)} \equiv \dfrac{A}{x-2}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$,

                                            evaluate A, B and C.

SOLUTIONS & HINTS—Ex. 109

1.   The given identity is :—

     $2ax^3+3bx^2-4cx-d \equiv 6x^3-4x^2+0x+3$

         [Writing $0x$ for the missing power on R. H. S.]

Coefficient of x^3 on L. H. S. should be equal to coefficient of x^3 on R. H. S.

$\therefore$ $2a=6$ or $a=3$.

Similarly, equating coefficients of other powers of x we get :—

$$3b=-4, \qquad \therefore \quad b=-\tfrac{4}{3}$$
$$-4c=0 \qquad \therefore \quad c=0$$
$$-d=3 \qquad \therefore \quad d=-3,$$

3. $(x+p)^2 \equiv x^2+3x+q$.

    or $x^2+2px+p^2 \equiv x^2+3x+q$

      Equate coefficients of x and the constant terms

$\therefore$ $2p=3$ and $p^2=q$.   Solve these equations.

5. $3x-3 \equiv A(x-2)+B(x-3)$.

First Method

R. H. S. $=Ax-2A+Bx-3B$

       $=(A+B)x-(2A+3B)$.

$\therefore$ we have :—

$3x-5 \equiv (A+B)x-(2A+3B)$

Equating cofficients of x and constant terms we get :—

     $A+B=3$ ......(i)

     $2A+3B=5$ ......(ii)

Solving equations (i) and (ii) we get $A=4$, $B=-1$ **Ans**.

Second Method

The given identity is true for all values of x.

Put $x=3$, then $9-5=A(3-2)+B(0)$

         or $4=A$

Put $x=2$, then $6-5=A(0)+B(2-3)$

           or $1=-B$ or $B=-1$

         $\therefore$ $A=4$. $B=-1$.   **Ans**.

    Note. Evidently the second method is more suitable for this example.

    7.   Put $x=\tfrac{1}{2}$ and $\tfrac{1}{3}$.

9. **R. H. S.** of the given identity is :—

$$Ax(x+1)+Bx(x+2)+C(x+1)(x+2)$$
$$=A(x^2+x)+B(x^2+2x)+C(x^2+3x+2)$$
$$=(A+B+C)x^2+(A+2B+3C)x+2C.$$

Hence we have :—

$$2x^2+9x+6\equiv(A+B+C)x^2+(A+2B+3C)x+2C.$$

Equating coefficients of like powers of x we get :—

$$A+ B+ C=2 \qquad\qquad \dots \text{(i)}$$
$$A+2B+3C=9 \qquad\qquad \dots \text{(ii)}$$
$$2C=6 \qquad\qquad \dots \text{(iii)}$$

From (iii) $c=3$,

Substituting this value of c in (i) and (ii) we have :—

$$A+ B+3=2 \quad \text{or} \quad A+ B=-1 \Bigg\}$$
$$\text{and} \quad A+2B+9=9 \quad \text{or} \quad A+2B=0 \Bigg\}$$

Solving these two equations we get $B=1$ and $A=-2$.

$$\therefore \quad A=-2, B=1, C=3. \quad \text{Ans.}$$

14. Put $x=1$, 2 and 3 in turn.

17. **R. H. S.** of the given identity

$$=\frac{A}{x+1}+\frac{B}{x-2}=\dots\frac{A(x-2)+B(x+1)}{(x+1)(x-2)}=\frac{A(x-2)+B(x+1)}{x^2-x-2}$$

$$\therefore \quad \text{We have} \quad \frac{3x}{x^2-x-2}\equiv\frac{A(x-2)+B(x+1)}{x^2-x-2}$$

$$\therefore \quad 3x\equiv A(x-2)+B(x+1)$$

$$[\because \quad \text{Denominators are the same}]$$

Put $x=2$, $\therefore$ $(6=A(0)+B(3)$, i.e. $3B=6$ or $B=2$

Put $x=-1$, $\therefore$ $-3=A(-3)+B(0)$

$$\therefore \quad -3A=-3 \quad \text{or} \quad A=1$$

$$\therefore \quad A=1, B=2. \quad \text{Ans.}$$

Paper 1—Ex. 110

1. Simplify : $\dfrac{a}{2}\left(\dfrac{1}{a-b}-\dfrac{1}{a+b}\right)\times\dfrac{a^2-b^2}{a^2b+ab^2}-\dfrac{1}{a+b}.$

2. Solve for x and y $\quad\left.\begin{array}{l}3^{x+1}=3\times9^y\\(\sqrt5)^{-y-2}=125\end{array}\right\}$

3. If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$, prove that $a,\,b,\,c,\,d$ are proportionals.

4. $\left.\begin{array}{l}x+z+y=0\\x^2+y^2+z^2=0\end{array}\right\}$ Eliminate x.

5. Solve graphically the equations $x+y+,10=0$ $2x-y+5=0.$

6. If $ax+by+cz=bx+cy+az=cx+ay+bz=0$, show that $a^3+b^3+c^3=3abc.$

Paper 2—Ex. 111

1. Find the square root of $\left(x^2+\dfrac{1}{x^2}\right)^2-4\left(x^2-\dfrac{1}{x^3}\right).$

2. Simplify : $\dfrac{18^{n+1}.25^{2m+n}.30^{1-2n}}{2^{1-n}.54^{m+1}\,9}$

3. If $x=\dfrac{2\sqrt{10}}{\sqrt5+\sqrt2}$, evaluate $\dfrac{x+\sqrt5}{x-\sqrt5}+\dfrac{x+\sqrt2}{x-\sqrt2}.$

4. $\left.\begin{array}{l}x+\dfrac{1}{x}=t^2\\[2mm]x-\dfrac{1}{x}=t\end{array}\right\}$ Eliminate t.

5. Solve graphically the equations :—
$$x-3y-6=\dfrac{3}{2},\ x+\dfrac{5}{2}y-2=0.$$

What is the intercept of the graph of the former on the axis of y ?

6. Solve the equation $16\left(\dfrac{2-3x}{2+3x}\right)^3=\dfrac{2+3x}{2-3x}.$

Paper 3—Ex. 112

1. Simplify : $\dfrac{\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}}{\dfrac{x-y}{x+y}-\dfrac{x+y}{x-y}} \times \dfrac{xy^3 - x^3 y}{x^2 + y^2}$.

2. Find the square root of $\dfrac{7}{2} - \dfrac{3}{2}\sqrt{5}$.

3. If $a : b : c : d$. show that $\dfrac{\sqrt{a^2 + ab + b^2}}{\sqrt{c^2 + cd + d^2}} = \dfrac{a^2 + b^2}{ac + bd}$.

4. $\left.\begin{array}{l} x^2 - \dfrac{1}{x^2} = ab \\[2mm] x - \dfrac{1}{x} = \frac{1}{2}b \end{array}\right\}$ Eliminate x.

5. Draw the graphs of the three equations $x - 2y + 3 = 0$, $2x + y - 4 = 0$ and $3x - y - 1 = 0$ and verify that they meet in a point. What are the co-ordinates of the common point ?

6. If $x + y + z = 0$, show that $(x+y)(y+z)(z+x) = -xyz$.

Paper 4—Ex. 113

1. Find the square root of $(2a-1)(2a-3)(2a-5)(2a-7) + 16$

2. Simplify $\dfrac{3125 \times 2^{\frac{1}{4}} \times 10^{-\frac{1}{4}}}{15^{\frac{3}{4}} \times 6^{-\frac{3}{4}} \times 4^{\frac{3}{8}}}$

3. If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$, show that $\dfrac{x^3 + y^3 + z^3}{a^3 + b^3 + c^3} = \dfrac{xyz}{abc}$

4. $\left.\begin{array}{l} x = 2t + \dfrac{1}{2t} \\[2mm] y^2 = 4t^2 + \dfrac{1}{4t^2} \end{array}\right\}$ Eliminate t

5. Solve the equation $\dfrac{1}{x+1} + \dfrac{2}{2x+3} = \dfrac{6}{3x+5}$

6. For what value of x will the expression $\dfrac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}}$ be equal to $\frac{1}{2}$?

Paper 5—Ex. 114

1. Simplify $\dfrac{x-y-z}{(x-y)(x-z)}+\dfrac{y-z-x}{(y-z)(y-x)}+\dfrac{z-x-y}{(z-x)(z-y)}$,

2. If $x=\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ and $y=\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$, find the value of x^2+y^2.

3. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, show that each of these ratios is

    equal to $\sqrt[3]{\dfrac{pa^3+qc^3+re^3}{pb^3+qd^3+rf^3}}$.

4. Eliminate t from $x=a+bt+ct^2$, $y=b+ct$.

5. Draw the graph of the equation $3x-4y-24=0$. Read. off and calculate its intercep's on the axes of co-ordinates. Also read off and calculate the abscissa of the point on the graph which has ordinate equal to -12.

6. If $s=\frac{1}{2}(a+b+c)$, show that $(s-a)(s-b)+(s-b)(s-c) + (s-c)(s-a)=s^2-\frac{1}{2}(a^2+b^2+c^2)$.

Paper 6—Ex. 115

1. Find the square root of :—
$$\frac{1}{x^4}+\frac{6}{x^3}+\frac{7}{x^2}+4x^2+4x-\frac{10}{x}-11.$$

2. Show that $x^0=1$ and simplify :—
$$\left(\frac{x^l}{x^m}\right)^{l2+lm+m2}+\text{two similar terms.}$$

3. If $ab=cd=ef$, show that :—
$$a^3+c^3+e^3 : b^{-3}+d^{-3}+f^{-3}=abcdef.$$

4. Eliminate x, y, z from :—
$$\frac{x}{y-z}=a, \quad \frac{y}{z-x}=b, \quad \frac{z}{x-y}=c.$$

5. Solve the equation $\dfrac{x-4}{x-5}+\dfrac{x-8}{x-9}=\dfrac{x-5}{x-6}+\dfrac{x-7}{x-8}$ and test the correctness of your result.

6. If $2x^2+9x+6\equiv Ax(x+1)+Bx(x+2)+C(x+1)(x+2)$, find the values of A, B and C.

Paper 7—Ex. 116

1. Simplify :— $\left\{ \dfrac{x+y}{x-y} + \dfrac{x^3+y^3}{x^3-y^3} \right\} \div \left\{ \dfrac{x-y}{x+y} - \dfrac{x^3-y^3}{x^3+y^3} \right\}$

2. If $a = 5 + 2\sqrt{6}$, evaluate $\sqrt{a} + \dfrac{1}{\sqrt{a}}$

3. If a, b, c be in continued proportion, then
$$\dfrac{a^3 - b^2 + c^2}{a^2 - b^2 + c^2} = b.[4]$$

4. Eliminate x and y from the equations :
$x - y = a$, $x^2 - y^2 = b^2$, $x^3 - y^3 = c^3$.

5. Draw the graphs of the functions $\dfrac{x-3}{2}$ and $\dfrac{8-x}{2}$

and read off their common value from the graph.

6. If $x - y - z = 0$, show that $x^3 - y^3 - z^3 - 3xyz = 0$.

Paper 8—Ex. 117

1. Find the square root of $(2a^2 - ab - b^2)(2a^2 + 5ab + 2b^2)$
$\times (a^2 + ab - 2b^2)$.

2. If m and n are positive integres, prove that $(x^m)^n = x^{mn}$.

If $a = \sqrt[3]{2} + \sqrt[3]{2^{-1}}$, show that $2a^3 - 6a - 5 = 0$.

3. If a, b, c, d be in continued proportion, prove that :—

$$\dfrac{a}{d} = \dfrac{(a-b)^3}{(b-c)^3}.$$

[P. U. 1922]

4. If $x + \dfrac{1}{x} = a$ and $x^3 + \dfrac{1}{x^3} = b^3 - 3b$, show that

$(a - b)(a^2 + ab + b^2 - 3) = 0$.

5 Solve the equation $\dfrac{x-a}{x-b} = \left(\dfrac{2x-a}{2x-b} \right)^2$

6. If $2x^3 - 3x - 4 \equiv A(x-2)(x-8) + B(x-8)(x-4) + C(x-4)(x-2)$, evaluate A, B and C.

Paper 9—Ex. 118

1. Simplify :— $\dfrac{8a^3+6a^2-12a-9}{21a^4-20a^2-24}$.

2. Solve the equation $\dfrac{x-1}{1+\sqrt{x}}=5-\dfrac{1-\sqrt{x}}{2}$.

3. If $\dfrac{a}{y+z-x}=\dfrac{b}{z+x-y}=\dfrac{c}{x+y-z}$. show that
$a(y-z)+b(z-x)+c(x-y)=0$.

4. $\left.\begin{array}{l}x(1-t^2)=b(1+t^2)\\ y(1-t^2)=2at\end{array}\right\}$ Eliminate t.

5. Draw the graph of the equation $2x-3y+18=0$. Where does the straight line $y+4=0$ cut it ?

6. If $a+b+c=0$, show that $a^4+b^4+c^4=2(ab+bc+ca)^2$.

Paper 10—Ex. 119

1. For what value of a will $9a^4-12a^3+22a^2-13a+12$ be a perfect square ?

2. If m and n are positive integers prove that $x^m \div x^n = x^{m-n}$.

Divide $8a^{-1}-b^{-3}-c^{-6}-6a^{-\frac{1}{3}}b^{-1}c^{-2}$ by $2a^{-\frac{1}{3}}-b^{-1}-c^{-2}$.

3. If x, y, z, be in continued proportion, show that $xy+yz$ is a mean proportional between x^2+y^2 and y^2+z^2.

4. Eliminate t from the equations :—
$$\left.\begin{array}{l}\dfrac{3t}{4}+\dfrac{2}{3t}=2a+3b\\[2mm]\dfrac{3t}{4}-\dfrac{2}{3t}=2a-3b\end{array}\right\}$$

5. Solve the equation $(x-2)^3+(x+3)^3+(x-4)^3=3(x-2)\times(x^2-x-12)$.

6. If $\dfrac{17-x}{x^2+x-6}\equiv\dfrac{A}{x-2}+\dfrac{B}{x+3}$, evaluate A and B.

1. (a) Solve :—
$$\frac{2(x-1)}{x-3} - \frac{3}{x+1} = 2.$$

(b) Solve :—
$$\left.\begin{aligned} \frac{2}{x} + \frac{3}{y} &= 2 \\ \frac{1}{x} - \frac{1}{2y} &= \tfrac{1}{3} \end{aligned}\right\}$$

Or,

(b) Thirty-six men and boys are employed to do a piece of work, and are paid Rs. 16, 4as. in all. If men are paid 8as. each and boys 6as. each, find the number of men and the number of boys.

2. (a) Draw the graphs of :—
$$x-y-7=0.$$
$$x-2y-11=0.$$

Read their point of intersection.

(b) Divide :—
$$27x^3 - 8y^3 - z^3 - 18xyz \text{ by } 3x-2y-z$$

in any manner you like.

3. (a) Extract the square root of :—
$$\left(x^2 + \frac{1}{x^2}\right)^2 - 4\left(x - \frac{1}{x}\right)^2 - 4.$$

(b) Find the H. C. F. of :—
$$x^4 - 5x^3 + 8x^2 - 5x + 1 \text{ and } 2x^4 - 7x^3 + 9x^2 - 5x + 1.$$

4. (a) Factorise :—
$$x^6 - 64.$$

(b) Factorise :—
$$x^8 + x^4 + 1.$$

Or,

Factorise : $\qquad 4b^2c^2-(b^2+c^2-a^2)^2$

and show that the expression

$$=16s(s-a)(s-b)(s-c),$$

where $\qquad 2s=a+b+c.$

5. (a) If n is a positive integer and m any number prove that :—

$$(a^m)^n=a^{mn}.$$

Evaluate :—

$$2^{3^2} \div (2^3)^2.$$

(b) Simplify :—

$$\frac{a(b+c)}{(a-b)(a-c)}+\frac{b(c+a)}{(b-c)(b-a)}+\frac{c(a+b)}{(c-a)(c-b)}.$$

6. (a) If $ax=by=cz$, show that :—

$$\frac{yz}{x^2}+\frac{zx}{y^2}+\frac{xy}{z^2}=\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}$$

(b) $\left.\begin{array}{l}\dfrac{x}{a}=\dfrac{1+t^2}{1-t^2}\\[2mm]\dfrac{y}{b}=\dfrac{2t}{1-t^2}\end{array}\right\}$ Eliminate t.

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1. (a) Solve :—

$$\frac{5x+8}{x-1}-\frac{x+9}{x+8}=4.$$

(b) Solve :—

$$\left.\begin{array}{l}\dfrac{8}{x}+5y=2\\[2mm]\dfrac{1}{x}-2y=8\end{array}\right\}$$

Or,

(c) A strip of steel is $18\frac{1}{4}$ inches long. How many pieces each $\frac{3}{4}$ inch long may be cut from it, if $\frac{1}{32}$ inch is wasted in each cutting.

2 (a) Draw graphs of —

$$x + y = 2$$
$$x + 10 = 3y$$

Read their points of intersection

(b) Evaluate $x^3 + x^{-3}$
    when $x = 3 - 2\sqrt{2}$

3. (a) Factorise $x^6 - 729$

(b) Find the square root of —

$$(x + 3y)(x^3 - x^2y - 8xy^2 + 12y^3).$$

4. Find the H. C. F and L. C M. of :—

$$4x^3 + 8x^2 - 3x - 9 \text{ and } 12x^3 + 28x^2 + 13x - 3$$

5. (a) Remove the bracket in $(a^3)^5$, giving proof of the process done.

Evaluate $2^{\frac{2}{3}} \div 2^{\frac{3}{2}}$

(b) Express $2(x^2 + y^2 + z^2 - yz - zx - xy)$ as the sum of three squares

Hence show that if the expression vanishes,

$$x = y = z.$$

6. (a) If $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d}$,

show that

$$\frac{b+a}{c+a} = \frac{c+d}{b+d}$$

(b) Eliminate t from :— $x = \dfrac{1 + t^2}{2at}$

$$y = \frac{1 - t^2}{2bt}$$

1942

1. (a) Solve :—

$$\frac{1}{x-1} + \frac{2}{x-2} = \frac{8}{x-3}$$

(b) Solve graphically the following equations :—

$$x + 2y = 5$$
$$2x - y = 5$$

Read the value of y from both the graphs when $x=7$.

2   A number consists of two digits whose sum is five. When the digits are reversed the number becomes greater by nine.   Find the number

Or,

The current of a stream runs at the rate of 5 miles an hour. A motor boat goes 10 miles upstream and back again to its starting point in 50 minutes. Find the speed of the motor boat in still water.

3.   (a) Find the square root of :—

$$x^4 + x^3 + \frac{83}{4}x^2 + 4x + 16$$

(b) Resolve into factors :—

(i) $2x^2 + 11x + 14$

(ii) $1 - 2b - 4b^2 + 8b^3$

4.   Simplify :—

$$\frac{1}{x+a} + \frac{a}{x^3-a^3} + \frac{x}{x^2+a^2}$$

5.   Eliminate t from :—

$$at^2 + bt + 1 = 0$$
$$ct^2 + dt + 1 = 0$$

6.   If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$ show that

$$\frac{x^3}{a^3} - \frac{y^3}{b^3} + \frac{z^3}{c^3} = \frac{xyz}{abc}.$$

7.   Evaluate $x^2 + \dfrac{1}{x^2}$

when $x = 6 - \sqrt{35}$

1943

1.   (a) Resolve into factors $x^5 - 5x^3 + 4x$

(b) Find the value of k when $x - 3$ is a factor of
$3x^2 + kx + 6$

(c) Evaluate $\dfrac{8^{2-n} \times 9^{2n-2}}{8^{3n}}$

2.   (a) Solve : $\dfrac{9x+7}{2}-\left(x-\dfrac{x-2}{7}\right)=36$

(b) Solve graphically the following equations :—
$$y=3x+4$$
$$y=x+8.$$

and read the values of x where the two graphs meet the line $y=4$.

3.   (a) There are two examination rooms A and B. If 10 candidates are sent from A to B, the number in each is the same, while if 20 are sent from B to A, the number in A is double the number in B. Find the number of candidates in each room.

Or

(b) When one is added to each of two numbers, their ratio becomes $1:2$ and when five is subtracted from each of them, their ratio becomes $5:11$. Find the numbers.

4.   If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$,

prove that each fraction is equal to $\sqrt{\dfrac{x^2+2y^2+3z^2}{a^2+2b^2+3c^2}}$

5.   Simplify : $\dfrac{2(2-x)}{x^2-1}-\dfrac{3}{x^2+x}-\dfrac{1}{x^2-x}$.

6.   Find the H. C. F. of :—
$x^3-x^2+4x-30$ and $2x^3-6x^2-3x+9$

7.   Find the value of $a^3+\dfrac{1}{a^3}$

when $a=2+\sqrt{3}$

1944

(a) Resolve into factors $6x^3-x^2y-35xy^2$.

Or

$1+2ab-(a^2+b^2)$

(b) Reduce to its simplest form $\left(\dfrac{a^2}{a}+\dfrac{a^2}{a}\right)+\left(\dfrac{a}{a}-\dfrac{a}{a}\right)$

(c) Find the value of :—

$$\dfrac{(27)^{\frac{2n}{3}}\times(8)^{-\frac{n}{6}}}{(18)^{-\frac{n}{2}}}$$

2. (a) Solve : $\dfrac{2x-9}{2} - \dfrac{7x-6}{6} = \dfrac{4x-3}{3}$.

(b) Solve graphically the following equations $y=2x-2$, $3y=6+2x$. Read the co-ordinates of the points where the lines intersect the axis of y.

3. In an examination paper one boy A got 4 marks less than $\frac{2}{3}$ of full marks and another boy B got 2 marks more than $\frac{1}{3}$ of full marks. The marks obtained by A were three times as many as obtained by B. What marks did each get ?

Or

Six years ago a man was three times as old as his son and in six years' time he will be twice as old as his son. Find their present ages.

Note. Do any three of the following :—

4. If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c} = \dfrac{u}{d} = k$, find in terms of k, the value of

$$\sqrt{\dfrac{x^2+2y^2-6z^2+5u^2}{a^2+2b^2-6c^2+3d^2}}$$

5. Simplify : $\dfrac{2x+5y}{x+y} - \dfrac{3x-2y}{x-y} + \dfrac{x^2+y^2}{x^2-y^2}$.

6. (a) Eliminate p from :—
$$x=ap^3 \text{ and } y=bp$$

(b) If $x=q+\dfrac{1}{q}$ and $y=q-\dfrac{1}{q}$,

show that $x^4+y^4-2x^2y^2=16$.

7. If $y=\dfrac{3x+2}{4x-5}$ find x in terms of y

[*Hint* :—Cross multiply and solve for x]

(b) If $x=\dfrac{p}{p+q}$ and $y=\dfrac{q}{p-q}$,

show that $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{p^2+q^2}{pq}$.

8. (a) Find the square root of :

$$x^2(x-6)+17x^2-8(8x-2).$$

(b) Find the value of $\left(\dfrac{x^2-y^2}{2}\right)^{\frac{1}{2}}$

when $x=116$ and $y=100$.

1945

1. Resolve into elementary factors : -

(i) x^6-y^6

(ii) $a^3-17a+26$.

2. Obtain an expression which will divide both $4x^3+7x^2-3x-15$ and $4x^2+3x-10$ without remainder.

Or,

Find the H. C. F. of :—

$$x^7+y^7 \text{ and } x^5+y^5.$$

3. Solve —

$$\frac{1}{2}\left[x-\frac{1}{3}\left\{x-\frac{1}{4}\left(x-\frac{x-\frac{1}{6}x}{5}\right)\right\}\right]=53.$$

Or,

$$\left.\begin{array}{l}\dfrac{x}{2}+\dfrac{y}{3}+\dfrac{z}{6}=12\\[2mm]\dfrac{y}{2}+\dfrac{z}{3}-\dfrac{x}{6}=8\\[2mm]\dfrac{z}{3}+\dfrac{x}{2}=10\end{array}\right\}$$

4. Solve the following equations graphically and verify your result :—

$$\left.\begin{array}{l}x=5-y\\[2mm]y=8-\dfrac{x}{3}\end{array}\right\}$$

5. Simplify :—

$$\dfrac{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}$$

6. Find the square root of :—

$$a^2 + \frac{1}{a^2} + 12\left(a + \frac{1}{a}\right) + 38.$$

7. (a) Find the value of $m^3 - \frac{1}{m^3}$ when

$$m - \frac{1}{m} = x.$$

(b) If $a + b + c = 0$, show that

$$a^2 - bc = b^2 - ca = c^2 - ab.$$

8. (a) Eliminate m from :—

$$m + \frac{1}{m} = .$$

$$m^3 + \frac{1}{m^3} = d.$$

(b) Find the continued product of :—

$\sqrt{x} + \sqrt{y}$, $\sqrt[3]{x} + \sqrt[3]{y}$, and $\sqrt[4]{x} - \sqrt[4]{y}$.

9. If a, b, c, and d be in continued proportion, show that

$$\frac{a}{d} = \frac{a^3 + b^3 + c^3}{b^3 + c^3 + d^3}.$$

10. Divide 81 into three parts such that half of the first part, one-third of the second part and one-fourth of the third part shall all be equal.

Or,

A certain number of two digits is equal to four times the sum of its digits. If 18 be added to the number, the digits are reversed. Find the number.

1946

1. (a) Solve for x :—

$$6 - \frac{x-1}{2} - \frac{x-2}{3} = \frac{8-x}{4}$$

(b) Solve for x and y :—

$$\frac{2}{x} + 8y = 15, \quad -\frac{5}{x} - 6y = 3$$

2.   (a) Find the square root of—
     $(2x-5)(8x^3-28x^2+22x-5)$.

(b) Simplify :—
$$\frac{x^4-8x}{2x^2+5x-3} \times \frac{2x-1}{x^2+2x+4} \div \frac{x^2-2x}{x+3}.$$

3.   (a) Find the L. C. M. of :—
     $x^3-2x^2-13x-10$ and $x^3-x^2-10x-8$.

(b) If $16^{x+1}=\dfrac{64}{4^x}$, find the value of x

4.   (a) If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$, prove that

$$\frac{x^2+y^2+z^2}{a^2+b^2+c^2}=\frac{(lx+my+nz)^2}{(la+mb+nc)^2}$$

(b) $\left. \begin{array}{l} x-\dfrac{1}{x}=a \\[2mm] x^3-\dfrac{1}{x^3}=b \end{array} \right\}$ Eliminate x.

5.   (a) If $a=\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ and $b=\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$
     find the value of a^2+b^2.

(b) Solve the following equations graphically and also find the intercept of the graph of the first equation on the y-axis.
$4x+8y=12,\ 2x+y=2$.

6.   (a) The age of a man is three times the sum of the ages of his two children, and five years hence his age will be double the sum of their ages. Find his present age.

(b) Prove that
$$\frac{1}{x+1}+\frac{2}{x^2+1}+\frac{4}{x^4-1}-\frac{1}{x-1}=0.$$

1947

1.   (a) Solve the equation :—
$$x-\frac{2x+8}{8}-\frac{1}{4}\left(x-\frac{2-x}{6}\right)+8=0.$$

(b) If $a+b+c=11$ and $ab+bc+ca=20$, find the value of $a^3+b^3+c^3=8abc$.

2. (a) Find the H. C. F. of :

$$8x^4 + 3x + 10 \text{ and } 10x^4 + 3x^3 + 8.$$

(b) Factorize :—

(i) $2x - 32x^5$

(ii) $15x(x^2 - 1) - 72x^2.$

3. (a) Find the square root of :

$$25x^2 + \frac{1}{x^2} - 20x + \frac{4}{x} - 6.$$

(b) If a, b, c, and d be in continued proportion, show that

$$a : d = a^3 + b^3 + c^3 : b^3 + c^3 + d^3.$$

4. (a) Simplify :—

$$\left(\frac{a^x}{a^y}\right)^{x+y} \times \left(\frac{a^y}{a^z}\right)^{y+z} \div 3(a^x.a^z)^{x-z}$$

(b) If $x = 3 + 2\sqrt{2}$, find the value of

$$x^{\frac{1}{2}} - x^{-\frac{1}{2}}$$

5. (a) Eliminate m from :—

$$m^2 + 1 = 2am$$
$$m^2 - 1 = 2bm.$$

(b) Solve graphically the equations :—

$$3x + 2y + 4 = 3$$
$$2x - 9 = y.$$

6. (a) If a room were 2 ft. longer and 3 ft. broader its area would be increased by 75 sq. ft. If it were 1 ft. shorter and 2 ft. broader the area would be increased by 16 sq. ft. Find its length and breadth.

(b) Simplify :—

$$\frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-c)(b-a)} + \frac{1}{c(c-a)(c-b)}$$

1948 (August)

Note :— Attempt Question VII and any five parts from the rest of the questions.

7.   (a) Find the factors of—
      (i) $a^4 + 4b^4$.   (ii) $6x^2 - 13x + 5$.   (iii) $x(x+2)(x+4)(x+6)+7$
     (b) Show that if $a+b+c=0$, then $a^3+b^3+c^3=3abc$.

8.   (a) Solve for $x, y, z,$—
$$\frac{a}{x} + \frac{b}{y} = 2c, \quad \frac{b}{y} + \frac{c}{z} = 2a, \quad \frac{c}{z} + \frac{a}{x} = 2b.$$
     (b) Solve. $\sqrt{x+3} - \sqrt{x} = \sqrt{4x+1}$.

9.   (a) Extract the square root of—
$$x^4 + \frac{1}{a^4} - \frac{2x}{3a}\left[x^2 + \frac{1}{a^2}\right] + \frac{19x^2}{9a^2}.$$
     (b) Find the H.C.F. and L.C.M. of—
$$2x^3 - 5x^2 + x + 2 \text{ and } 4x^3 + x^2 - 4x - 1.$$

10.   (a) If $\dfrac{x}{y+z} = a$, $\dfrac{y}{z+x} = b$, $\dfrac{z}{x+y} = c$.

     Show that $\dfrac{1}{1+a} + \dfrac{1}{1+b} + \dfrac{1}{1+c} = 2$.

     (b) Eliminate x from $x - \dfrac{1}{x} = a$, $x^4 + \dfrac{1}{x^4} = b^4$,

11.   (a) If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$,

     show that each of these ratio is equal to $\sqrt{\dfrac{a^3+c^3+e^3}{b^3+d^3+f^3}}$,

     as well as equal to $\dfrac{la+mc+ne}{lb+md+nf}$,

     (b) Show that $\left[\dfrac{x^a}{x^b}\right]^{a+b} \times \left[\dfrac{x^b}{x^c}\right]^{b+c} \times \left[\dfrac{x^c}{x^a}\right]^{c-a} \div \dfrac{1}{x^{2ac}}$.

     is equal to x^{2a^2} Find the result when $a=0$.

12.   (a) Three times the present age of father is equal to eight times the present age of his son. Eight years hence father will be twice as old as his son then. What are their ages at present ?

     (b) Solve graphically and verify your result —
$$x = \frac{y-1}{3}, \quad \frac{x+5}{2} + y = 0.$$

ANSWERS

Exercise 1 (Page 3)

1. 24.	2. 36.	3. 12.	4. 50.	5. 48.
6. 6.	7. 128.	8. 125.	9. 96.	10. 108.
11. 5.	12. 8.	13. 9.	14. 625.	15. 1.
16. 6.	17. 2.	18. 36.	19. 12.	20. $9\frac{1}{3}$.
21. 72.	22. 0.	23. 0.	24. 0.	25. 81.
26. $\frac{2}{3}$.	27. 0.	28. 3.	29. 0.	30. $1\frac{3}{6}$.	31. 8.
32. 6.	33. 0.	34. $\frac{2}{3}$.	35. $\frac{3}{4}$.	36. $\frac{3}{2}$.	37. $\frac{2}{3}$.
38. 0.	39. 12.	40. $2\frac{2}{3}$.	41. 0.	42. 32.
43. $\frac{1}{2}$	44. 0.	45. 24.

Exercise 2 (Page 5)

1. 6.	2. 6.	3. 15.	4. 3.	5. 29.	6. 8.
7. 107.	8. 11.	9. 38.	10. 46.	11. ·14.
12. 5.	13. 27.	14. 6.	15. 2.	16. $\frac{1}{2}$.	17. 56.
18. 6.	19. 44.	20. 22.	21. 0.	22. $\frac{5}{6}$.	23. 1.
24. 1.	35. 2.	36. 6.	37. 1.	38. 7.	39. 4.
40. 4.	41. $\frac{1}{3}$.	42. 2.

Exercise 3 (Page 9)

1. (a) Yes.   (b) No; 3.5 means 3×5 which is equal to 15.
   (c) $3.5 = 15$ and $3\cdot5 = 3\frac{1}{2}$.
2. No; 30.   3. $x \times y$, xy, $x. y$, $y \times x$, yx, $y. x$.
4. 3.   5.   1.
6. Four times $x = 4x$ or $4 \times x$; x to the fourth $= x^4$
   $= x \times x \times x \times x$, values 8 and 16.
7. (a) x^2. (b) x^n.   8.   (a) 5.   (b) $5a$.   (c) 1.
9. (a) 5, a and b.   (b) 5, a, b, $5a$, $5b$, ab, $5ab$, 1.
10. (a) x^4; 4.   (b) 1.   11. $\sqrt[5]{2a}$.
12. (a) No.   (b) No.   (c) Yes; $c^2d = c \times c \times d$ and cd^2
    $= c \times d \times d$.
13. One; Simple Expression or Monomial.
14. Three; Trinomial.

15   Two.   One only.   [Note that 'terms' are separated by the signs + and − and not by the sign × or ÷] The first exp. is a Binomial and the second a Monomial.

16.   The first three signs of equality are wrong.

Exercise 4 (Page 16)

1.   −2.   2.   −7.   3.   −9.   4.   −6.   5.   2.

6.   0.   7.   0.   8.   −8.   9.   0.   10.   −1.   11.   5.

12.   −8.   13.   −Rs 25 : loss of Rs. 25.

14.   +Rs. 50 ; gain of Rs. 50.

15.   −13 miles ; 13 miles west.   16.   −4 ft.   17. $3a + 5b$.

18. $2a + 5ab$.   19. $2a$.   20. $2a^2$.   21. $3a^3$.

22. $-3ab$ or $-3ba$.   23. $3a-2$.   24 $x^3y + y^2x$

25. $-2x^4$.   26. $x^3 + x^4$.   27.   0.   28. $9k$.

29. $-l - l^2 - l^3$.   30 $2lm + 3mn$.   31. $a + ab + abc$.

32. $3xy - 5x$ 33. $xyz - x - y - z$.   34. $b^2 - 2b$

35. $a^3 - 3a$.   36. $3abc$.   37. $x^2y^2 - x^2 - y^2$

38. $a^2b - ab^2 - ab - a - b$.   39. $1 + 2a + 3b + 4c + 5d$.

40. $-a + 2a^2 + 3a^3 + 4a^4$.   41. $3x$.   42. $3a^2$.   43 $3b$.

44. $10\,l$.   45. $-6p$.   46. $-7q$.   47.   0.   48. $-2a^2$.

49.   • $-10l^3$.   50. $-10a^2b$.   51. $-x^2$.   52 $\frac{3}{5}kcb^2$.

53. $\frac{7}{24}xy$.   54. $-3a^3$.   55. $-k^3$.   56. $\frac{3}{8}l^4$.

57. $-3a + 4ab$.   58 $-2a^2 + 2a$.   59. $-2c - 7xy$.

60. $-2ab$ or $-2ba$.   61. $13l + 3$.   62. $-5l - 4l^2 + 3l^3$.

63. $-1 - 3a - 3ab - b$.   64. $-4 - 4x - 5x^2 - 7x^3$.

Exercise 5 (Page 19)

1. $a + b$.   2. $-3a - 3b$.   3. $2a - 3b$.   4. $x - z$.

5. $-c + z + x$.   6. $a - 4b + c$.   7 $-r$.   8. $x - 4y$

9. $2p - q + 3r - s$.   10. $5a - b + 3ab - bc$.

11. $l - m - 4n - p$.   12.   0.   13. $-xy - yz - zx$.

14. $2l + 3k + 4l - 9$.   15. $a + b + ab + bc + a^2 + b^2$

16. $-2a + c$.   17. $-3a - 3b$.   18. $-4a^2 - 7a + 1$

19. $-4x^2 - 2xy - 4y^2$.   20 $-2ab + bc + 5ca$.

21. $a^3 - 3ab^2 - 2b^3$.   22. $2z$.   23   0.   24. $8x + 5y - 2z$

25. $39x - 5y + 4z$ 26. $3ab - 3bc + 3ca$.   27. xy

28. $5a^2 + b^2$ 29. $6a^3$.   30. $2ab + 2ca$.

31. $ab + bc + ca$.   32. $-2a^3 + a^2 + 4a + 2$.

33. $9a^3 + 6a^2 - 32a - 18$.   34 $15a^3 - 4a^2 + 8a - 1$

35. $.2x^3+2y^3+2z^3.$

36. $2x^3+2y^3+2z^3+2t^3.$

37. $2a^3+2a^2+2z+6.$

38. $6a^3-4b^2-4ab-8bc-6cz.$

39. $-3a^3+8a^2+3b+6b^2.$

40. $3a^2b+ab^2.$

41. $2x^2y.$

42. $2a^3+2a^2h+14ab^2+6b^3.$

43. $\frac{5}{6}x+\frac{1}{6}y.$

44. $a+\frac{2}{3}b+\frac{3}{4}c.$

45. $\frac{4}{3}a^2-\frac{1}{2}ab+\frac{1}{3}b^2-\frac{1}{2}bc.$

46. $-\frac{1}{3}y^2.$

47. $-\frac{1}{2}a^3-\frac{2}{3}a^2b+ab^2-\frac{1}{2}b^3.$

48. $-\frac{1}{2}x^3-\frac{1}{4}x^2y+\frac{1}{3}xy^2+\frac{1}{2}y^3.$

Exercise 6 (Page 23)

1. $8.$ 2. $-7.$ 3 $7.$ 4. $-3.$ 5. $4a.$

6. $-12x.$ 7. $7k.$ 8. $6t.$ 9. $-5b.$ 10. $6c.$

11. $-8xy,$ 12. $5dc-3d.$ 13. $6fg+3f.$

14. $-abc+ab.$ 15. $-5a^2-2a.$ 16. $9x^3+6x^2.$

17. $-8xy.$ 18. $8-8a.$ 19. $3a-a^2.$

20. $-5ab^2+4a^2b.$ 21. $5x+3y.$ 22. $-3a+9b.$

23. $-4c+d.$ 24. $5xy-2x-2y.$ 25. $4x-4y.$

26. $4a^2+2a.$ 27. $-2a+3b.$ 28. $a^3+a^2-a-1.$

29. $2ab+2xy-x-y.$ 30. $a^5-a^3-a^2.$ 31. $-2a+2b.$

32. $-x-2z.$ 33. $-3p+4q+2r.$ 34. $-2x^2-x+1.$

35. $5x^3+x^2-x-7.$ 36. $-2b^3-2c^3+3abc.$

37. $6ab^2-4a^2b-2a^2b^2.$ •38. $-2a^4-2a^3-4a^2-3a.$

39. $3ab+4b-2a-3.$ 40. $8x^4-4y^4-5x^2+xy+y^2.$

41. $\frac{3}{6}x-\frac{5}{6}y.$ 42. $-\frac{2}{3}a+2ab-\frac{1}{2}a.$

43. $-\frac{1}{4}a^2-\frac{1}{4}a.$ 44. $a^4-a^2b^2-\frac{3}{4}a^2-1.$

45. $x^2+\frac{3}{2}y^2-\frac{1}{2}x-\frac{1}{2}y.$ 46. $4b-8a.$ 47. $2xy-3x+1.$

48. $x^2-3x-1.$ 49. $2abc-ab+bc-b+c.$ 50. $2.$

51. $-x^2-x-3.$ 52. $5ab-a+b-2.$ 53. $-3a^2+a+2.$

54. $-2a^2-2a.$ 55. $abcd-a-b-c-d.$

56 $-ab-2b-2a.$ 57. $-2a-8b+4c.$ 58. $-a-b.$

59. $2a-ab-3.$ 60. $3x+y+z.$ 61. $-2x^2-2x-6.$

62. $b^2-ab+4b.$ 63. $-a^2-2ab+b^2.$ 64. $y^3-4y^2+6.$

Exercise 7 (Page 31)

1. $-40.$ 2 $-42.$ 3. $45.$ 4. $-6ab.$ 5. $85xy.$

6. $-8xyz.$ 7. $a^6.$ 8. $x^{10}.$ 9. $x^3y^3.$ 10. $a^4b^4c^4.$

11. $-12a^4b^3.$ 12. $-80a^3b^5c.$ 13. $42a^2b^2c^2.$

14. $-10a^3b^3c^3.$ 15. $0.$ 16. $64x^9y^{12}z^9.$ 17. $24a^2b^3c^2.$

18. $-15a^3b^3c^3.$ 19. $-12a^2b^3c^3.$ 20. $6x^6y^6z.$

21. $60x^5y^2.$ 22. $-24l^9m^9n^9.$ 23. $8ab^2x^2y.$

34. $-4a^3b^3c^2$.   **25.** $-\frac{1}{6}x^2y^3z$.   **26.** $-\frac{1}{2}x^2y^5z^3$.

27. $-a^2b^2c^2$.   **28.** $24x^2y^4z^3$.   **29.** $60x^3y^3z^3$.

30. $x^3y^3z^3t^3$.   **31** $8a^3b^3$.   **32.** $-27x^6y^3$.   **33.** $256x^4y^4z^9$.

34. $-a^{10}b^{15}$.   **35.** $-4a^3$.   **36.** $27x^6$.   **37.** $432x^2y^3$

38. $486a^6$.   **39** $5a^4b^4c^4$.   **40.** $-8x^2y^6z^{12}$.   **41.** -40.

42. 96.   **43.** -72.   **44.** -192.   **45.** -288.   **46.** -20.

47 -105.   **48.** 30.   **49.** -1.   **50.** -37.   **51** 19.

52. 38.

Exercise 8 (Page 33)

1. $5a-10b$.   **2.** $-8x+20y$.   **3.** $-6x^2+8x^2y$.

4. $-9x^3+12x^2$.   **5.** $a^2bc-ab^2c-abc^2$.

6. $-15a^4+20a^3+25a^2$.   **7.** $6x^4y-9x^3y^2+3x^2y^3$.

8. $16x^5y^3-4x^4y^4+4x^2y^6$.   **9.** $3a^3b^2x^3z^2+3a^2b^3y^3z^2-3a^2b^2cz^5$.

10. $-a^4b^3x^7y^5+3a^2b^6x^4y^7+4a^2b^3x^5y^6$.

11. $2abcdx^5-6b^2cdx^4+6bc^2dx^3-2bcd^2x^2$.

12. $-3ax^4y^3+6hx^3y^4-3bx^2y^5+6gx^3y^3+6fx^2y^4-3cx^2y^3$.

13. $-6x^2y+1\frac{5}{2}xy^2-9xyz$.   **14.** $\frac{9}{2}x^5y^2-3x^4y^3-\frac{3}{4}x^3y^4$.

15. $-\frac{2}{3}x^3y^6-x^6y^8+12x^4y^{10}$.

16. $6a^7b^3-4a^6b^3+\frac{7}{2}a^4b^3+\frac{4}{5}a^2b^3$.   **17.** $-6a+2b-1$.

18. $-xy+14x$.   **19** $2ab^3$.   **20.** $5a^3-11a^2b+6b^2$.

21. $-4a^4-4a^3+a+20$.   **22.** 0.   **23.** $-3x^2y^2-12y$.

24. 0.

Exercise 9 (Page 35)

1. $-2x^3-5x^2+6x+7$.   **2.** $-x^4+x^3+10x^2-7x+20$.

3. $x^3-9x^2+100x+50$.   **4.** $x^4-5x^3-x^2-7$.

5. $4x^4+5x^3+6x^2-10x-4$.

6. (i) $-a^4+a^3x+5a^2x^2+ax^3-6x^4$.

    (ii) $-6x^4+ax^3+5a^2x^2+a^3x-a^4$

7. (i) $x^4-a^3x+20a^2x^2-ax^3+x^4$.

    (ii) $x^4-ax^3+20a^2x^2-a^3x+a^4$.

8. (i) $-6a^5-xa^4+100a^2x^3+6ax^4-x^5$.

    (ii) $-x^5+6ax^4+100a^2x^3-xa^4-6a^5$.

9. (i) $-a^6+xa^5-x^3a^3+5a^2x^4+30x^6$.

    (ii) $30x^6+5a^2x^4-x^3a^3+xa^5-a^6$.

10 (i) $a^4-6a^3x-4a^2x^2-5ax^3+x^4$.

    (ii) $x^4-5ax^3-4a^2x^2-6a^3x+a^4$.

11. $a^2-(b+c)a+(b^2+c^2-bc)$
12. $a^3-3abc+(b^3+c^3)$ 13 $a^3+3a^2b-3ab^2-(b^3+c^3)$
14. $(x^2-4y^2-4xy)+(6y-8x)z-3z^2$
15. $(8x^3+y^3)\times 18xyz-27z^3$.
16   $(x^2y^2-y^3)+4xyz-(x^2+y^2)z^2-z^3$

Exercise 10 (Page 37)

1. x^2-x-6 2 x^2-x-20.   3 $x^2-13x+42$
4. $a^2+17a+70$.   5 a^2-64.   6. $k^2-12k+27$
7. $-x^2+11x-30$.   8. $-a^2+6a+40$.
9. $-k^2+24k-144$.   10. $-t^2-10t+11$
11. $-6x^2+23x-20$.   12. $20t^2-9t-20$.
13. $-24m^2+62m-35$.   14. $30-89a+63a^2$.
15   $1-64w^2$.   16. $-18+33v+121v^2$.
17. $-56x^2+37xy-6y^2$.   18 $-27x^2+90xy-27y^2$.
19. $15a^2-38ab+7b^2$.   20. $-56u^2+19uv+15v^2$.
21. $-10x^4+37x^2-30$.   22. $18x^4-27x^2-35$.
23. $-18x^4+27x^2a-10a^2$ 24. z^2-49y^4.
25. $-30x^2y^2+38axy-12a^2$.   26. $-2x^2y^2+10xyab-12a^2b^3$
27. $-10p^2q^2+17pqrs-3r^2s^2$.   28. $-42x^4+20x^3-2x^2$.
29. $8x^3-27$ 30. $3x^3-10x^2-7x+20$.
31. $18x^3-15x^2+32x-5$.   32. $8a^3+125$.
33. $12x^3-17x^2y+3xy^2+2y^3$.   4. x^3-9a^2x.
35. $-10b^3-ab^2+26a^2b-7a^3$ 36. $8x^3+27y^3$.
37. $27b^3-64a^3$.   38. x^6+y^6.
39. $x^4-4x^2+8x+16$.   40. x^4-2x+1
41. $6x^4-96$.   42. x^4-a^4.
43. $x^6-x^4y^2+x^2y^4-y^6$ 44. $a^6+a^3b^3$
45. $81x^4-256a^4$ 46 $1-2a-31a^2+72a^3-30a^4$.
47. $a^4+2a^3-7a^2-8a+12$.   48 $-x^4+4x^3y-x^2y^2-4xy^3-y^4$.
49. $a^4-2a^2b^2+b^4$ 50 $x^4-25x^2y^2-10xy^3-y^4$
51   $a^4+4a^2x^2+16x^4$.
52. $x^7+5x^6+4x^4+7x^3+10x^2+6x+2$; 10548062
53   $x^5-41x-120$.   54 $a^5+151a-264$
55   $2a^5-18a^4+39a^3-25a^2+a+1$
56   $4a^5+12a^4+17a^3+12a^2-4a$. 538240
57   $a^5-2a^4-4a^3+19a^2-31a+15$ 58 x^5+4xy^4
59   $x^6+1008xy^5+720y^6$.
60   $4x^6-5x^5y+8x^4y^2-10x^3y^3-8x^2y^4-5xy^5-4y^6$
61   $x^3+8y^3-27z^3+18xyz$.   62 $x^3+y^3+z^3-3xyz$

63. $a^3 + a^2b + ab^2 + b^3 + 2b^2x - ax^2 + bx^2.$

64. $x^3 + y^3 + 3xy - 2x - 2y + 1.$ 65. $a^3 - ab^2 + 2ab + b - 1.$

66. $x^3 + 27y^3 - 1 + 9xy.$ 67. $a^4 - 2a^2b^2 + b^4 + 4abc^2 - c^4.$

68. $a^3(b-c) - a(b^3 - c^3) + (b^3c - bc^3).$ 69. $x^4 - a^4.$

70. $16x^4 - 81a^4.$ 71. $x^8 + a^4x^4 + a^8.$

72. $x^4 - 5a^2x^2 + 4a^4.$ 73. $a^6 - 14a^4 + 49a^2 - 36.$

74. $a^5 - 3a^4b + a^3b^2 - a^2b^3 - 18b^5.$ 75. $\frac{1}{4}x^3 + \frac{1}{7}\frac{1}{2}x - \frac{1}{12}.$

76. $\frac{1}{4}a^3 - \frac{5}{6}a^2 + \frac{1}{12}a + \frac{1}{2}.$ 77. $\frac{1}{3}x^3 - \frac{4}{72}x^2y + \frac{1}{2}xy^2 - \frac{1}{6}y^3.$

78. $\frac{1}{4}x^4 - \frac{43}{36}x^2y^2 + \frac{9}{16}y^4.$ 79. $11.$ 80. $-6.$

81. $-12.$ 82. $13.$

Exercise 11 (Page 44)

[See Answers to Ex. 10]

Exercise 12 (Page 46)

1. -8 2. $7.$ 3. $-9.$ 4. $2b.$ 5 $-7y.$

6. $-8z.$ 7. a^4 8. $-x^7.$ 9. $-xy^2.$ 10. $-a^3b^2c.$

11. $4ab^2.$ 12. $-6ab^2c.$ 13. $-7bc.$ 14. $10a^3b^3c^3.$

15. $-4xy^2z^3.$ 16. $-8xy^2z^9.$ 17. $-\frac{2}{3}bxy.$ 18. $-6a^2bc.$

19. $1\frac{4}{5}xz.$ 20. $-\frac{2}{3}y^3x.$

Exercise 13 (Page 47)

1. $a - 2b.$ 2. $2x - 5y.$ 3. $-3x + 4xy.$ 4. $3x^2 - 4x.$

5. $a - b - c.$ 6. $3a^2 - 4a - 5.$ 7. $2x^2 - 3xy + y^2.$

8. $-4x^3 + x^2y - y^3.$ 9. $ax^3 + by^3 - cz^3.$

10. $a^2x^3 - 3b^3y^2 - 4xy.$ 11. $ax^3 - 3bx^2 + 3cx - d.$

12. $ax^2 - 2hxy + by^2 - 2gx - 2fy + c.$ 13. $4x - 5y + 6z.$

14. $6x^2 - 4xy - y^2.$ 15. $\frac{1}{9}x^4 + \frac{2}{3}x^2y^2 - 8y^4.$

16. $-1\frac{1}{2}a^4 + 5a^3 - 3\frac{3}{5}a - 1.$

Exercise 14 (Page 49)

1. $x - 3.$ 2. $x + 4.$ 3. $x - 6.$ 4. $a + 7.$ 5. $a + 8.$

6. $k - 3.$ 7. $x - 5.$ 8. $a + 4.$ 9. $k - 12.$ 10. $t + 11.$

11. $3x - 4.$ 12. $5t + 4.$ 13. $6m - 5.$ 14. $5 - 9a.$

15. $8w + 1.$ 16. $3 - 11v.$ 17. $8x - 3y.$ 18. $3x - 9y.$

19. $8a - 7b.$ 20. $-5v + 7u.$ 21. $2x^2 - 5.$ 22. $3x^2 - 7.$

23. $3x^3 - 2a.$ 24. $7y^2 + z.$ 25. $x^2 - 2x + 5.$

26. $-3x^2 + 2x - 5.$ 27. $4x^2 - 3xy - y^2.$ 28. $2b^2 + 3ab - a^2.$

29. $x^5 - x^4y + xy^4 - y^5$

30. $4x^2 + 6x + 9$.

31. $4a^2 - 10a + 25$.

32. $x^2 - 3ax$.

33. $4x^2 - 6xy + 9y^2$

34. $16a^2 + 12ab + 9b^2$.

35. $x^4 - x^2y^2 + y^4$

36. $x^3 - 2x^2 + 8$.

37. $x^3 + x^2 + x - 1$

38. $2x^3 + 4x^2 + 8x + 16$.

39. $x^3 - ax^2 + a^2x - a^3$

40. $-a^5 + a^4b - a^3b^2$

41. $27x^3 - 36ax^2 + 48a^2x - 64a^3$

42. $2 - a - a^2$

43. $1 + 4a - 10a^2$

44. $x^2 - 3xy - y^2$

45. $a^2 - 2ab + b^2$

46. $x^2 - 3xy - y^2$.

47. $a^2 - 2ax + 4x^2$

48. $x^4 - 3x^2 + 2x + 1$

49. $a^3 - 7a^2 + 5a + 1$

50. $a^3 - 7a + 5$.

51. $x^3 - 4x^2 + 11x - 24$.

52. $a^3 + 4a^2 + 5a - 24$.

53. $2a^3 - 3a^2 + 2a$

54. $x^3 + 2x^2y + 2xy^2$.

55. $x^3 + 6x^2y + 24xy^2 + 60y^3$ 56. $x^2 + 4y^2 + 9z^2 - 2xy + 6yz + 3zx$.

57. $x^2 + y^2 + z^2 - xy - yz - zx$

58. $a^2 - ax + bx + b^2$

59. $x^2 + y^2 - xy + x + y - 1$.

60. $a^2 + 1 - ab + a$.

61. $x^2 + 9y^2 + 1 - 3xy + 3y + x$.

62. $a^2 - 2ab + b^2 + c^2$.

63. $a^2b - b^2a + b^2c - bc^2 + c^2a - ca^2$

64. $\frac{1}{2}a^2 + \frac{1}{3}x + \frac{1}{4}$.

65. $\frac{1}{2}a^2 - 2a + \frac{3}{2}$.

66. $\frac{1}{2}x - \frac{1}{3}y$.

67. $\frac{1}{2}x^2 - \frac{2}{3}xy - \frac{1}{3}y^2$.

68. $2x^2 + x - 1 + \dfrac{3x + 4}{x^2 - x + 3}$.

69. $10x^2 - 3x - 12 + \dfrac{7x - 45}{3x^2 + 2x - 4}$.

70. $x + 2$.

71. $2x + 1$.

72. $2m - 3$.

73. $-7x + 7$.

74. 3.

75. 2.

76. 4.

77. -1.

78. $a^3 + 2a^2b + 2ab^2 + b^2$.

79. Quotient $x^2 + x + 1$; other factor of 14541 is 111.

80. Quotient $x^2 + x + 3$.

Exercise 15 (Page 57)

[See Answers to Ex. 14]

Exercise 16 (Page 58)

1. $2x$ 2. $2z$. 3. $2x + y - z$. 4. $2x - 3y$.

5. $-2y + 2z$. 6. $6a + 3b - c$ 7. $4x - 3y$. 8. $x - 2y + 2z - t$.

9. $2z$. 10. x. 11. $x + 6y - 4z$. 12. $3x - 2y - z$.

13. $-4x - 4y - 2z$. 14. $-2x + y$. 15. $-2x - y - z$.

16. $4x - 2y$. 17. $5 - a$. 18. $6x - 2z$. 19. $5a - 6b + 7ab$.

20. $10+13x.$ **21.** $-139.$ **22.** $2x^4+3x^2.$ **23.** $a^3-b^3.$
24. $a^3-b^3+c^3+3abc.$

Exercise 17 (Page 60)

1. $x+(y-z).$ 2. $x+(-y+z).$ 3. $x+(-y-z).$
4. $x+(y-z).$ 5. $2x+3(y+z).$ 6. $4x+a(-y+z).$
7. $5a+b^2(-c-d).$ 8. $3l+5ab(m-n).$
9. $ax-b(y-z).$ 10. $3x^2-4a^2(y+z).$
11. $abc-cd(a-b).$ 12. $6a^2-5b(-3a-4b).$
13. $x(x+y)+(y^2-3x).$ 14. $x(x-2y)+(1-y^2-x).$
15. $ac(b+d)+ab(c-d).$ 16. $a(ab-c^2)+c(-b^2-a^2-bc).$
17. $(a^3-d^3)-(b^3+c^3).$ 18. $a^2bc(1+bc)-bca(b-c).$
19. $3x(x^2-6y^2)-2x^2(3-2y).$ 20. $(20-a^4)-5a^2(2+a).$
21. $x^4(a+2)+x^3(3-b)+x^2(1-c).$
22. $x(3+a)+x^3(-1-c)+x^4(-1-b)$
23. $(1-5d)+2x(-1-2c)+3x^2(-1-b)+2x^3(-2-a).$
24. $(1-4c)+x(1-2a)+x^2(1+a^2)+x^3(b-1).$
25. $-a(x-x^3-1)-a^2(x^4-1)-a^3(-3-x).$
26. $-a(1+x-2bc)-a^2(3c+1)-a^3(1-4b).$

Exercise 18 (Page 62)

1. $4x+13y-6z.$ 2. $2x-11y+z.$ 3. $10a-7b+2c.$
4. $-31a.$ 5. $-l-2.$ 6. $8p+7q-11r.$
7. $4a^2+12ab+9b^2.$ 8. $4a^2-12ab+9b^2.$ 9. $4a^2-9b^2.$
10. $16a^2$ 11. $36b^2.$ 12. $8a^3+36a^2b+54ab^2+27b^3.$
13. $8a^3-36a^2b+54ab^2-27b^3.$
14. $24a^3+36a^2b-54ab^2-81b^3.$
15. $24a^3-36a^2b-54ab^2+81b^3.$ 16. $64a^3.$ 17. $216b^3.$
18. $16a^3++108ab^2.$ 19. $72a^2b+54b^3.$
20. $96a^3b+216ab^3.$ 21. $-8x^2y^2.$ 22. $9x^2y^2.$ 23. $24x^3y^3.$
24. $-4x^4.$ 25. $-64y^4.$ 26. $36x^5y.$ 27. $-144x^2y^4.$
28. $-160x^2y^4.$ 29. $4x^4-12x^3y-7x^2y^2+24xy^3+16y^4$
30. $576x^6y^6.$ 31. $16xy.$ 32. $-18.$ 33. $x^6.$

34. $-48y^6.$ 35. $-\dfrac{2x^5}{y}.$ 36. $\dfrac{9y}{4x}.$

Test Paper 1 (Ex. 19) (Page 64)

1. 42.
2. $8x^3 - 2x.$
3. 0.
4. $x^5 + 2x^4 + 5x^3 + 5x^2 + 8x + 3 : 125583.$
5. $a^3 - 3a^2 + 2a - 1.$
6. Rs. $(2-a)$; (i) $a-1$, (ii) $1-a.$

Test Paper 2 (Ex. 20) (Page 64)

1. $+2.$
2. $2x^3 - 2x^3 + x.$
3. $13c.$
4. $a^4 - a^3 - 8a^2 + 11a - 3.$
5. $7x^2 + 5x - 3 + \dfrac{x+1}{4x^2 - 13x + 6}.$
6. $\frac{23}{60}x - \frac{943}{60}.$

Test Paper 3 (Ex. 21) (Page 64)

1. $-2.$
2. $-y^3 + 4y.$
3. $-8x + 15y.$
4. $5k^5 - 15k^4 + 4k^3 + 7k^2 - 7k + 2.$
5. $x^2 + 2x - 3.$
6. $a^2 - 2a + 1.$

Test Paper 4 (Ex. 22) (Page 65)

1. $3\frac{2}{3}.$
2. $x^2 + 8xy.$
3. $4z.$
4. $x^5 + x^4y - x^3y^2 + x^3 - 2xy^2 + y^3.$
5. $2a^2 + 3a + 1 : 231.$
6. $3x^2 - x - 1.$

Test Paper 5 (Ex. 23) (Page 65)

1. 1
2. $-8a^3 + 6a^2 + 5a.$
3. $12x + 2z - 2t.$
4. $1 - a^3 - 8x^3 - 6ax$
5. $a^4 + 2a^3 + 3a^2 + 2a + 1.$
6. $3 \times a$, $3.a$, $3a$, $a \times 3$, $a.3$; 4.6 means "four decimal six", 4.6 means "four into six"

Test Paper 6 (Ex. 24) (Page 65)

1. $54a^3 + 18ab^2.$
2. $4a.$
3. $1\frac{1}{8}x + 1 ; 0.$
4. $x^3 - 8y^3 - 27z^3 - 18xyz.$
5. $x^2 + 2xy + 3y^2.$
6. $2y^2$ a^5 ; $x^{35} . x^{100}.$

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Exercise 25 (Page 69)

1. 2 9	2. 4.	3. 8.	4. $-4.$	5. 0.	6. $-3.$
7 $-10.$	8. 0.	9. 3.	10. 5.	11. $-3.$	
12. $-6.$	13. 1.	14. 0.	15. 0.	16. 0.	
17. 27	18. $-32.$	19. $-2.$	20 $\frac{1}{2}.$	21. $-\frac{3}{5}.$	
22. $-\frac{2}{3}.$	23. $-\frac{2}{3}.$	24. $\frac{8}{7}.$	25. $-\frac{1}{3}.$		
26. 0.	27. $-4.$	28. 5.	29. $-5.$	30. 0	
31. $-\frac{3}{4}.$	32. $\frac{3}{4}.$	33 1.	34. $-3.$	35 $\frac{3}{3}.$	

36. $-\frac{2}{3}$. **37.** $-\frac{2}{3}$. **38.** 0. **39.** 4. **40.** 3.
41. 15. **42.** $4\frac{1}{2}$.

Exercise 26 (Page 73)

1. 1. **2.** 3. **3.** -2. **4.** 5. **5.** -4. **6.** -6.
7. 7. **8.** 8. **9.** -9. **10.** -10. **11.** $\frac{1}{4}$.
12. $-3\frac{1}{2}$. **13.** $\frac{1}{2}$. **14.** 4. **15.** -8. **16.** $-\frac{10}{13}$.
17. 2. **18.** 8. **19.** 0. **20.** 4. **21.** 0. **22.** $\frac{8}{9}$.
23. 2. **24.** 1. **25.** 0. **26.** 8. **27.** -1. **28.** 8.
29. 10. **30.** -5. **31.** 0. **32.** 1. **33.** -1.
34. 2. **35.** 3. **36.** 4. **37.** 5. **38.** $1\frac{1}{7}$. **39.** $\frac{3}{2}$.
40. 4. **41.** 1. **42.** 1. **43.** 7. **44.** 1. **45.** $\frac{1}{4}$.
46. 3. **47.** 7. **48.** $5\frac{1}{2}$. **49.** 2. **50.** $13\frac{6}{7}$.
51. $-\frac{2}{5}$. **52.** 5.

Exercise 27 (Page 80)

1. $a-b$.

2. $100-x$

3. $\frac{y}{x}$.

4. ab.

5. $5x+y$.

6. $100-x$.

7. $3b-2a$.

8. $50-k$.

9. $\frac{y}{x}$.

10. $192x$.

11. $240a+12b$.

12. $\frac{n}{x}$ rupees.

13. $\frac{25x}{4}$.

14. $(42-x)$ years.

15. $\frac{10m}{n}$

16. $\frac{k}{3}$.

17. $\frac{22}{15}x$ ft. p. s.

18. $x-\frac{31}{16}y$.

19. $40-x-y$.

20. $a, a+1, a+2, a+8$.

21. $k-2, k-1, k$.

22. $2n-4, 2n-2, 2n$.

23. $2n-3, 2n-1, 2n+1$.

24. $\frac{100(b-a)}{a}\%$.

25. $\frac{(100+y)x}{100}$ rupees.

26. Rs. $(500+10x)$

27. $\dfrac{108x}{y}$ inches.   28. $\dfrac{108x^2}{y}$ ft.

29. $(330-a-b)$ degrees.   30. $\dfrac{180-x}{2}$ degrees.

31. $10a+5$ 32. $100x+10y+z$.

Exercise 28 (Page 83)

1. 120.   2. 50.   3. 160.   4. 60.   5. 50.   6. 40
7. 35.   8. 43, 31.   9. 78, 22.   10. 112, 88.
11. 162 ; 63.   12. 72, 48.   13. 10, 15.
14. 19, 13, 7, 42.   15. 20, 30, 8, 120.   16. 30, 50, 7, 126.
17. 50. 40, 38, 30.   18. 50, 32, 28, 30.
19. Rs. 1000. Rs. 2000, Rs. 3000, Rs. 5000.
20. £ 60, £48, £35.
21. Rs. 2 00, Rs. 3000, Rs. 2600, Rs. 1800.   22. 29, 30, 31.
23. 16, 18, 20, 22, 24.   24. 25, 27, 29, 31, 33, 35.
25. 20, 21.   26. 12, 14.   27. 19, 21.
28. 60 annas, 120 pies. 29. 61, 372.   30. 40, 10, 20.
31. 14.   32. 8.   33. 32, years, 16 years.
34. 40 years. 10 years.   35. 50 years, 10 years.
36. 30 years.   37. 60 years.   38. Rs. 40.
39. Rs. 42.   40. Rs. 60, Rs. 100.   41. Rs. 80, Rs. 40.
42. 46.   43. 83.   44. 68.   45. 432.   46. 25.
47. Rs. 1200.   48. 600.   49. Rs. 1400 ; Rs. 625.
50. 60 tables, 48 chairs.   51. Rs. 5260, Rs. 3800.
52. 54, 48.   53. Rs. 1250 ; Rs. 2750.
54. 14 miles.   55. 80.   56. 20.
57. 28 ft., 20 ft.   58. 25 ft., 18 ft.   59. 5 miles.
60. 360.   61   64°, 96°. 128°, 72°.   64. 144°, 72°, 36°, 108°.

Exercise 29 (Page 100)

1. $x=2, y=1$.   2. $x=3, y=2$.   3. $x=1, y=1$.
4. $x=2. y=2$.   5. $x=3, y=3$.   6. $x=-4, y=2$.
7. $x=2 \; y=-3$.   8. $x=4, y=-2$.   9. $x=-1, y=2$.
10. $x=5, y=-1$. 11. $x=-4, y=5$. 12. $x=6, y=0$.
13. $x=4, y=-4$. 14. $x=3, y=4$.   15. $x=2, y=1$.
16. $x=5, y=-2$. 17. $x=1, y=-1$. 18. $x=4, y=2$.
19. $x=3, y=1$ 20. $x=5, y=-2$. 21. $x=-4, y=-3$.

22. $x=2, y=2$.   23. $x=-4, y=-3$. 24. $x=6, y=-1$
25. $x=1, y=\frac{3}{2}$.   26. $x=\frac{5}{2}$, $y=\frac{3}{2}$. 27. $x=3, y=1$
28. $x=6, y=-2$. 29. $x=2, y=2$.   30. $x=3, y=3$
31. $x=1, y=-1$. 32. $x=4, y=4$.   33 $x=1, y=1$
34. $x=3, y=2$.   35. $x=17, y=-1$.
36. $x=3, y=1$.   37. $x=5, y=-3$
38. $x=1, y=6$.
39. $x=4, y=4$.   40. $x=-3, y=-2$.   41. $x=13, y=3$.
42. $x=2, y=5$.   43. $x=3, y=7$ 44. $x=\frac{3}{2}, y=\frac{3}{2}$.
45. $x=\frac{1}{2}, y=\frac{1}{2}$.   46. $x=\frac{1}{4}, y=-4$.   47 $x=3, y=1$.
48. $x=\frac{3}{2}, y=-\frac{1}{3}$.   49. $x=\frac{1}{2}, y=3$.   50. $x=1, y=\frac{1}{3}$
51. $x=\frac{1}{15}, y=18$.   52. $x=1, y=1$.   53. $x=3, y=5$
54. $x=4, y=10$.   55. $x=2, y=-1$.   56. $x=\frac{1}{2}, y=\frac{1}{3}$.
57. $x=\frac{1}{3}, y=\frac{1}{3}$.   58. $x=\frac{1}{2}, y=\frac{1}{4}$.

Exercise 30 (Page 113)

1. $x=1, y=2$.   2. $x=2, y=-1$.   3. $x=4, y=-3$.
4. $x=5, y=-2$.   5. $x=6, y=-1$.   6. $x=10, y=0$.
7—40. See Ex. 29, answers to questions No. 7 to 26 and
45 to 58.
41. $x=\frac{1}{2}, y=-\frac{2}{5}$.   42. $x=\frac{1}{4}, y=\frac{1}{4}$.   43. $x=\frac{1}{3}, y=-1$.
44. $x=\frac{1}{3}, y=-\frac{1}{4}$.   45. $x=8, y=32$.   46. $x=\frac{1}{4}, y=-\frac{1}{8}$
47. $x=-\frac{1}{2}, y=1$.   48. $x=-1, y=-2$. 49. $x=\frac{3}{4}, y=\frac{3}{8}$
50. $x=\frac{2}{5}, y=\frac{3}{2}$.   51. $x=\frac{3}{2}, y=-\frac{1}{2}$.   52. $x=-\frac{2}{3}, y=1$.
53. $x=-1, y=\frac{5}{2}$.   54. $x=0, y=0$.   55. $x=2, y=1$.
56. $x=1, y=2$.   57. $x=3, y=2$.   58. $x=2, y=4$.
59. $x=1, y=-1$.   60. $x=\frac{1}{3}, y=-\frac{1}{2}$.

Exercise 31 (Page 120)

1. $x=1, y=2, z=3$.   2. $x=3, y=2, z=1$.
3. $x=2, y=2, z=2$.   4. $x=1, y=1, z=1$.
5. $x=2, y=-1, z=-3$.   6. $x=2, y=1, z=3$.
7. $x=3, y=4, z=6$.   8. $x=2, y=1, z=3$.
9. $x=2, y=3, z=5$.   10. $x=2, y=3, z=4$.
11. $x=3, y=4, z=5$.   12. $x=3, y=-1, z=-2$
13. $x=-1, y=-3, z=4$.   14. $x=8, y=5, z=4$.
15. $x=5, y=-3, z=-1$.   16. $x=7, y=5, z=3$.
17. $x=3, y=-2, z=-1$.   18. $x=3, y=0, z=1$

19. $x=3, y=2, z=1.$ 20. $x=0, y=2, z=3.$

21. $x=3, y=7, z=-2.$ 22. $x=1, y=1. z=0.$

23. $x=2, y=3, z=4.$ 24. $x=8, y=4, z=5.$

25. $x=2, y=-2, z=5:$ 26. $x=3, y=4. z=5.$

27. $x=1, y=2, z=3.$ 28. $x=1, y=\frac{1}{3}, z=\frac{1}{3}.$

29. $x=\frac{1}{3}, y=\frac{1}{4}. z=\frac{1}{5}.$ 30. $x=1, y=1, z=1.$

31. $x=4, y=-3, z=2.$ 32. $x=\frac{2}{3}, y=\frac{3}{2}, z=\frac{6}{5}$

33. $x=\frac{1}{2}, y=\frac{1}{3}, z=\frac{1}{4}.$ 34. $x=3. y=-2, z=1.$

35. $x=1, y=-1, z=2, t=-2.$ 36. $x=2, y=0. z=1, t=-1.$

37. $x=3, y=1, z=-1, t=0.$

38. $a=1, b=2, c=3, d=4, e=5.$

39. $p=2, q=1, r=0, s=4, t=3.$

40. $l=3. m=0, n=2, p=0, q=1.$

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Exercise 32 (Page 128)

1. $x=8, y=6, z=4.$ 2. $x=2, y=3, z=4.$

3. $x=5, y=3, z=1.$ 4. $x=6, y=8, z=10.$

5. $x=1\frac{1}{2}, y=2, z=2\frac{1}{2}.$ 6. $x=-\frac{1}{2}, y=1, z=-\frac{3}{2}.$

7. $x=2\frac{1}{2}, y=4\frac{1}{2}, z=4.$ 8. $x=1. y=-\frac{5}{3}, z=\frac{5}{3}.$

9. $x=1, y=\frac{2}{3}, z=\frac{1}{2}.$ 10. $x=\frac{1}{2}, y=\frac{1}{3}, z=\frac{1}{4}.$

11. $x=2, y=3, z=4.$ 12. $x=3, y=4, z=5.$

13. $x=3, y=4, z=-2.$ 14. $x=4, y=10, z=14.$

15. $x=7, y=-5, z=-3.$ 16. $x=9, y=-7, z=5.$

17. $x=6, y=4, z=3.$ 18. $x=10, x=6, z=8.$

19. $x=2, y=-3, z=4.$ 20. $x=\frac{1}{2}, y=-\frac{2}{3}, z=-\frac{3}{4}.$

21. $x=6, y=4, z=2.$ 22. $x=4, y=8, z=10.$

23. $x=8, y=-6, z=-4.$ 24. $x=\frac{1}{2}, y=\frac{1}{6}, z=\frac{1}{6}.$

25. $x=\frac{1}{4}, y=\frac{1}{6}, z=\frac{1}{12}.$ 26. $x=\frac{1}{6}, y=-\frac{1}{12}, z=-\frac{1}{20}.$

27. $x=\frac{24}{5}, y=\frac{24}{7}, z=-24.$ 28. $x=3, y=4, z=5.$

29. $x=1, y=2, z=3.$ 30. $x=2, y=5, z=6.$

31. $x=2, y=-3, z=4.$ 32. $x=1, y=2, z=3.$

33. $x=5, y=3, z=2.$ 34. $x=2, y=4, z=6.$

35. $x=3, y=4, z=5$ or $x=-3, y=-4, z=-5.$

36. $x=4. y=1, z=\frac{1}{2},$ or $x=-4, y=-1, z=-\frac{1}{2}.$

37. $x=\frac{2}{3}, y=\frac{3}{4}, z=-\frac{4}{5},$ or $x=-\frac{2}{3}, y=-\frac{3}{4}, z=\frac{4}{5}.$

38. $x=3, y=4, z=7,$ or $x=-3, y=-4, z=-7.$

39. $x=3, y=4, z=5,$ or $x=-3, y=-4, z=-5.$

40. $x=\frac{1}{2}, y=\frac{1}{3}, z=\frac{1}{4},$ or $x=-\frac{1}{2}, y=-\frac{1}{3}, z=-\frac{1}{4}.$

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Exercise 33 (Page 133)

1. 33, 21.   2. 78, 22.   3. 50, 34.   4. 69, 51.
5. 11, 15.   6. 24, 32.   7. 72, 180.   8. 330, 195 and 135.
9. 36, 45.   10. 48, 54.   11. 108 : 45 and 63.
12. 15, 12.   13. 16, 11.   14. 50, 16.   15. 115, 16.
16. $\frac{18}{37}$.   17. $\frac{48}{60}$.   18. $\frac{60}{72}$.   19. 45, 75.   20. 56, 40
21. $\frac{11}{24}$.   22. $\frac{18}{27}$.   23. $\frac{1}{4}$.   24. $\frac{3}{8}$.   25. $\frac{5}{13}$.
26. 45.   27. 36.   28. 72.   29. 78 or 87.   30. 48.
31. 672.   32. 534.   38. Rs. 12, Rs. 8.
34. Rs. 40, Rs. 25.   35. Horse £46, cow £32.
36. Rs. 1, 15a.   37. Rs. 240.   38. 36 years, 11 years
39. 48 years 18 years.   40. 36 years, 14 years.
41. 55 years, 21 years.   42. 60 years.
43. Krishna 18 years, Hari 12 years.
44. 22 years, 12 years.   45. 15, 22, 40.   46. 45, 60, 72.
47. Rs. 400, Rs. 200, Rs. 150.   48. Rs. 120, Rs. 50, Rs. 40.
49. Rs. 280, Rs. 120, Rs. 140, Rs. 40.
50. Rs. 297, Rs. 345, Rs. 99, Rs. 69.
51. Rs. 232, Rs. 332.   52. A Rs 64, B Rs. 96.
53. Rs. 280, Rs. 200.   54. A Rs. 50, B Rs. 41.
55. Rs. 100.   56. Tea Re. 1, sugar 8as.
57. Tea 2s. 6d., coffee 1s. 8d.   58. 4.
59. A 12 days, B 24 days.   60. $2\frac{2}{5}$ days.
61. Rs. 3, 3a. Rs. 2, 5a. Rs. 4, 1a.
62. Sovereigns 3, crowns 12.   63. 12 ; Rs. 60.
64. 25 yards, 20 yards.   65. 2 m.p.h. and 2½ m.p.h.
66. 3 m.p.h. ; $4\frac{2}{7}$ m.p.h.   67. 12 miles ; 3 m.p.h.
68. Stream 3 m.p h., boat 8 m p.h.   69. Rs. 450, Rs. 250.
70. Rs. 750.   71. A$=30°$, B$=78°$, C$=72°$.
72. A$=80°$, B$=90°$, C$=70°$, D$=120°$.
73. A$=75°$, B$=100°$, C$=105°$, D$=80°$ 74. 20 and 8.

Test Paper 1—Ex. 34 (Page 146)

1. -1.   2. $a^5+4a^4+48a-32$.   3. $x=41$.
4. $x=21, y=12$.   5. $x=5, y=-5, z=5$.   6. 48, 3.

Test Paper 2—Ex 35 (Page 146)

1. $-28x$.   2. $1+2a-8a^3-16a^4-32a^5$.   3. $x=3$.
4. $a=5, b=-2$.   5. $a=12, b=-60, c=60$.   6. 3s. 1½d.

Test Paper 3—Ex. 36 (Page 148)

1. 5.　　2. $6-\dfrac{x^4}{2}-\dfrac{x^6}{6}$.　　3. $x=3$.　　4. 8.

5. $a=10, b=18, c=14$.　　6. 81, 18.

Test Paper 4—Ex. 37 (Page 148)

1. $-63x+60y-45z$.　　2. $a+2b+c$.　　3. $x=4$.
4. $x=\tfrac{1}{4}, y=\tfrac{1}{3}$.　　5. $x=3, y=2, z=1$.
6. Sugar $8d.$, tea $5d.$ per lb.

Test Paper 5—Ex. 38 (Page 149)

1. 33.　　2. $x^6+3x^5y-3x^4y^2-11x^3y^3+6x^2y^4+12xy^5-8y^6$.
3. 6.　　4. $x=-\tfrac{4}{5}, y=-\tfrac{5}{6}$.　　5. $x=1, y=\tfrac{1}{2}, z=\tfrac{1}{3}$.
6. 4 miles.

Test Paper 6—Ex. 39 (Page 150)

1. $\tfrac{105}{64}a-\tfrac{105}{8}$; 0.　　2. $-x+2$.　　3. 1.　　4. $p=9, q=2$
5. $a=2, b=3, c=4$.　　6. 7 miles ; $3\tfrac{1}{2}$ miles per hour.

Exercise 40 (Page 154)

1. $9x^2+24xy+16y^2$.
2. $16a^2+40ab+25b^2$.
3. $4r^2+20xy+25y^2$.
4. $9x^2+42xy+49y^2$.
5. $25p^2+60pq+36q^2$.
6. $49l^2+14l+1$.
7. $64a^2-48ab+9b^2$.
8. $25x^2-70xy+49y^2$.
9. $16n^2-72nt+81t^2$.
10. $64k^2-144kl+81l^2$
11. $s^2-30st+225t^2$.
12. $1-40t+400t^2$.
13. $a^2+\dfrac{1}{a^2}+2$.
14. $x^2+\dfrac{1}{x^2}+2$.
15. $b^2+\dfrac{1}{b^2}-2$.
16. $4c^2+\dfrac{1}{4c^2}-2$.
17. $k^2+\dfrac{9}{k^2}+6$.
18. $4t^2+\dfrac{9}{t^2}-12$.
19. $9a^2b^2-24abc^2+16c^4$.
20. $16x^2y^2-40xyz^2+25z^4$.
21. $k^4+12k^2l^2+36l^4$.
22. $9a^2b^2-24abcd+16c^2d^2$
23. $49-112m^2+64m^4$.
24. $1+18xyz+81x^2y^2z^2$.
25. $25a^6b^8-80a^4b^9+64a^2b^{10}$.
26. $16x^8y^4-40x^7y^5+25x^6y^6$.
27. $64x^4y^6+48x^2y^7z^2+9y^8z^4$.
28. $1-24k^{12}+144k^{24}$
29. $100a^{20}-220a^{10}b^{11}+121b^{22}$
30. $a^{24}b^8-8a^{12}b^{10}c+16b^{12}c^2$.
31. $a^2+b^2+c^2-2ab-2ac+2bc$.
32. $x^2+y^2+z^2-2xy+2xz-2yz$.
33. $p^2+q^2+r^2-2pq-2pr+2qr$.

34. $l^2+m^2+n^2+2lm-2ln-2mn.$

35. $g^2+h^2+k^2-2gh+2gk-2hk.$

36. $x^2+y^2+z^2+2xy+2xz+2yz.$

37. $4x^2+9y^2+16z^4-12xy+16xz^2-24yz^2.$

38. $9a^2+16b^2+25c^4-24ab+30ac^2-40bc^2.$

39. $1-6k+k^2+24k^3+16k^4.$ **40.** $x^4-2x^3y+3x^2y^2-2xy^3+y^4.$

41. $p^4-4p^3q-2p^2q^2+12pq^3+9q^4.$

42. $\dfrac{4a^4}{9}-\dfrac{4a^3}{3}+3a^2-3a+\tfrac{9}{4}.$

43. $a^2+4b^2+9c^2+16d^2+25-4ab+6ac-8ad+10a-12bc+$
$16bd-20b-24cd+30c-40d.$

44. $4x^2+y^2+9z^2+t^2-4xy+12xz-4xt-6yz+2yt-6zt.$

45. $9x^6-12x^5+10x^4-10x^3+5x^2-2x+1.$

46. $16a^8+24a^7-7a^6-4a^5+2a^4-10a^3+5a^2-2a+1.$

47. $904009.$ **48.** $990025.$ **49.** $633616.$ **50.** $494209.$

51. $160400\tfrac{1}{4}.$ **52.** $359700\tfrac{1}{16}.$ **53.** $24xy^2.$ **54.** $20x^2y.$

55. $36b^2.$ **56.** $6c^5.$ **57.** $2a^4b^4.$ **58.** $220x^8yz.$

59. $2.$ **60.** $2.$ **61.** $1.$ **62.** $2.$ **63.** $9y^4.$ **64.** $4.$

65. $9x^4.$ **66.** $4a^4b^4.$ **67.** $9.$ **68.** $16b^4.$

69. $\dfrac{1}{x^2}.$ **70.** $\dfrac{1}{x^4}.$ **71.** $\dfrac{4}{a^4b^4}.$ **72.** $\dfrac{25}{a^8}.$

73. $400.$ **74.** $1600.$ **75.** $16.$ **76.** $16.$ **77.** $81.$

78. $9.$ **79.** $100.$ **80.** $0.$ **81.** $1.$ **82.** $2500.$

83. $3600.$ **84.** $16.$ **85.** $-8a^2+8b^2.$ **86.** $8x^2+34y^2-8xy.$

87. $25a^2+41b^2+16ab.$ **88.** $4ab-4ac.$ **89.** $16b^2.$

90. $36x^2.$ **91.** $16a^2.$ **92.** $a^2+10ab+25b^2.$ **93.** $64x^2.$

94. $l^2.$ **95.** $11.$ **96.** $20.$ **97.** $15.$ **98.** $58.$

99. 4 or $-4.$ **100.** 1 or $-1.$ **101.** 4 or $-4.$

102. 6 or $-6.$ **103.** 6 or $-6.$ **104.** 7 or $-7.$

105. 5 or $-5.$ **106.** 7 or $-7.$ **107.** 5 or $-5.$

108. $12 ; 3.$ **109.** $9 ; 1.$ **110.** $32\tfrac{1}{2}.$ **111.** $26.$

112. $14\tfrac{7}{8}.$ **113.** $31.$ **114.** $49.$ **115.** $160.$

116. $70.$ **117.** $192.$ **118.** 6 or $-6.$ **119.** 7 or $-7.$

120. 8 or $-8.$ **121.** $-23.$ **122.** $188.$ **123.** $7.$

124. $23.$ **125.** $47.$ **126.** $38.$ **127.** $10.$ **128.** $7.$

129. $23.$ **130.** $119.$ **131.** 3 or $-3.$ **132.** 4 or $-4.$

133. 5 or $-5.$ **134.** 6 or $-6.$ **135.** 7 or $-7.$

136. 3 or $-3.$ **137.** 4 or $-4.$ **138.** 2 or $-2.$

139. 3 or $-3.$ **148.** $7.$ **151.** $169b^2.$ **152.** $121m^2.$

153. $9b^2$.   **157.** $\left(x+\dfrac{1}{x}+2 \right)^2$.   **158.** $\left(x-\dfrac{1}{x}+1 \right)^2$

159. $\left(a^2+\dfrac{1}{a^2}-2 \right)^2$.   **160.** $\left(x^2+\dfrac{1}{x^2}-3 \right)^2$.

Exercise 41 (Page 168)

1 $4a^2-9$.   **2.** $9x^2-25$.   **3.** $16x^2-49y^2$.

4 $\dfrac{64}{l^2}-m^2$.   **5.** $100p^2-1$.   **6.** $81r^2-225$.

7. $25a^6b^2-36c^{10}$.   **8.** $9a^8-16b^6c^2$.   **9.** $25x^{13}-49y^8z^6$.

10. $9y^6z^6-64x^{16}$.   **11.** $81k^{18}-100l^{20}$.

12. $25r^{24}-\dfrac{9p^6}{q^6}$.   **13.** $4x^2+9y^2+12xy-16z^2$.

14. $9a^2-24a+16-25b^2$.   **15.** $4a^2+20ab+25b^2-36c^2$.

16. $4m^2-9p^2+30pq-25q^2$.   **17.** $a^2+b^2+2ab-c^2-d^2-2cd$.

18. $x^2-9y^2-16z^2+24yz$.   **19.** $4x^2+25z^2-20xz-y^2$.

20. $x^4+x^2y^2+y^4$.   **21.** a^4-a^2+2a-1

22. x^8+x^4+1.   **23.** $c^2z^4-a^2x^4+2abx^2y^2-b^2y^4$.

24. $x^2+2xl+l^2-y^2-2yz-z^2$

25. $a^2x^2-b^2y^2-c^2z^2-1+2bcyz-2by+2cz$.

26. a^4-b^4.   **27.** x^4-y^4.

28. $16a^4-81b^4$.   **29.** 16^4-625y^4.

30. $x^8-\dfrac{1}{x^8}$ **31.** $256a^8-b^8$.

32. $l^{16}-m^{16}$.   **33.** $1+a^4+a^8$.

34. $a^8+a^4b^4+b^8$.   **35.** x^5-1

36. $a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2$.

37. $12ac-8ab$ **38.** $24xy+32xz$.

39. $20x^3+140x$.   **40.** $4a^2b^2-4b^2\sqrt{ab}$

41. $8xy-24xz+8xp$.   **42.** $5a^4-12a^3b+16a^2b^2$.

43. 8999984.   **44.** 809775.

45. 489999.64.   **46.** 899.9744.

47 1776000.   **48.** 9128000.

49 7.89.   **50.** 2055.6.

Exercise 42 (Page 172)

1 x^2+5x+6.   **2.** $x^2+12x+85$.   **3.** $a^2+10a+24$.

4. $m^2+17m+72$.   **5.** $x^2-4x-45$.   **6.** x^2-x-56.

7. $a^2-7a-30.$ 8. $p^2+4p-5.$ 9. $b^2-14b+48.$

10. $t^2-12t+11.$ 11. $-x^2+15x-56.$

12. $-x^2+11x-30.$ 13. $x^2+x-90.$ 14. $x^2+21x+110$

15. $4x^2-24x+35.$ 16. $9x^2+3x-72.$ 17. $-16a^2+32a$ 7.

18. $\dfrac{25}{k^2}-\dfrac{35}{k}+6.$

19. $9x^2+16y^2+24xy+6x+8y-35.$

20. $4a^2+25b^2-20ab-18a+45b-10.$

21. $x^4-7x^3+16x^2-21x+9.$ 22. $9a^4+8a^3-26a^2-4a+16.$

23. $25a^2-25a-16b^2-28b-6.$

24. $16x^4-4x^3+14x^2-5x+3.$

Exercise 43 (Page 174)

1. $x^3+9x^2+26x+24.$ 2. $x^3+8x^2+19x+12.$
3. $x^3+18x^2+52x+60.$ 4. $x^3+12x^2+39x+28.$
5. $x^3-5x^2-18x+72.$ 6. $x^3-x^2-17x-15.$
7. $x^3-9x^2+8x+60.$ 8. $x^3-6x^2-18x+42.$
9. $x^3-6x^2+11x-6.$ 10. $x^3-15x^2+74x-120.$
11. $x^3-48x-42.$ 12. $x^3+8x^2+x-42.$
13. $a^3-14a^2-119a+1716.$ 14. $m^3-127m+546.$
15. $k^3+5k^2+204k-1440.$ 16. $8a^3+86a^2+46a+15.$
17. $27k^3-63k^2+42k-8.$

18. $\dfrac{a^3}{b^3}-8\dfrac{a^2}{b^2}+11\dfrac{a}{b}+20.$

19, $x^3y^3-14x^2y^2+61xy-84.$ 20. $125a^3-325a^2b+144b^3.$
21. missing terms are $-1, -5, -7.$
22. missing terms are $2, -5, -1.$
23. missing terms are $8, -2, -1.$
24. missing terms are $-a, 1.$

Exercise 44 (Page 176)

1. $8x^3+36x^2y+54xy^2+27y^3.$
2. $125x^3+300x^2z+240xz^2+64z^3.$
3. $27m^3+189m^2n+441mn^2+318n^3.$
4. $a^3x^3+8a^2x^2by+8axb^2y^2+b^3y^3.$
5. $x^6+6x^4y^2+12x^2y^4+8y^6.$ 6. $\frac{1}{8}x^3+\frac{1}{4}x^2y+\frac{1}{6}xy^2+\frac{1}{9}y^3.$
7. $27x^6-54x^4y+36x^2y^2-8y^3$

8. $a^6-9a^4b+27a^2b^2-27b^3$.   **9.** $8a^3b^3-60a^2b^2+150ab-125$.

10. $\frac{1}{27}m^6-\frac{1}{3}m^4n+m^2n^2-n^3$.

11. $\frac{1}{27}x^3-\frac{1}{12}x^2a^2+\frac{1}{16}xa^4-\frac{1}{64}a^6$.

12. $8x^3-4x+\dfrac{2}{3x}-\dfrac{1}{27x^3}$.

14. $a^3-b^3+c^3-3ab(a-b)-3bc(-b+c)+3ca(c+a)-6abc$.

15. $a^3+b^3-c^3+3ab(a+b)-3bc(b-c)-3ca(-c+a)-6abc$.

16. $-a^3+b^3+c^3-3ab(-a+b)+3bc(b+c)-3ca(c-a)-6abc$.

17. $a^3-b^3-c^3-3ab(a-b)-3bc(b+c)-3ca(-c+a)+6abc$

18. $8a^3-27b^3+c^3-18ab(2a-3b)-9bc(-3b+c)+6ca(c+2a)$
$$-36abc.$$

19. $27x^3-8y^3-1-18xy(3x-2y)-6y(2y+1)-9x(-1+3x)$
$$+36xy.$$

20. $a^3-8b^3-27c^3-6a^2b+12ab^2-9a^2c+27ac^2-36b^2c-54bc^2$
$$+36abc.$$

21. $27x^3+8y^3-64z^3+54x^2y+36y^2x-108x^2z+144xz^2-48y^2z$
$$+96z^2y-144xyz.$$

22. $x^6-3x^5y+6x^4y^3-7x^3y^3+6x^2y^4-3xy^5+y^6$.

23. 4.   **24.** 20.   **25.** -72.   **26.** p^3-3pq.

27. $-10k^3$.   **28.** 208.   **29.** -17.   **30.** $7\frac{1}{8}$.

31. p^3+3pq.   **32.** $26l^3$.   **33.** 2.   **34.** 18.   **35.** -52.

36. 110.   **37.** $8k^3-\frac{y}{2}k$.   **38.** $-27l^3-9l$.   **39.** 234.

40. $\dfrac{a^3}{8}+\dfrac{5a}{2}$.   **41.** $-\dfrac{k^3}{27}-8k$.

42. 198.   **43.** 18.   **44.** 76 or -76.   **45.** 36 or -36.

46. 0.   **47.** 110 or -110.   **48.** 8.   **49.** 2.   **50.** k^3.

51. 8.   **52.** $-2l^3$.   **53.** 64.   **54.** 125.   **55.** 216.

58. 27000.   **59.** 8000.   **60.** 0.   **61.** 64115.

62. 348009.   **63.** $64a^3$.   **64.** $125a^3$.   **65.** $64q^3$.

66. $-x^3$.   **67.** $8y^3$.   **68.** $8p^3$.

69. $27m^3+27m^2n+9mn^2+n^3$.

70. $27x^3-81x^2y+81xy^2-27y^3$.

Exercise 45 (Page 184)

1. x^3+1.   **2.** x^3+8.   **3.** $8x^3+27$.   **4.** $27a^3+64b^3$.

5. $125a^3+8b^3$.   **6.** $a^3b^3+27c^6$.   **7.** $8a^3-27b^3$.

8. $27x^3-64y^3$.   **9.** $x^3y^3-z^6$.   **10.** a^6-b^6.

11. $27a^6-64b^3c^3$.   **12.** $27p^9-8$.

13. $x^3 + \dfrac{1}{x^3}$.

14. $\dfrac{a^3}{27} - \dfrac{27}{a^3}$.

15. $\tfrac{1}{8}a^3 + \tfrac{1}{27}b^3$.

16. $8x^6 - \dfrac{27}{x^3}$.

17. $x^9 - 1$.   18. $a^9 + b^9$.   19. $x^6 - 1$ 20. $c^6 - 64$.

21. $1 - x^6$.   22 $x^6 - 64y^6$

23. $729m^6 - 64n^6$.   24. $a^{12} - \dfrac{1}{a^{12}}$.

25. $a^3 + 3a^2b + 3ab^2 + b^3 + 1$.   26. $8x^3 + 12x^2y + 6xy^2 + y^3 - 1$.

27. $x^3 + 3x^2 + 3x + 1 - 8y^3$.   28. $x^3 + \dfrac{1}{x^3} + 3x + \dfrac{3}{x} + 1$.

Exercise 46 (Page 186)

1. $x^3 + y^3 - z^3 + 3xyz$.   2. $p^3 - q^3 + r^3 + 3pqr$.
3. $l^3 - m^3 - n^3 - 3lmn$.   4. $-a^3 + b^3 + c^3 + 3abc$.
5. $8a^3 - 27b^3 + 64c^3 + 72abc$.   6. $a^3 - 8b^3 + 27c^3 + 18abc$.
7. $64x^3 - 125y^3 - 1 - 60xy$.   8. $27p^3 - 8q^3 + 125 + 90pq$.
9. $x^3y^3 + y^3z^3 - z^3x^3 + 3x^2y^2z^2$.   10. $a^6 - b^6 - c^6 - 3a^2b^2c^2$.
11. 2.   12. 50.   13. -16.   14. 9.
15. -18.   16. 28 or -28.   17. 98 or -98.
18. 10003.   19. 11606.   20. 266760.   21. 0.
22. 0.   25. 174.   26. 550.   33. 4.
34. 6.   35 10.   36. 0.

Exercise 47 (Page 191)

1. $2(2m+3)$.   2. $5(2x-5)$.
3. $m(a+c)$.   4. $bc(a-d)$.
5. $b^2(a+1)$.   6. $a(p+q-1)$.
7. $5x(2y-3z)$.   8. $a^2b^2(b+a)$.
9. $xyz(z-y)$.   10. $p^2q^3(a-bpq)$.
11. $3m(2m-5n+9)$.   12. $3(x^2+3xy+1)$.
13. $7m(3-2n+4m)$.   14. $2y^2(x^3-2x^2+4xy-8y^2)$.
15. $4xyz(3x+4y-6z)$.   16. $11a^5y^6(13x^5+11y^2z)$
17. $13a^5b^5c^5(4b^2c^2-6a^2c^2-7a^2b^2)$.
18. $16x^6y^5z^2(6x^3y^3z-7x^2+8z)$.   19. $(b+c)(a-b)$.
20. $(y+z)(x-y)$.   21. $(2q+r)(p+1)$.
22. $(3c-d)(ab-1)$.   23. $(b-4c)(c^2-b)$.

24. $(5x-1)(7m+2)$. 25. $(x^2+y^2)(5ab-6mn)$.
26. $(a-b+c)(10a^2-3b)$. 27. $(2a+b)(x-y-z)$.
28. $(x^3+y^3)(2a-3b+1)$. 29. $(b-c)(ab-ac+b)$.
30. $(y+z)(xy+xz-y)$. 31. $(2y-z)(6xy-3xz-1)$.
32. $(4x-y)^2(8ax-2ay-b)$. 33. $(y+z)^2(3xy+3xz+1)$.
34. $(a-b)(a^2-2ab+b^2+2a-2b-3)$.
35. $(x+y)(3x^2+6xy+3y^2+5x+5y+1)$.
36. $3x(3a-4b)(ax-8bx-3y)$.
37. $b^2(p-2q)(c^3p-2c^3q+a^2b)$. 38. $5m(a-3b)(m-a+3b)$.
39. $8b(2b-5c)(2a^2b-6a+3b)$. 40. $5(p^2-qr)^2(3p^2-3qr-2)$.
41. $x(x+y)(x^2+2xy+y^2-3y)$. 42. $2(x-y)(3a+b)$.
43. $(x-y)^2(x-z)^2(z-y)$.
44. $2a(a-1)^2(b-1)^2(2a-2+8b^2-3b)$.

Exercise 48 (Page 193)

1. $(2a+3b)^2$. 2. $(3a+4b)^2$. 3. $(5x+2y)^2$.

4. $(6x+5)^2$. 5. $(12a+1)^2$. 6. $\left(3a+\dfrac{2}{a}\right)^2$.

7. $(2x^2+3y^2)^2$. 8. $\left(x^2+\dfrac{3}{x^2}\right)^2$. 9. $(5x-3y)^2$.

10. $(6x-5y)^2$. 11. $(3x^2-4)^2$. 12. $(5a^2-3b^2)^2$.

13. $(x^3-4)^2$. 14. $\left(x^2-\dfrac{5}{x^3}\right)^2$. 15. $3a(4a-5b)^2$.

16. $4b(2a+5b)^2$. 17. $5ab(3a-b)^2$. 18. $6xy(x+3y)^2$.
19. $6x^2(x-1)^2$. 20. $5xy^2(2y+1)^2$. 21. $2a^2(a^2+b^2)^2$.
22. $3m^2(m^2-2n^2)^2$.

Exercise 49 (Page 194)

1. $(2a+3b)(2a-3b)$. 2. $(3x+4y)(3x-4y)$.
3. $(5c+6d)(5c-6d)$. 4. $(10+3z)(10-3z)$.

5. $(7+8k)(7-8k)$. 6. $\left(11+\dfrac{1}{c^3}\right)\left(11-\dfrac{1}{c^3}\right)$.

7. $(ab+4c^2)(ab-4c^2)$. 8. $\left(p^2+\dfrac{3}{q^3}\right)\left(p^2-\dfrac{3}{q^3}\right)$.

9. $(x^2+9y^2)(x+3y)(x-3y)$. 10. $(1+x^2)(1+x)(1-x)$.
11. $(4a^2+1)(2a+1)(2a-1)$. 12. $(c^2+16)(c+4)(c-4)$.
13. $(a^2+25b^2)(a+5b)(a-5b)$.

14. $\left(a^4 + \dfrac{1}{a^4}\right)\left(a^2 + \dfrac{1}{a^2}\right)\left(a + \dfrac{1}{a}\right)\left(a - \dfrac{1}{a}\right)$.

15. $a(2a+5x)(2a-5x)$.

16. $2(3ab+4)(3ab-4)$.

17. $pq(1+q^2p)(1-q^2p)$.

18. $2ab^3(3a+11b^2)(3a-11b^2)$.

19. $4x^5a^3(9x^6+11)(9x^6-11)$.

20. $3xy(x^2+9y^2)(x+3y)(x-3y)$.

21. $(x^2+2y-3z)(x^2-2y+3z)$.

22. $(2a^2+2b-c)(2a^2-2b+c)$.

23. $(2x+1+3x^2)(2x+1-3x^2)$.

24. $(8x-4+5z)(8x-4-5z)$.

25. $(a^2+bc+ab+ac)(a^2+bc-ab-ac)$

26. $(2xy+3x-1)(2xy-3x+5)$.

27. $(a+b+c)(a+b-c)$

28. $(2a+b+2c)(2a+b-2c)$.

29. $(5x-1+6y)(5x-1-6y)$.

30. $(x+y-3)(x-y+3)$.

31. $(8a+5b+2c)(8a-5b-2c)$.

32. $(7x-y+1)(7x-y-1)$.

33. $(2a-b+c-3d)(2a-b-c+3d)$.

34. $(p-5q+3r-s)(p-5q-3r+s)$.

Exercise 50 (Page 196)

1. $(2x+3y)(4x^2-6xy+9y^2)$.

2. $(y+4z)(y^2-4yz+16z^2)$.

3. $(5a+b)(25a^2-5ab+b^2)$.

4. $(2z+7)(4z^2-14z+49)$.

5. $(3b^2+5)(9b^4-15b^2+25)$.

6. $\left(x^3 + \dfrac{4}{y^3}\right)\left(x^6 - \dfrac{4x^3}{y^3} + \dfrac{16}{y^6}\right)$.

7. $(4a-3b)(16a^2+12ab+9b^2)$.

8. $(5p-2q)(25p^2+10pq+4q^2)$.

9. $(9a^2-2)(81a^4+18a^2+4)$.

10. $(x^4-10)(x^8+10x^4+100)$.

11. $(2z^6-5)(4z^{10}+10z^5+25)$.

12. $\left(8 - \dfrac{x^3}{y^3}\right)\left(64 + \dfrac{8x^3}{y^2} + \dfrac{x^3}{y^4}\right)$.

13. $(a+b)(a-b)(a^2-ab+b^2)(a^2+ab+b^2)$.

14. $\left(x + \dfrac{1}{x}\right)\left(x - \dfrac{1}{x}\right)\left(x^2 + \dfrac{1}{x^2} + 1\right)\left(x^2 + \dfrac{1}{x^2} - 1\right)$.

15. $(z+2)(z-2)(z^2-2z+4)(z^2+2z+4)$.

16. $(3a+1)(3a-1)(9a^2-3a+1)(9a^2+3a+1)$

17. $(x-2)(x^2+2x+4)(x^6+8x^3+64)$.

18. $(x-y)(x+y)(x^2+y^2)(x^2-xy+y^2)(x^2+xy+y^2)(x^4-x^2y^2+y^4)$.

19. $a(3a^2-b^2)(9a^4+3a^2b^2+b^4)$.

20. $2p(2p^3-3q^3)(4p^6+6p^3q^3+9q^6)$

21    $a^2(1+b^2)(1-b^2+b^4).$
22    $4a(1+2a)(1-2a+4a^2)(1-8a^3+64a^6).$
23    $(2a-3b+4c)(4a^2+9b^2+16c^2-12ab-8ac+12bc).$
24    $(3x+2y-5z)(9x^2+4y^2+25z^2+12xy+15xz+10yz).$
25    $(x-y-1)(x^2+y^2+1-2xy+x-y).$
26. $(3b-a-1)(9b^2+a^2+1+3ab-3b+2a).$
27    $(2p+4q-5r)(4p^2+16q^2+25r^2+16pq+10pr+20qr).$
28. $(3a-7x+7)(9a^2+49x^2+49+21ax-21a-98x).$

Exercise 51 (Page 197)

1 $(x+2y)^3$ 2. $(3x+y)^3.$ 3. $(5a+2)^2.$

4. $\left(m+\dfrac{1}{m}\right)^3$ 5. $(3a-2b)^3.$ 6. $(4k-3)^3.$

7 $(ax-by)^3.$ 8. $(a^2-3b^2)^3.$ 9. $2x(2x+y)^3.$

10. $4b(a-3b)^3.$ 11. $(a+b)^3(a-b)^3.$

12. $(x-y)^3(x^2+xy+y^2)^3.$

Exercise 52 (Page 202)

1. $(x+3)(x+4).$	2. $(x+6)(x+2).$
3. $(x+12)(x+1).$	4. $(x+8)(x+1).$
5. $(x+5)(x+2).$	6. $(x+7)(x+2).$
7. $(x+10)(x+2).$	8. $(x+4)(x+5).$
9. $(x-3)(x-5).$	10. $(x-15)(x-1).$
11. $(x-8)(x-2).$	12. $(x-3)(x-6).$
13. $(x-2)(x-9).$	14. $(x-18)(x-1).$
15. $(x-3)(x-8).$	16. $(x-2)(x-12).$
17. $(x+8)(x-3).$	18. $(x+12)(x-2).$
19. $(x+6)(x-4).$	20. $(x+24)(x-1).$
21. $(x+13)(x-2).$	22. $(a+14)(a-2).$
23. $(a+7)(a-4).$	24. $(k+28)(l-1).$
25. $(m+5)(m-6).$	26. $(p+2)(p-15).$
27. $(q+3)(q-10).$	28. $(l+1)(l-30).$
29. $(a+4)(a-8).$	30. $(b+2)(b-16).$
31. $(x+3)(x-11).$	32. $(x+2)(x-17).$
33. $(x+5)(x+7).$	34. $(x+18)(x-2).$
35. $(x+4)(x-9).$	36. $(k-12)(k-3).$
37. $(a+18)(a+2).$	38. $(a+12)(a-3)$
39. $(k+9)(k-4).$	40. $(m-4)(m-9).$

41. $(a+4)(a+10)$.
43. $(a+20)(a-2)$.
45. $(k+2)(k-21)$.
47. $(x-16)(x-3)$.
49. $(x+4)(12-x)$.
51. $(x-2)(25-x)$.
53. $(a-2)(26-a)$.
55. $(a+4)(18-a)$.
57. $(x-6y)(x-7y)$.
59. $(a-8b)(a+7b)$.
61. $(a-8b)(a-7b)$.
63. $(a-8b)(a-4b)$.
65. $(p-16q)(p+3q)$.
67. $(x-15y)(x+9y)$.
69. $(x-36y)(x+5y)$.
71. $x+3)(x-3)(x^2+4)$.

42. $(a-8)(a+5)$.
44. $(a-40)(a-1)$.
46. $(k-7)(k+6)$.
48. $(x+8)(x-6)$.
50. $(x+16)(3-x)$
52. $(x+10)(5-x)$.
54. $(a+13)(4-a)$.
56. $(m+9)(12-m)$.
58. $(x+7y)(x-6y)$.
60. $(a+8b)(a+7b)$.
62. $(a+8b)(a-7b)$.
64. $(m-15n)(m+4n)$.
66. $(x+11y)(x+8y)$.
68. $(p-16q)(p+15q)$.
70. $(a+16b)(a-14b)$.
72. $(a+2)(a-2)(a^2+15)$.

73. $(a+2)(a^2-2a+4)(a^3+11)$.
74. $(p-3)(p^2+3p+9)(p+1)(p^3-p+1)$.
75. $(a^2+5)(a^2-5)(a^4+3)$.
76. $(a^4+2b^4)(a^2+9b^2)(a+3b)(a-3b)$.
77. $(m^4-15n^2)(m^4+13n^2)$.
78. $(p^6-15q^4)(p^6+14q^4)$.
79. $2x(x+10)(x-8)$.
80. $3x(x+5)(x-16)$.
81. $4a(a-8)(a-12)$.
82. $a^2(a-12)(a+15)$.
83. $2k^2(k+9l)(k+15l)$.
84. $3x^2(x-6y)(x-14y)$.
85. $x^2(x^2-124)(x^2+1)$.
86. $3z^2(z^2+15)(z+2)(z-2)$.
87. $(a+b-6)(a+b+4)$.
88. $(2a-b-6)(2a-b+5)$.
89. $(x+3y-6)(x+3y-15)$.
90. $(2x-y-3z)(2x-y+18z)$.
91. $(3-4a+14b)(3-4a-6b)$.
92. $(x-1)(x+2)(x-3)(x+4)$.
93. $(a-2)^2(a^2-4a+12)$.
94. $(x+2)(x+3)(x+7)(x-2)$.
95. $(x-2y)(x-3y)(x-6y)(x-8y)$.
96. $(a-5b)(a+2b)(a^2-3ab+5b^2)$.

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Exercise 53 (Page 208)

1. $(3x+2)(2x+3)$.
3. $(3x+1)(x+4)$.
5. $(3x+1)(2x+5)$.
7. $(3x-2)(x-4)$.

2. $(2x+1)(x+3)$
4. $(2x+5)(x+4)$.
6. $(2x+7)(x+2)$.
8. $(2x-5)(3x-2)$.

9. $(x-4)(8x-5)$.
10. $(3x-2)(2x-8)$.
11. $(4x-1)(3x-2)$.
12. $(3x-1)(5x-1)$.
13. $(4x+1)(2x-5)$.
14. $(x-4)(2x+5)$.
15. $(3x-1)(2x+7)$.
16. $(4x+3)(2x-5)$.
17. $(5x-2)(x+7)$.
18. $(3x-2)(2x+5)$.
19. $(4-3m)(4m+1)$.
20. $(3-4k)\ 3k+4)$.
21. $(6a+1)(2-5a)$.
22. $(1-7a)(4a+3)$.
23. $(3a+4b)(5a-2b)$.
24. $(4a+b)(5a-3b)$.
25. $(5a-4b)(2a+3b)$.
26. $(6x-5y)(2x+y)$.
27. $(7x+y)(3x-4y)$.
28. $(5x-2y)(4x-3y)$.
29. $(4x-5y)(3x+2y)$.
30. $(5x-y)(5x-2y)$.
31. $(x-25y)(2x+y)$.
32. $(8a-b)(a-8b)$.
33. $2x(5x-1)(x+3)$.
34. $3a(a-4)(3a-5)$.
35. $a^2(2a-3)(3a+4)$.
36. $2a^2(3a-4)(2a+3)$.
37. $6xy(4x-5y)(x-3y)$.
38. $5a^3(a-3b)(2a+3b)$.
39. $(3x+y)(x+3y)(3x-y)(x-3y)$.
40. $(2x^3+3y^2)(3x+y)(3x-y)$.
41. $(m^2n^3-2p^2)(3m^2n^2-4p^2)$.
42. $\left(\dfrac{x^2}{2}+5\right)\left(\dfrac{x^3}{2}-1\right)$.
43. $(x+1)(x^2-x+1)(7x^3-8)$.
44. $(x-2)(2x-1)(x^3+2x+4)(4x^2+2x+1)$.
45. $(x+3)(2x-1)(x^2-3x+9)(4x^2+2x+1)$.
46. $(2a^3+b^2)(2a^2-b^2)(a^2+2b^2)(a^2-2b^2)$.
47. $(x+1)(5x+3)(5x^3+8x-5)$.
48. $(x+5)(3x-1)(3x^2+14x-8)$.
49. $(2x-3)(3x+4)(x-1)(6x+5)$.
50. $(x-1)(x-3)(x+2)(x+4)$.
51. $(a^3+a-21)(a^2+a-5)$
52. $(a+4)^2(a^2+8a+6)$.
53. $(a-5)(a+1)(a-2)^3$.
54. $(2a^3+2b^2+ab)(a+b)^3$.
55. $x^2(7x+18)(2x+3)$.

Exercise 54 (Page 213)

1. $(2a-b+3c)(2a-b-3c)$.
2. $(x+3y+4z)(x+3y-4z)$.
3. $(4x+5y-1)(4x-5y-1)$.
4. $(x^2+2x+2y^2)(x^2-2x+2y^3)$.
5. $(5a+8b-1)(5a-8b+1)$.
6. $(8y+4x-3)(3y-4x+3)$.
7. $(x+y-6z-5)(x-y+6z-5)$.
8. $(2a^3+b-2c-3)(2a^3-b+2c-3)$.
9. $(x+y+z-1)(x+y-z-1)$.

10. $(2a-b-c+2)(2a-b-c-2)$.
11. $(2c+a-b-1)(2c-a+b+1)$.
12. $(4+2x+y-z)(4-2x-y+z)$.
13. $(a^2+2a+2)(a^2-2a+2)$.
14. $(a^2+4a+8)(a^2-4a+8)$.
15. $(a^2+6a+18)(a^2-6a+18)$.
16. $(2x^2+6xy+9y^2)(2x^2-6xy+9y^2)$.
17. $(8a^2b^2+12ab+9)(8a^2b^2-12ab+9)$.
18. $(2a^4+10a^2b+25b^2)(2a^4-10a^2b+25b^2)$.
19. $4x(x^2+2xy+2y^2)(x^2-2xy+2y^2)$.
20. $a^2(8a^2+4a+1)(8a^2-4a+1)$.
21. $18x(2x^2+2xy+y^2)(2x^2-2xy+y^2)$.
22. $3y^2(9y^2+6yz+2z^2)(9y^2-6yz+2z^2)$.
23. $(3x^2+2x-2)(3x^2-2x-2)$.
24. $(2x^2+3x-5)(2x^2-3x-5)$.
25. $(3x^2+3x-4)(3x^2-3x-4)$.
26. $(4a^2+4ab-b^2)(4a^2-4ab-b^2)$.
27. $(10p^2+9pq-3q^2)(10p^2-9pq-3q^2)$.
28. $(6a^2+9ab-7b^2)(6a^2-9ab-7b^2)$.
29. $ab(a^2+ab-b^2)(a^2-ab-b^2)$.
30. $2p^3(p^2+3pq-q^2)(p^2-3pq-q^2)$.
31. $(x^2+x+1)(x^2-x+1)$.   32. $(x^2+xy+y^2)(x^2-xy+y^2)$.
33. $(2a^2+2a+3)(2a^2-2a+3)$.   34. $(4a^2+a+1)(4a^2-a+1)$.
35. $(x^2+2xy+3y^2)(x^2-2xy+3y^2)$
36. $(5x^2+3xy+y^2)(5x^2-3xy+y^2)$.
37. $(2a^2+4ab+5b^2)(2a^2-4ab+5b^2)$.
38. $(5a^2+2ab+3b^2)(5a^2-2ab+3b^2)$.
39. $(a^2+5a+3)(a^2-5a+3)$.   40. $(3a^2+5a+4)(3a^2-5a+4)$.
41. $(5x^2+6xy+3y^2)(5x^2-6xy+3y^2)$.
42. $(4x^2+7xy+5y^2)(4x^2-7xy+5y^2)$.
43. $(2a+1)(2a-1)(3a+1)(3a-1)$.
44. $(a+1)(a-1)(3a+1)(3a-1)$.
45. $(x+3y)(x-3y)(2x+1)(2x-1)$.
46. $(a+b)(a-b)(5a+2b)(5a-2b)$.
47. $(x^2+x+1)(x^2-x+1)(x^4-x^2+1)$.
48. $(a^2+ab+b^2)(a^2-ab+b^2)(a^4-a^2b^2+b^4)$
49. $(a^2+3a+1)(a^2-3a+1)(a^4+7a^2+1)$.
50. $(a^2+2ab+3b^2)(a^2-2ab+3b^2)(a^4-2a^2b^2+9b^4)$.
51. $(x-z)(x+z-4y)$.   52. $(3x-4y)(3x+4y-2)$.
53. $(a+c)(a-c-2b)$.   54. $(2a-3c)(2a+3c-5b)$.
55. $(x^2+2y^2+2xy+1)(x^2+2y^2-2xy+1)$.

56. $(2x^2+2x-y+1)(2x^2-2x-y+1)$.
57. $(a+3b-c)(a+b+c)$ 58. $(2a-b-3)(2a-5b+3)$.
59. $(x+4y+3)(x-6y-3)$.   60. $(2x+y-5)(2x-7y+5)$.

Exerciee 55 (Page 217)

1. $(b+c)(a+d)$.
2. $(a+b)(x+y)$.
3. $(x-4)(a+b)$.
4. $(a+b)(b-c)$.
5. $(x^2+3)(x-1)$.
6. $(x-a)(x^2-3a)$.
7. $(3b-2)(2a-3)$.
8. $(c-1)(ac+by)$.
9. $(a+2b)(a-c)$.
10. $(a^2-3)(2a-1)$.
11. $(p+1)(q+1)$.
12. $(5a-2)(a^2+1)$.
13. $(a+1)(a-1)(2a+3)$.
14. $(2x+3)(2x-3)(x-5)$.
15. $(a-1)(a^2+a+1)(2a+1)$.
16. $(x+y)^2(x^2-xy+y^2)$.
17. $(mn+1)(ln+m)$.
18. $(bx-ay)(ax-by)$.
19. $(ab-xy)(ax-by)$.
20. $(p^2+q^3)(r^3+s^2)$.
21. $(x^2+y^2)(a^2+b^2)$.
22. $(a+b)(a-b)(x+y)(x-y)$
23. $(a+2b)(a+2b+3)$.
24. $(2a-3b)(2a-8b+4)$.
25. $(3a-1)(3a+1-2b)$.
26. $(a+3)(a-5+3b)$.
27. $(2x-1)(2b-4x^2-2x-1)$.
28. $(2x+3)(a-3x+2)$.
29. $(2x-y)(4x^2-4xy+y^2-5)$.
30. $(8x+y)(3x+y+4)$.
31. $(4x-5y)(4x-5y+3)$.
32. $(3x-4y)(3x+4y-5)$.
33. $(x-4)(x-y+6)$.
34. $(2a-5b)(1-4a^2-10ab-25b^2)$
35. $(3a-y)(2-2a-5y)$.
36. $(a-2b)(a^2+4b^2-4ab-3)$.
37. $(2x+3y)(2x+3y+5)$.
38. $(a+1)(a-1)(a^2-1+b)$.
39. $(a+b)(a-b)(a^2+b^2-c)$.
40. $(m-1)(m+2n+12)$.
41. $(x+y)(2-x^2+xy-y^2)$.
42. $(2x-8y)(3-x-4y)$.
43. $(x+y)(x^2+2xy+y^2+4)$.
44. $(x+1)(x^3+x+1)$.
45. $(a-b)(a^2-ab+b^2)$.
46. $(a-1)(y^2-y+1)$.
47. $(p+q)(2a-b-c)$.
48. $(x+1)(x^2+x+1)(x^2-x+1)$.

Exercise 56 (Page 219)

1. $(x-y-z)(x^2+y^2+z^2+xy+xz-yz)$.
2. $(a-b-1)(a^2+b^2+1+ab+a-b)$.
3. $(a+b+1)(a^2+b^2+1-ab-a-b)$.
4. $(a+2b-1)(a^2+4b^2+1-2ab+a+2b)$.
5. $(a-3b+4c)(a^2+9b^2+16c^2+3ab-4ac+12bc)$.
6. $(5x-2y-1)(25x^2+4y^2+1+10xy+5x-2y)$.
7. $(3x+4y-5z)(9x^2+16y^2+25z^2-12xy+15xz+20yz)$.
8. $(1-6a-3b)(1+36a^2+9b^2+6a+3b-18ab)$.

9. $(x^2-3y^2+10)(x^4+64y^4+100+8x^2y^2-10x^2+80y^2)$.

10. $\left(x-\dfrac{1}{x}+y\right)\left(x^2+\dfrac{1}{x^2}+y^2+1-xy+\dfrac{y}{x}\right)$.

11. $\left(a-b+\dfrac{2}{a}\right)\left(a^2+b^2+\dfrac{4}{a^2}+ab-2+\dfrac{2b}{a}\right)$.

12. $\left(a-\dfrac{1}{a}+2\right)\left(a^2+\dfrac{1}{a^2}+5-2a+\dfrac{2}{a}\right)$.

13. $\left(x-\dfrac{1}{x}+1\right)\left(x^2+\dfrac{1}{x^2}+2+\dfrac{1}{x}-x\right)$.

14. $\left(a-\dfrac{1}{a}-3\right)\left(a^2+\dfrac{1}{a^2}+10+3a-\dfrac{3}{a}\right)$.

15. $\left(a-\dfrac{1}{a}-2\right)\left(a^2+\dfrac{1}{a^2}+5+2a-\dfrac{2}{a}\right)$.

16. $\left(2x+\dfrac{1}{x}+2\right)\left(4x^2+\dfrac{1}{x^2}+2-4x-\dfrac{2}{x}\right)$.

17. $3(x-2y)(2y-3z)(3z-x)$.

18. $3(2a-3b)(a+b)(2b-3a)$.

19. $3(a+2b-3c)(b+2c-3a)(c+2a-3b)$.

20. $3(a+b)(a-b)(b+c)(b-c)(c+a)(c-a)$.

21. $3xyz(x-y)(y-z)(z-x)$.

22. $2(x+y+z)(x^2+y^3+z^2-xy-xz-yz)$.

23. $2(2a+3)(a^2+3a+3)$.

24. $2(c-b)(3a^2+b^2+c^2-3ac-3ab+bc)$.

Exercise 57 (Page 222)

1. $(x^2+5x+7)(x^2+5x+3)$.
2. $(x^3+6x+6)(x^2+6x+7)$.
3. $(x^2+8x+6)(x+4)^2$
4. $(x^2+6x+2)(x^2+6x+3)$
5. $(x^2+5x+3)^2$.
6. $(x^2+7x+2)(x+2)(x+5)$.
7. $(x^2+2x-7)(x^2+2x-4)$.
8. $(x^2-3x-5)(x^2-3x-17)$.
9. $(x^2-3x-16)(x^2-3x-12)$.
10. $(a^2+3a-4)^2$
11. $(x^2-x-10)(x^2-x-16)$.
12. $(a^2+11a+3)(a+5)(a+6)$
13. $(a^2-5a-40)(a^2-5a-20)$.
14. $(x^2-10x+4)(x^2-10x+20)$
15. $(x^2-3x-14)(x-4)(x+1)$.
16. $8(x^2+4x+5)(2x^2+8x+1)$
17. $4(x-1)(2x-1)(2x^2-3x-3)$.
18. $(2x^2-7x+9)(2x^2-7x-8)$.
19. $(x-3)(2x+3)(2x^2-3x+7)$
20. $(x+1)(3x-7)(3x^2-4x+3)$
21. $(3x^2+x-1)(9x^2+3x+1)$.
22. $(x^2+x-3)(x^2+x+1)$.
23. $8(a+1)^2(2a^2+4a-3)$.
24. $(4a^2+6a-13)(4a^2+6a+5)$

25. $(a^3-11a+12)(a^2-4a+12.$ 26. $(a^2-8a-16)^2$
27. $(x-12)(x+1)(x^2+8x-12).$
28. $(a+1)(3a-2)(3x^2+3a-2).$

Exercise 58 (Page 227)

1. 2.   2. 2.   3. 55.   4. 142   5. 9.
6. 31.   7. Yes.   8. Yes.   9. Yes. 10. No.
11. Yes.   12. Yes.   13. Yes.   14. No.
23. $k=3$ 24. $a=-1.$
25. $a=-7$ 26. $a=6, b=11.$
27. $m=2, n=-11$ 28. $k=\frac{5}{2}, l=-1.$

Exercise 59 (Page 231)

1. $(x-1)(x^2-8x+10)$
2. $(a-1)(a^2-7a+18).$
3. $(x-1)(3x^2+11x+8)$
4. $(x-2)(x^2+2x+3).$
5. $(x-2)(x^2+3x+7)$
6. $(x-2)(2x^2+x+2)$
7. $(x+3)(x^2-3x+4)$
8. $(x-3)(x^3+3x+4)$
9. $(2x-1)(4x^2+2x+3)$
10. $(2x+1)(5x^2+2x-1)$
11. $(x+1)(x+2)(x+3).$
12. $(x+1)(x+4)(x+5).$
13. $(x-2)(x+3)(x-7)$
14. $(x+3)(x-2)(x+7)$
15. $(x-1)(x+2)(x+3).$
16. $(x+2)(x-3)(x+5).$
17. $(x+1)(x+3)(4x-3).$
18. $(x-1)(x+3)(2x+1).$
19. $(2a+b)(2a-b)(3a+b).$
20. $(x+y)(x+5y)(x-6y).$
21. $(x+2y)(x+3y)(x-5y).$
22. $(x+y)(x-2y)^2.$
23. $(3a+b)(3a^2-ab-b^2).$
24. $(2a-b)(2a^2+3ab+2b^2)$
25. $(x-2)(x-3)(x^2+x+1).$
26. $(a+3)(a-4)(a^2-5a+3).$
27. $(x+1)(x-2)(x+3)(x-4).$
28. $(a+1)(a-4)(a-3)(2a+1).$
29. $(a-1)(a+1)(a-2)(2a-3).$
30. $(a+1)(a-1)(a+2)(4a-1).$

Exercise 60 (Page 235)

1. $a+b+c$
2. $a^2+b^2+c^2.$
3. $ab+bc+ca$
4. $a^3b+b^3c+c^3a$
5. $-(a-b)(b-c)(c-a)$
6. $(a+b)(b+c)(c+a).$
7. $x^2(y+z)+y^2(z+x)+z^2(x+y)$
8. $x^2(y^3-z^3)+y^2(z^3-x^3)+z^2(x^3-y^3)$
9. $a^2b+b^2c+c^2a$
10. $ab^3-bc^3-ca^3$

11. $ab^2c + bc^2a + ca^2b.$

12. $a(b^2-c^2) + b(c^2-a^2) + c(a^2-b^2)$

13. $a^3(b-c) + b^3(c-a) + c^3(a-b).$

14. $\dfrac{ab}{a-b} + \dfrac{bc}{b-c} + \dfrac{ca}{c-u}.$

15. $a^2 + b^2 + c^2.$ 16. $bc + ca + a\jmath.$

17. $(a^2-bc) + (b^2-ca) + (c^2-ab)$

18. $a^3(b-c) + b^3(c-a) + c^3(a-b)$

19. $a^2b^3 + a\ c^3 + b^2a^3 + b^2c^v + c^2 - + c^2b^3.$

20. $a(b-c)(c-a) + b(c-a)(a-b) + c(a-b)(b-c).$

21. $\dfrac{a^2}{(a-b)(a-c)} + \dfrac{b^2}{(b-c)(b-a)} + \dfrac{c^2}{(c-a)(c-b)}.$

22. $\dfrac{a^3}{bc(b-c)} + \dfrac{b^3}{ca(c-a)} + \dfrac{c^3}{ab(a-b)}.$

Exercise 61 (Page 237)

1. $-(a-b)(b-c)(c-a).$ 2. $-(a-b)(b-c)(c-a).$

3. $(a-b)(b-c)(c-a).$

4. $-(a-b)(b-c)(c-a)(a+b+c)$

5. $(a-b)(b-c)(c-a)(a+b+c)$

6. $3(a-b(b-c)(c-a).$

7. $-(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca).$

8. $-(a-b)(b-c)(c-a)(ab+bc+ca).$

9. $-(x-y)(y-z)(z-x)(x^2+y^2+y^2+xy+yz+zx).$

10. $(x-y)(y-z)(z-x)(xy-yz+zx).$

11. $(a+b)(b+c)(c+a).$ 12. $(x+y)(y+z)(z+x).$

13. $(a+b)(b+c)(c+a).$ 14. $(a+b)(b+c)(c+a).$

15. $(a-b)(b+c)(c+a).$ 16. $-(a-b)(b-c)(c-a).$

17. (i) $(a+b+c)(ab+bc+ca)$ (ii) same (iii) same.

18. $(a+b+c)(ab+bc+ca).$ 19. $(x+y+z)(x^2+y^2+z^2).$

20. $(a+b+c)(a^2+b^2+c^2).$

21. $-(a+b)(a-b)(b+c)(b-c)(c+a)(c-a).$

22. $2(a-b)(b-c)(c-a)(a+b+c)$

Exercise 62 (Page 241)

1. $a.$ 2. $bc.$ 3. $x^2.$ 4. $xy^2.$ 5 $a^2b^2.$

6. $2ab^3.$ 7. $3abc.$ 8. $4x^2yz.$ 9. $5x^2y^2z^2$

10. $7a^3b^4c^5$.   11 $6a^2b^3$ 12. xy^2z.   13. $5x^3y^3z^2$.
14. $25ab$.

Exercise 63 (Page 242)

1. b.   2. $b(a-b)$.   3. $ab(a^2+b^2)$.   4. $a(a+b)$
5. $a(a-b)$.   6. $a(a-b)^2$.   7. $(x+y)^2$.   8. $5y^2$.
9. 6.   10 1.   11. b.   12. $6a^2$.   13. $a+b$.
14. $2(x+1)$.   15. $a(a-b)$.   16. $2a-3b$.   17. 1.
18. $3ab(a-b)$.   19. $2(a-3b)$.   20. $(x-y)^3$.
21. $2(x^2+a^2)$.   22. $a+2b$.   23. $2(a+2)$.   24. $a-2$.
25. $a-9$.   26. $a-3$.   27. $3a+1$.   2. $4xy(x-y)$.
29. $2a(a^2-b^2)$.   30. x^2-xy+y^2.   31. $4(c^3-y^2)$.
32. a^2+2.   33. $a(a-x)$.   34 $a-b$.   35 $5(x^2-x+1)$.
36. $a^2(3a+2)$.

Exercise 64 (Page 250)

1. $x-1$.   2. $x-3$.   3. $a-1$.   4. x^2-3x+2.
5. x^2-3x+7.   6. x^2+3x+2.   7. $x+2$.
8. $x^2-13x+5$.   9. $x-3$.   10. x^2+2x+1.
11. $2x^2-3x+2$.   12. $x+2$.   13. $3a-7$.
14. x^3+x-1.   15. x^2-3x-2.   15. $2x^2+3x+2$.
17. $5x^2-1$.   18. $3x^2+1$.   19. x^2-3x-1.
20. x^2-3x-4.   21. $a(2a-3)$.   22. $a(a-3)$.
23. $x-2$.   24. x^2+3x+5.   25. $x^2 2x-5)$.   26. $x-y$.
27. $2xy(x-y)$.   28. $2x-y$.   29. $2x-3y$.   30. $x+y$.
31. $x-1$.   32. a^2-3a+7 33. a^2+2a+3.
34. a^2-a-1.   35. a^2+a+1.   36. $x(x^2-2x+1)$.
37   $x+4$ 38. $x-1$.   39. $2x^2-3$.   40. $2x^2+3x+7$.
41   -3.   42. 4.

Exercise 65 (Page 254)

1. a^2b.   2 a^2b^3.   3 a^4b^4.   4. a^3b^3 5 $6a^2b^4$
6. $12a^6b^6$.   7. $24abcd$.   8. $12abc$.   9 $60xyz$.
10. $12a^2b^2c^2$.   11 $a^6b^3c^5$.   12. $56x^4y^5$ 13. $210x^3y^3z^3$.
14. $264x^3y^3z^4$

Exercise 66 (Page 255)

1. $2(a+b)^2$.   2. $6(a-b)^2$.   3. $a^3(a+b)(a-b)$.
4. $12a^3b(a+b,^2$.   5. $24abc(a-b)(b-c)$.
6. $60a^2b^2(a^2+b^2)(a+b)$.   7. a^5+a^4.   8. $36b^4+12b^3$
9. $4(a^3+9a^2+20a)$.   10. $4(a+b)(a-b)^2$.
11. $6xy(x^3+y^3)$ 12. $6(x+y)(x^3-y^3)$.
13. $24a^2bc(a-b)(a^3+b^3)$
14. $2x(x+2)(x+3)(x+12)$.
15. $x(x+1)(x-1)(x-2)$.   16. $x(x-1)(x-2)(x-4)$.
17. $2x(x-3)(x-1)(x+2)$.   18. $4x(x+5)(x-4)(x-6)$.
19. $6(a+1)(a+2)(2a+1)$.   20. $2a(a+2)(a-2))3a-7)$.
21. $60xy(x+y)^2(x-y)^2(x^2+xy+y^2)$.
22. $24a^2(a+3)(a-3)(a^2-3a+9)(a^2+3a+9)$.
23. $12x(x-a)(x+a)^2(x^2+ax+a^2)(x^2-ax+a^2)^2$.
24. $(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$.

Exercise 67 (Page 257)

1. $(3x-4)(3x^3+14x^2+13x-8)$.
   or $(3x+8)(3x^3+2x^2-11x+4)$.
2. $(x-1)(4x^3-9x^2-15x+18)$.
3. $4(a+4)(a^3+5a^2+7a+2)$.   4. $(a-1)(a^3-9a^2+26a-24)$.
5. $3(a-3)(a^3+8a^2+17a+10)$.
6. $2(x+7)(2x^3-9x^2+10x-3)$ 7. $(a-2)(a^3-12a^2+47a-60.)$
8. $(3a^2+a-2)(6a^3+a^2-5a-2)$.
9. $(2a-3)(12a^3-35a^2-23a+60)$.
10. $(8a-8)(a^4-5a^3-6a^2+35a-7)$.
11. $2a^6+6a^5+13a^4-42a^3-162a^2-381a-234$.
12. $2(a^2-2a+5)(3a^4-34a^2+51a-20)$.
13. $(4x+3)(6x^4-5x^2-6)$.   14. $(x^2-x-3)(4x^3-29x^2+25)$.
15. $(x+y)^2 x^4+x^2y^2+y^4)$.
16. $(5x^2+3xy+y^2)(64x^4-3xy^3+5y^4)$.
17. $(x+1)(x-1)(x+2)(x-2)$. 18. $a(a+1)(a+2)(a+3)(a-1)$.
19. $a(a^2+5a+10)(a^3-19a-30)$.
20. $(8a-b)(a+b)^2(a-b)^2$.   21. $a^4-10a^3+35a^2-50a+24$.
22. $a^2-12a+35$.   23. $15a^3-a^2-5a-1$.

Paper 1.—Ex. 68 (Page 260)

1. $x=5$.   2. 84.   3. 28.

4. $(a^2+4a+8)(a^2-4a+8)$; $(a^2+8)(a^2-8)$.   5. x^2+y.
6. $(a-3)(a^3+a^3-3a+9)$

Paper 2—Ex. 69 (Page 260)

1. $x=3, y=4$.
2. $2\frac{1}{2}$ miles.
4. $(a+1)(a-1)(a^2+a+1)(a^2-a+1)(a^2+1)(a^4-a^2+1)$.
5. $x(x+2y)$.
6. $(x-2y)(x^3-15x^2y+48xy^2+64y^3)$.

Paper 3—Ex. 70 (Page 261)

1. 900.
2. Rs. 2400 ; Rs. 4000.
3. $x^{12}-y^{12}$.
4. $x^2(3x-4y)(2x+3y)$.
5. $a-b-c$
6. $(x-4)(x^4-2x^3-3x^2+8x-4)$

Paper 4—Ex. 71 (Page 261)

1. $x=6, y=8, z=5$.
2. 14 as ; 10 as.
3. $2x^2y^2+2y^2z^2+2z^2x^2-x^4-y^4-z^4$.
4. $(x-1)(x-2)(x^2+x+1)(x^2+2x+4)$.
5. $a^2(3a+2)$.
6. $(a-2b)(a^4+a^2b^2+b^4)$.

Paper 5—Ex. 72 (Page 262)

1. $x=5$.
2. 38, 42, 20, 80.
3. 0.
4. (i) $(3a^2+2a+1)(3a^2-2a+1)$.
   (ii) $(a^2+3a+4)(a^2+3a-12)$.
5. a^6-b^6.
6. $4a^2-3a+1$.

Paper 6—Ex. 73 (Page 262)

1. $x=\frac{1}{3}, y=-\frac{1}{3}$.
2. $\frac{7}{8}$.
3. $27x^3+27x^2y+9xy^2+y^3$.
4. (i) $(3a-4b)(3a-4b-2)$;   (ii) $-(a-b)(b-c)(c-a)$.
5. $x^3y(x-y)^2(x^6-y^6)$.
6. $(3a-2)(2a^4-2a^3+3a^2-a+1)$.

Paper 7—Ex. 74 (Page 263)

1. $x=-2\frac{1}{2}$
2. 40 years ; 80 years.
3. $3(x-y)(y-z)(z-x)$,
4. $a=-8, b=-5$.
5. $a^2(a+1)(a-2)(2a+1)$.
6. 2.

Paper 8—Ex. 75 (Page 263)

1. $x=1, y=3, z=5.$ 2. A Rs. 200, B Rs. 160.   3. 50.
4. (i) $(x-2y-1)(x^2+4y^2+1+2xy+x-2y).$
   (ii) $(x+2)(x+3)(x-4).$
5. $24a^4b^2(a-b)^3(a+b)(a^2+ab+b^2).$
6. $x^3-10x^2+28x-15.$

———

Exercise 76 (Page 274)

1. $\dfrac{2}{3c}.$

2. $\dfrac{y}{2z}.$

3. $\dfrac{4y}{3}.$

4. $\dfrac{3ac}{5}.$

5. $-\dfrac{2a^2}{3b^3}.$

6. $-\dfrac{3x^2}{4z^2}.$

7. $\dfrac{4n}{5mp}.$

8. $-\dfrac{c}{a^2b}.$

9. $-\dfrac{yz^3}{2x}.$

10. $\dfrac{nq}{mp^3}.$

11. $\dfrac{3x^2}{5ay^4}.$

12. $-\dfrac{2p^2m^2}{3k}.$

13. $-\dfrac{2cd^2}{3b}.$

14. $-\dfrac{8a^3}{bc^2}.$

15. $\dfrac{9ax^3z^2}{bcy}.$

16. $\dfrac{7b^4}{5c^3y}.$

17. $\dfrac{x}{2a^2}.$

18. $-\dfrac{y^3z^2}{nx^4}.$

19. $-\dfrac{7b}{4a}.$

20. $y^2.$

21. $\dfrac{a-x}{x}.$

22. $\dfrac{b-c}{a}.$

23. $-\dfrac{a-1}{a}$ or $\dfrac{1-a}{a}.$

24. $\dfrac{b+c}{b}.$

25. $\dfrac{x+y}{x^2+xy+y^2}.$

26. $\dfrac{x^2-xy+y^2}{x}.$

27. $a+b.$

28. $\dfrac{1}{x-y}.$

29. $-(a^2+b^2).$

30. $-\dfrac{1+a}{1+a+a^2}.$

31. $\dfrac{x^3+x^2+x+1}{x^2+x+1}.$

32. $\dfrac{a^2+1}{a^4+a^2+1}.$

33. $x^2+y^2.$

34. $\dfrac{1}{a^2-b^2}.$

35. $\dfrac{a+3}{a+2}.$

36. $\dfrac{2a-3}{a-4}.$

37. $x^2-2x+2.$

38. $\dfrac{a^2-4a+3}{a}.$

39. $\dfrac{x-y+z}{y+z-x}.$

40. $\dfrac{a^2+1}{a^2+3}.$

41. $\dfrac{x+1}{x-5}$.

42. $\dfrac{a-8}{x+2}$.

43. $\dfrac{5a+2}{7a-4}$.

$\dfrac{3a^2-a-2}{3a^2+a-2}$.

45. $\dfrac{4a-b}{8a^2+b^2}$.

46. $\dfrac{x+2y}{x--2y}$.

47. $\dfrac{a^2-3a+3}{a^2-3a+2}$.

48. $\dfrac{5x^2+3xy+y^2}{16x^2+12xy+5y^2}$.

49. $\dfrac{a^2+ab+b^2}{a}$.

50. $-4(1+a+a^2)$. **51.** 1.

52. $\dfrac{x+2}{x-1}$.

53. 1.

54. $\dfrac{2a^2-a}{2a-8}$.

55. $\dfrac{a+1}{a+5}$.

56. 1.

57. x.

58. $\dfrac{x-5}{x-1}$.

59. $\frac{1}{2}$.

60. xy.

61 $\dfrac{2+a}{a}$.

62. 1.

63. $\dfrac{25x}{12}$.

64. $-\dfrac{17a}{16}$.

65. 0.

66. $-\dfrac{x}{6}$.

67. $\dfrac{5}{3x}$.

68. $-\dfrac{61}{24a}$.

69. $\dfrac{a}{12x}$.

70. $\dfrac{a+b+c}{abc}$.

71. $\dfrac{a^2+b^2+c^2}{abc}$.

72. $\dfrac{6x^2+4y^2+3z^2}{12xyz}$.

73. $\dfrac{2(a-1)}{5}$.

74. $-\dfrac{13(a-2)}{12}$.

75. $-\dfrac{17a}{36}$.

76. $\dfrac{6x-6y-z}{4}$.

77. $\dfrac{x^2+3y^2}{2xy}$.

78. 0.

79. $\dfrac{3b+2c}{bc}$.

80. $\dfrac{x^3+y^3+z^3-3xyz}{xyz}$.

81. $\dfrac{a^3+b^3}{a^2b^2}$.

82. $\dfrac{x+y^2-xy}{x^3y^3}$.

83. $\dfrac{1}{(a-4)(a-5)}$.

84. $\dfrac{2a+5}{(a+2)(a+8)}$.

85. $\dfrac{2(a+6)}{(a-6)(a+2)}$.

86. $\dfrac{2bc}{(a+b)(a-b)}$.

87. $\dfrac{1}{a-b}$.

88. $\dfrac{xy}{x^2-y^2}$.

89. $\dfrac{5a+9}{a^2-9}$.

90. $\dfrac{8xy}{x^2-4y^2}$.

91. $\dfrac{2ab}{a^2-b^2}$.

92. $\dfrac{2a^3}{1-a^4}.$

93. $\dfrac{4a}{a+x}.$

94. $\dfrac{2x^2+xy+y^2}{y^2-x^3}$

95. $\dfrac{y}{x-y}.$

96. $\dfrac{a}{2(a+y)}.$

97. $\dfrac{a^2}{a^3-1}.$

98. $-\dfrac{a^2+2a}{a^3+1}.$

99. $\dfrac{2y^3}{x^4+x^2y^2+y^4}.$

100. $\dfrac{2b^3}{a^2-b^2}.$

101. $\dfrac{x}{1+x}.$

102. $2(1+x^2).$

103. $\dfrac{2}{1-x^2}.$

104. $-\dfrac{2ab}{a^4+a^2b^2+b^4}.$

105. $\dfrac{2}{(a-1)(a-2)(a-3)(a-4)}.$

106. $\dfrac{a}{(a-1)(a-2)(a-3)}$

107. $0.$

108. $0.$

109. $\dfrac{2a^2}{a^3+b^3}.$

110. $\dfrac{2b^2}{a^3-b^3}$

111. $\dfrac{2(a+1)}{a^2+a+1}.$

112. $\dfrac{8b^7}{a^8-b^8}.$

113. $\dfrac{2a^2b^2}{a^4-b^4}.$

114. $\dfrac{8x^4}{1-x^8}.$

115. $\dfrac{16}{1-a^{16}}.$

116. $\dfrac{4x^3}{x^8+x^4+1}.$

117. $\dfrac{a+2}{(a+1)(a+3)}.$

118. $\dfrac{52-x}{(x+4)(x-4)(x-3)}.$

119. $\dfrac{2}{(a-1)(a+1)^2}.$

120. $\dfrac{48y^3}{(x^2-y^2)(x^2-9y^2)}.$

121. $\dfrac{24y^3}{x(x^2-y^2)(x^2-4y^2)}.$

122. $0.$

123. $\dfrac{2x^2}{x^2-a^2}$

124. $\dfrac{y(x+y)}{z^2-y^2}.$

125. $\dfrac{4y^3}{x^4-y^4}.$

126. $0.$

Exercise 77 (Page 276)

1. $0.$

2. $0.$

3. $\dfrac{bc+ca+ab-a^2-b^2-c^2}{(a-b)(b-c)(c-a)}.$

4. $0.$

5. $\dfrac{-2(x^2+y^2+z^2-xy-yz-zx)}{(x-y)(y-z)(z-x)}.$

6. $0.$

7. $0.$

8. $1.$

9. $1.$

10. $a+b+c.$

11. $x^2+y^2+z^2+xy+yz+zx.$

12. $-1.$

13. $-(a+b+c).$

14. $\dfrac{1}{abc}.$

15. $0.$

16. $\dfrac{1}{abc}$.

17. 2.

18. 0.

19. 1

20. $\dfrac{1}{(x-a)(x-b)(x-c)}$.

21. $\dfrac{x}{(x-a)(x-b)(x-c)}$.

22. $\dfrac{x^2}{(x+a)(x+b)(x+c)}$.

23. $\dfrac{1}{abc}$.

Exercise 78 (Page 279)

1. $\dfrac{ay-bx}{ay+bx}$.

2. $\dfrac{a^3-b^2}{a^2+b^2}$.

3. $\dfrac{1}{x+1}$.

4. $\dfrac{a+b}{b-a}$.

5. $\dfrac{a+c}{b-a}$.

6. $\dfrac{z}{xz+y}$.

7. $\dfrac{a^2-b^2}{a^2+b^2}$.

8. $\dfrac{b^2}{b^2-a^2}$.

9. $\dfrac{a(a+3)}{a+4}$.

10. $\dfrac{2(x-3)}{x+6}$.

11. $\dfrac{-2(a+1)}{a(a+3)}$.

12. $-\dfrac{a^2(2a+3)}{a+2}$.

13. $\dfrac{2(x+y)}{(x-y)}$.

14. $\dfrac{xy}{x^2+y^3}$.

15. $\dfrac{2xy}{x^2+y^2}$.

16. $\dfrac{1}{x^2+y^2}$.

17. $2a^3-a^2$.

18. 1.

19. $\dfrac{b}{a}$.

20. $a-b$.

21. $x+1$.

22. $\dfrac{x^3-1}{x^3-2x}$.

23. 1.

24. $\dfrac{x^2-3x+1}{x^2-4x+1}$.

25. -2.

26. $\dfrac{1}{x}$.

27. $\dfrac{2x^2+y^2}{x^2}$.

28. $\dfrac{1}{a+b}$.

29. 1.

30. 1.

Exercise 79 (Page 284)

1. $9a^2b^6$.

2. $16a^6b^2$.

3. $25a^6b^4$.

4. $36a^{10}b^8$.

5. $\frac{1}{9}x^4y^6$.

6. $\frac{9}{16}x^{12}y^{10}$.

7. $\dfrac{9x^2y^4z^{10}}{25a^8}$.

8. $\dfrac{25a^{20}b^{14}}{49c^{30}d^{40}}$.

9. $8x^3y^6$.

10. $-27x^6y^9$.

11. $64x^{12}y^{15}$.

12. $-\frac{1}{125}x^6y^{15}z^{21}$.

13. $-\dfrac{216a^{12}b^{18}}{343c^9d^{21}}$
14. $\dfrac{1000a^{30}b^{45}c^{60}}{27x^9y^{36}}$
15. $81x^8y^{12}$

16. $-1024a^{10}b^{25}$
17. $\dfrac{1}{64a^{18}b^{30}}$
18. $-\dfrac{2187a^{35}}{b^{42}}$

19. $-\dfrac{256x^{24}}{y^{32}}$
20. $\dfrac{x^{10}}{y^{50}z^{60}}$

Exercise 80 (Page 288)

1. $32a^5-240a^4b+720a^3b^2-1080a^2b^3+810ab^4-243b^5.$
2. $a^6+5a^4+10a^3+10a^2+5a+1.$
3. $1-6a+15a^2-20a^3+15a^4-6a^5+a^6.$
4. $32x^5-80x^4+80x^3-40x^2+10x-1.$
5. $1-18m+135m^2-540m^3+1215m^4-1458m^5+729m^6.$
6. $1-10a+45a^2-120a^3+210a^4-252a^5+210a^6-120a^7$
$$+45a^8-10a^9+a^{10}.$$
7. $243x^5-810x^4y+1080x^3y^2-720x^2y^3+240xy^4-32y^5.$
8. $729a^6+2916a^5+4860a^4+4320a^3+2160a^2+576a+64.$

9. $x^5+5x^3+10x+\dfrac{10}{x}+\dfrac{5}{x^3}+\dfrac{1}{x^5}.$

10. $a^7-7a^5+21a^3-35a+\dfrac{35}{a}-\dfrac{21}{a^3}+\dfrac{7}{a^5}-\dfrac{1}{a^7}.$

11. $8a^3+8a.$
12. $2a^5+20a^3+10a.$
13. $12p^5q+40p^3q^3+12q^5.$
14. $864x^3y+1536xy^3.$
15. $-18.$
16. $-67.$
17. $10.$
18. $25.$

Exercise 81 (Page 290)

1. $+3xy^3$ or $-3xy^3$
2. $\pm 2x^2y.$
3. $\pm 4a^4b^5.$
4. $\pm 5a^2bc^3.$
5. $\pm\dfrac{6}{x^{18}}.$
6. $\pm\dfrac{x^6b^3}{5y^7}.$
7. Impossible.
8. Impossible
9. $-2a^3b^5.$
10. $3a^2bc^3.$
11. $-\dfrac{x^4y^6}{5}.$
12. $-\dfrac{2y^8}{9z^{10}}.$
13. $\pm 2a^2b^3.$
14. $-3a^2b^3.$
15. $\pm 2xy^3z^4.$
16. Impossible.
17. $\dfrac{2}{x^8y^9}$
18. $-\dfrac{z^4}{y^3z^4}.$

Exercise 82 (Page 294)

1. $2x+8y$.

2. $x+2y$.

3. $a-5b$.

4. $9a-b$.

5. x^2+4y^2.

6. a^3-3.

7. $\dfrac{x}{y}-2$.

8. $\dfrac{x}{2}-3$.

9. $x-\dfrac{1}{x}$.

10. $\dfrac{x}{y}+\dfrac{y}{2x}$.

11. $\dfrac{x}{2y}+2$.

12. $\dfrac{2x^3}{y^2}-\dfrac{y^3}{4x^2}$.

13. $2a+8b-4c$.

14. $x-y-z$.

15. $x-2y+3z$.

16. $x+\dfrac{y}{4}-2$.

17. $3-\dfrac{b}{8}-c$.

18. $(x-3)(x+2)(x+4)$.

19. $(x+3)(x+2)(x-5)$.

20. $(2x+1)(3x-2)(2x-1)$.

21. $(x-2y)(2x+y)(3x-y)$.

22. (i) $a^4+a^2b^2+b^4$ (ii) $(2x-8)(2x+1)$.

23. x^2+5x+5.

24. $x^2+10x+20$.

25. $x^2-3x-11$.

26. $2m^2-3m-1$.

27. $x+\dfrac{1}{x}-4$.

28. $x+\dfrac{1}{x}-3$.

29. $2\left(a+\dfrac{1}{a}\right)+3$.

30. $3\left(a+\dfrac{1}{a}\right)-1$.

31. $4\left(a-\dfrac{1}{a}\right)+1$.

32. $5\left(x-\dfrac{1}{x}\right)-3$.

33. $x-\dfrac{1}{x}-2$.

34. $a-\dfrac{1}{a}+3$.

35. $a+\dfrac{1}{a}-4$.

36. $x+\dfrac{1}{x}-5$.

37. $x^2+\dfrac{1}{x^2}+2$.

38. $x^2-\dfrac{1}{x^2}-3$.

39. $x^2+\dfrac{1}{x^2}-2$.

40. $x^2+\dfrac{1}{x^2}+4$.

41. $a^2+\dfrac{1}{a^2}+2$.

42. $m^2+\dfrac{1}{m^2}-6$.

43. $x^4-\dfrac{1}{x^4}-2$.

44. $x^4+\dfrac{1}{x^4}+5$.

45. $m^{12}+\dfrac{1}{m^{12}}-1$.

46. $2x^2-x+1$.

47. x^2+x+1.

48. $2a^2-3a+5$.

49. $3a^2-2a-1$.

50. $1-2a+a^2$.

51. $2-x-2x^2$.

52. x^3-2x-2.

53. $x^2-ax+2a^2$.

54. $5x^2-3ax+4a^2$.

55. $a^3-6a^2+12a-8$.

56. $m^3-11m+17$.

57. $x^3+2x^2y-2xy^2-y^3$.

58. $1-x+x^2-x^3+x^4$.

59. $a^3(2a^2+a-2)$.

60. $2x^3(1-5x+x^2)$.

61. $x^2-\frac{3}{2}x-\frac{1}{3}$.

62. $x^2-\frac{x}{2}-1$.

63. $a^3-\frac{3a}{4}+\frac{1}{3}$.

64. $a^3-\frac{2a}{3}+\frac{1}{7}$.

65. $\frac{x^2}{2}-2x+\frac{a}{3}$.

66. $\frac{a^2}{5}-\frac{ax}{3}+\frac{x^2}{2}$.

67. $x^2-\frac{1}{2}+\frac{2}{x}$.

68. $2x-\frac{3}{2}+\frac{1}{x}$.

69. $x^2+4+\frac{4}{x^2}$.

70. $2x^2+5-\frac{7}{x^2}$.

71. $2x^2+8+\frac{8}{x^2}$.

72. $x^3+3x+\frac{3}{x}+\frac{1}{x^3}$.

73. $a^3-2a+\frac{1}{a^3}$.

74. $2a+1-\frac{3}{a}-\frac{1}{a^2}$.

75. $\frac{x}{a}+3-\frac{a}{x}$.

76. $\frac{a}{b}-1+\frac{b}{a}$.

77. $\frac{3a}{b}-\frac{1}{5}+\frac{2b}{3a}$.

78. $\frac{x^2}{2}+\frac{x}{y}-\frac{y}{x}$.

79. $\frac{4y}{x^2}-4+\frac{x^2}{y}$.

80. $\sqrt{2}(x^2+xy+y^2)$.

81. $\sqrt{2}(a^2-ab+b^2)$.

82. $2a-1$.

83. $2x-3y$.

84. $x+1$.

85. $3x+8$.

86. $2x-3$.

87. $6x+8$.

88. 8.

89. -5.

90. 1.

91. $\frac{1}{3}$.

92. $a=2, b=8$.

Exercise 83 (a) (Page 313)

1. 81.

2. 64.

3. 32.

4. 625.

5. $2\frac{1}{4}3$.

6. $\frac{1}{64}$.

7. $\frac{1}{64}$.

8. $3\frac{1}{121}$.

9. 2.

10. 3.

11. 4.

12. 4.

13. $\frac{1}{2}$.

14. $\frac{1}{3}$.

15. $\frac{1}{4}$.

16. $\frac{1}{14}$.

17. 9.

18. 125.

19. 4.

20. 248.

21. $\frac{1}{9}$.

22. $\frac{1}{256}$.

23. $\frac{1}{8}$.

24. $\frac{1}{121}$.

25. $\frac{8}{125}$.

26. 1296

27. $\frac{5}{4}$.

28. $2\frac{7}{8}$.

29. $\frac{8}{a^{\frac{1}{4}}}$.

30. $\frac{4}{x^{\frac{1}{3}}}$.

31. $\frac{x^{\frac{1}{3}}}{5}$.

32. $\frac{-5a^3}{6}$.

33. $--yz$.

34. $\frac{8y t^3}{-z^2 x^3}$.

35. $\frac{5}{\sqrt[4]{x^3}}$.

36. $\frac{8}{\sqrt[5]{a^4}}$.

37. $\frac{-3a^7\sqrt{b^3}}{4}$.

38. $\frac{1}{a}$.

39. $\frac{-x^{\frac{2}{3}}}{2}$.

40. $\frac{1}{3a^{\frac{5}{2}}}$.

41. $\frac{-4}{5a^{\frac{1}{3}}}$.

42. $\frac{2.\sqrt{x}}{-2x^{\frac{a}{b}}}$.

Exercise 83 (b) (Page 315)

1. $\sqrt[4]{x^5}$.

2. $\sqrt{x^3}$.

3. $\frac{1}{\sqrt[5]{x}}$.

4. $\frac{1}{\sqrt[12]{a^5}}$.

5. $\frac{1}{\sqrt[9]{a^7}}$.

6. $\frac{1}{\sqrt[10]{b^7}}$.

7. $\frac{1}{\sqrt{x}}$.

8. x.

9. $\frac{1}{\sqrt{a}}$.

10. $\frac{2}{3}\sqrt{a^3}$.

11. x^{15}.

12. x^{12}.

13. $\sqrt{x^3}$.

14. x.

15. $\frac{1}{a^3}$.

16. $\frac{1}{\sqrt{a}}$.

17. $\frac{1}{\sqrt[5]{a}}$.

18. $\sqrt[3]{a^2}$.

19. $\frac{1}{\sqrt{x}}$.

20. $\frac{1}{\sqrt[5]{x}}$.

21. $\frac{1}{\sqrt[3]{x}}$.

22. $\frac{1}{\sqrt[4]{x^3}}$.

23. $\frac{1}{\sqrt[5]{a}}$.

24. $\frac{1}{a}$.

25. 8.

26. 320.

27. $\frac{1}{9}$.

28. 0.

Exercise 83 (c) (Page 317)

1. $\frac{a^{\frac{4}{3}}}{b}$.

2. $\frac{a^{\frac{1}{3}}}{y^{\frac{1}{4}}}$.

3. $\frac{1}{2a^{\frac{1}{3}}b^{\frac{1}{4}}}$.

4. $\frac{4}{9a^3b^2}$.

5. $\frac{4by}{3}$.

6. $a^{1\frac{1}{2}}$.

7. $\frac{1}{x^{\frac{1}{3}}y^{\frac{1}{4}}}$.

8. x^{b+1}.

9. $\dfrac{1}{a^{\frac{2}{3}}}$. 10. $\dfrac{1}{y^{\frac{1}{4}}}$. 11. $\dfrac{1}{y^{\frac{1}{3}}}$. 12. $\dfrac{1}{a^5}$

13. $\dfrac{x^2}{a^3}$. 14. $a^{\frac{1}{7}}b^{-\frac{8\cdot5}{6}}$. 15. 1 16. 1.

17. 1. 18. 1. 19. 1

20. $x^{a^3+b^3+c^3-3abc}$

21. $\cdot10$. 22. $\frac{9}{5}$

23. $\frac{2}{3}$. 24. 25. 25. 4. 26. 10.

27. 1. 28. $2\frac{1}{2}$. 29. 8. 30. $\frac{1}{4}$.

Exercise 83 (d) (Page 320)

1. $x-y$.

2. $x^{\frac{2}{3}}+y^{\frac{2}{3}}$

3. $a^{\frac{4}{3}}+a^{\frac{2}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}$

4. $x^{\frac{3}{2}}+y^{\frac{3}{2}}+z^{\frac{3}{2}}-3x^{\frac{1}{2}}y^{\frac{1}{2}}z^{\frac{1}{2}}$

5. $a^{\frac{2}{3}}-4a^{\frac{1}{3}}+3-6a^{-\frac{1}{6}}$

6. $a^{-1}-1$.

7. $a^2-3a^{\frac{2}{3}}+3a^{-\frac{2}{3}}-a^{-2}$

8. $9x^{\frac{4}{5}}-9x^{\frac{2}{5}}-25+18x^{-\frac{2}{5}}+6x^{-\frac{4}{5}}$.

9. $x^{\frac{1}{6}}-y^{-\frac{1}{6}}$.

10. $7x^{\frac{2}{3}}-2x^{\frac{1}{3}}+1$.

11. $x^{\frac{1}{2}}+x^{\frac{1}{3}}y^{\frac{1}{6}}+x^{\frac{1}{6}}y^{\frac{1}{3}}+y^{\frac{1}{2}}$

12. $16a^{-\frac{2}{3}}-12a^{-\frac{1}{3}}b^{-\frac{2}{3}}+9b^{-\frac{4}{3}}$.

13. $1-a^{-\frac{1}{4}}+a^{\frac{3}{4}}$.

14. $xy^{-1}+yx^{-1}+1$.

15. $a^{\frac{2}{3}}+b^{\frac{2}{3}}+c^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}-a^{\frac{1}{3}}c^{\frac{1}{3}}+b^{\frac{1}{3}}c^{\frac{1}{3}}$.

16. $x^6+3x^2+3x^{-2}+x^{-6}$.

17. $x^{\frac{2}{3}}-7+x^{-1}$.

18. $x^{\frac{5}{6}}-2x^{\frac{1}{2}}+x^{\frac{1}{3}}$.

19. $a^{\frac{2}{3}}-a^{-\frac{1}{6}}-1$

20. $a-2-a^{-1}$

21. $a^{\frac{1}{2}}-1-a^{-\frac{1}{2}}$.

Exercise 83 (e) (Page 323)

1. $\frac{1}{4}x^{\frac{2}{3}}-x^{\frac{1}{3}}+x^{-\frac{2}{3}}$

2. $4x+2+\frac{1}{4}x^{-1}$.

3. $x^{2m}+2x^{m+n}+x^{2n}$.

4. $x^n-n+n^2x^{-n}$

5. $9a-12a^{\frac{1}{2}}-2+4a^{-\frac{1}{2}}+a^{-1}$.

6. $a^{\frac{2}{3}}-2a^{\frac{1}{3}}+3-2a^{-\frac{1}{3}}+a^{-\frac{2}{3}}$.

7. $x^2-2x^{\frac{3}{2}}y^{-\frac{1}{4}}+xy^{-\frac{1}{2}}+2x^{\frac{3}{2}}y^{-\frac{1}{2}}-2x^{\frac{1}{2}}y^{-\frac{3}{4}}+y^{-1}$.

8. $x^{2a}-x^{a}-\frac{7}{4}+x^{-a}+x^{-2a}$.   9. $x+9x^{\frac{1}{3}}+27x^{-\frac{1}{3}}+27x^{-1}$.

10. $x^2-3x^{\frac{2}{3}}+3x^{-\frac{2}{3}}-x^{-2}$ 11. $a^{3m}-3a^{m}+3a^{-m}-a^{-3m}$.

12. $a^x+a^{\frac{2a}{3}}+\frac{1}{3}a^{-\frac{x}{3}}+2\frac{1}{7}$.   13. $x-y$.

14. $x^{\frac{4}{3}}-y^{\frac{4}{3}}$.

15. $x+y$.

16. $x-y^{-1}$

17. $a-a^{-1}-4$.

18. $a^{\frac{1}{2}}-3b^{\frac{1}{2}}$

19. x^m+4.

20. $x^{2a}-2x^{a}+4$.

21. $1+2a^{-1}+4a^{-2}$.

22. $x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{1}{3}}z^{\frac{1}{3}}-z^{\frac{1}{3}}x^{\frac{1}{3}}$.

26. $\dfrac{x(x+3)}{x+4}$.

27. $-\dfrac{x+1}{x^2(x+3)}$.

28. $\dfrac{2xy}{y^2-x^2}$.

29. 1.

30. $\dfrac{8xy(x^2+y^2)}{(x^2-y^2)^2}$.

Exercise 83 (f) (Page 326)

1. $x=13$.   2. $x=3$.   3. $x=4$.   4. $x=3$.
5. $x=3$.   6. $x=6$.   7. $x=2\frac{3}{7}$.   8. $x=-6$.
9. $x=\frac{2}{3}$.   10. $x=3$.   11. $x=\frac{2}{5}$.   12. $x=1$.
13. $x=-2$.   14. $x=-8$.   15. $x=1$.   16. $x=0$-
17. $x=3$.   18. $x=-3$.   19. $x=1$.   20. $x=2$.
21. $x=3, y=2$.   22. $x=3, y=\frac{1}{2}$.   23. $x=2, y=1$.
24. $x=1, y=2, z=3$.   25. $x=0, y=1$.   26. $x=1, y=2$.
27. $x=2, y=1$.   28. $x=1, y=0$.

Exercise 84 (Page 331)

1. $5\sqrt{3}$.   2. $4\sqrt{5}$.   3. $2\sqrt{6}$.   4. $2\sqrt{2}$.
5. $3\sqrt[3]{2}$.   6. $5\sqrt[3]{3}$.   7. $5\sqrt[3]{5}$.   8. $-9\sqrt[3]{3}$.
9. $6a\sqrt{ab}$.   10. $3a^2b\sqrt{2a}$.   11. $-3ab\sqrt[3]{ea}$ 12. $(x+y)\sqrt{a}$.
13. $\sqrt{72}$.   14. $\sqrt{175}$.   15. $\sqrt{147}$.   16. $\sqrt{a^2b^2}$.
17. $\sqrt{18}$.   18. $\sqrt{\dfrac{m^2x}{n^2}}$.   19. $\sqrt[3]{135}$.   20. $\sqrt[3]{\dfrac{64}{9}}$.

21. $\sqrt[5]{288}$. 22. $\sqrt[3]{\dfrac{a^{10}}{b^2}}$ 23. $9\sqrt{2}$. 24. $5\sqrt{10}$.

25. 0. 26. $\tfrac{1}{4}\sqrt{3}$. 27. 0. 28. $6\sqrt{6}$.

29. $3\sqrt[3]{5}$. 30. $2\sqrt[3]{2}$. 31. $3x^2\sqrt{2x}$. 32. $-x\sqrt[3]{x}$.

33. $\sqrt[4]{16}$. 34. $\sqrt[3]{125}$. 35. $\sqrt[5]{1024}$. 36. $\sqrt[n]{a^n}$.

37. $\sqrt[12]{125}$. 38. $\sqrt[20]{\sqrt{248}}$. 39. $\sqrt[24]{256}$. 40. $\sqrt[30]{3125}$.

41. $\sqrt[12]{27}$, $\sqrt[12]{25}$. 42. $\sqrt[6]{8}$, $\sqrt[6]{25}$.

43. $\sqrt[23]{16}$, $\sqrt[20]{243}$. 44. $\sqrt[12]{86}$, $\sqrt[12]{27}$, $\sqrt[12]{16}$.

45. $\sqrt[4]{3}$. 46. $\sqrt[6]{10}$. 47. $\sqrt[3]{5}$.

48. $\sqrt[6]{4}$, $\sqrt[6]{7}$, $\sqrt[6]{3}$. 49. $\sqrt[3]{6}$, $\sqrt[4]{10}$, $\sqrt{3}$.

50. $\sqrt[6]{10}$, $\sqrt[4]{3}$, $\sqrt[6]{5}$. 51. 6. 52. 15.

53. 30. 54. $a^2b^2c^2$. 55. $12\sqrt{14}$. 56. $12\sqrt{3}$.

57. $42\sqrt{6}$. 58. 660. 59. $\sqrt[6]{500}$. 60. $\sqrt[9]{54}$.

61. $\sqrt[24]{432}$. 62. 7. 63. $\sqrt[8]{128}$. 64. $\sqrt[4]{a^3b^3}$.

65. $8\sqrt{3}$. 66. 6. 67. $3\sqrt{3}$. 68. $\sqrt[12]{48}$.

69. $\sqrt[6]{\dfrac{135}{64}}$. 70. $\sqrt[12]{\dfrac{1}{80}}$.

Exercise 85 (Page 335)

1. $5+4\sqrt{2}$. 2. $1+\sqrt{3}$. 3. $19-11\sqrt{2}$.

4. $-1+13\sqrt{3}$. 5. $36+7\sqrt{10}$. 6. $24-9\sqrt{6}$.

7. $6+\sqrt{10}$. 8. $30+2\sqrt{35}$.

9. $6\sqrt{10}+3\sqrt{14}+2\sqrt{15}+\sqrt{21}$.

10. $4\sqrt{15}-4\sqrt{10}+12\sqrt{2}-8\sqrt{3}$.

11. 33. 12. 30. 13. -21. 14. 95.

15. $88-30\sqrt{7}$. 16. $30+12\sqrt{6}$. 17. $69-12\sqrt{30}$.

18. $54+24\sqrt{2}$. 19. $2a-2\sqrt{a^2-x^2}$.

20. $1+8x-4\sqrt{x+4x^2}$. 21. $13x^3+5y^2-12\sqrt{x^4-y^4}$.

22. $8a^2-2\sqrt{16a^4-1}$. 23. $-6+2\sqrt{15}$. 24. $16+6\sqrt{10}$.

25. $85+4\sqrt{10}$. 26. $29-4\sqrt{5}$. 27. 4.

28. 4. 29. 24. 30. 59.

Exercise 86 (Page 338)

1. $\sqrt{3}$

2. $\sqrt{3}$

3. $\sqrt{5}$.

4. $\sqrt{a}$.

5. $\sqrt{3}-\sqrt{2}$.

6. $2\sqrt{5}+\sqrt{3}$.

7. $4-3\sqrt{5}$.

8. $3\sqrt{5}+5\sqrt{7}$.

9. $\dfrac{5\sqrt{2}}{2}$.

10. $\dfrac{6\sqrt{5}}{5}$

11. $\dfrac{3\sqrt{10}}{2}$

12. $2\sqrt{3}$.

13. $\dfrac{\sqrt{6}}{3}$.

14. $\dfrac{\sqrt{mn}}{xn}$.

15. $\dfrac{\sqrt{3}+1}{2}$.

16. $\sqrt{5}+2$.

17. $\sqrt{6}+\sqrt{5}$.

18. $-(4+3\sqrt{2})$.

19. $\sqrt{6}+2$.

20. $5\sqrt{3}+3\sqrt{5}$,

21. $\dfrac{8+5\sqrt{2}}{2}$.

22. $5+2\sqrt{6}$

23. $\dfrac{24-\sqrt{15}}{2}$.

24. $\dfrac{\sqrt{7}-\sqrt{2}}{5}$.

25. $3\sqrt{2}-2\sqrt{3}$.

26. $\dfrac{25+17\sqrt{10}}{15}$.

27. $\frac{5}{2}+\frac{1}{2}\sqrt{15}$.

28. $\frac{1}{3}\sqrt{10}+\frac{2}{3}$.

29. $2\frac{3}{8}\sqrt{2}+\frac{5}{8}\sqrt{10}$

30. $18-7\sqrt{6}$.

31. $\dfrac{1}{x^2}(2a^2-x^2+2a\sqrt{a^2-x^2})$.

32. $\sqrt{x^2+a^2}-a$.

33. $2a-1+2\sqrt{a^2-a}$.

34. $\dfrac{x^2+\sqrt{x^4-y^4}}{y^2}$.

35. $\dfrac{1+\sqrt{2}-\sqrt{3}}{2}$.

36. $\dfrac{2\sqrt{3}+\sqrt{30}+3\sqrt{2}}{12}$.

37. $-(2+\sqrt{2}+\sqrt{6})$.

38. $\sqrt{2}$.

39. 3.

40. $5(\sqrt{3}-\sqrt{2})$.

41. $\dfrac{25+\sqrt{3}}{22}$.

42. $\frac{13}{11}$.

43. $\frac{8}{3}\sqrt{6}$.

44. $\dfrac{2(a+b)}{a-b}$

45. $5+\sqrt{2}+\sqrt{5}$.

46. 0.

47. $\dfrac{4ab}{a^2-b^2}$

Exercise 87 (Page 344)

1. $2+\sqrt{3}$

2. $1+\sqrt{2}$.

3. $\sqrt{3}+\sqrt{2}$.

4. $\sqrt{5}+\sqrt{3}$

5. $2\sqrt{2}+\sqrt{3}$.

6. $2\sqrt{3}+3\sqrt{2}$.

7. $8-\sqrt{5}$

8. $\sqrt{5}-\sqrt{2}$.

9. $\sqrt{10}-3\sqrt{2}$.

10. $4\sqrt{2}-3$.   11. $2\sqrt{3}-\sqrt{6}$.   12. $3\sqrt{3}-2\sqrt{2}$.
13. $\sqrt{\tfrac{3}{2}}-\sqrt{\tfrac{1}{2}}$.   14. $\sqrt{\tfrac{3}{2}}-\sqrt{\tfrac{1}{2}}$.   15. $\sqrt{\tfrac{7}{2}}+\sqrt{\tfrac{1}{2}}$.
16. $\sqrt{\tfrac{7}{2}}+\sqrt{\tfrac{3}{2}}$.   17. $\tfrac{1}{2}(3-\sqrt{5})$.   18. $\tfrac{1}{2}\sqrt{5}+1$.
19. $\tfrac{1}{2}(\sqrt{3}+\sqrt{3})$.   20. $\sqrt{\tfrac{5}{2}}-\sqrt{\tfrac{3}{5}}$.   21. $\sqrt[4]{2}(\sqrt{3}+1)$.
22. $\sqrt[4]{3}(1-\sqrt{2})$.   23. $\sqrt[4]{5}(\sqrt{3}-\sqrt{2})$.   24. $\sqrt[4]{7}(\sqrt{\tfrac{7}{2}}+\sqrt{\tfrac{3}{2}})$.
25. $\sqrt[4]{6}(\sqrt{\tfrac{5}{2}}-\sqrt{\tfrac{1}{2}})$.   26. $\sqrt[4]{5}(\sqrt{\tfrac{11}{2}}+\sqrt{\tfrac{1}{2}})$.   27. $x=10,\ y=3$.
28. $x=8,\ y=6$.   29. $a=10,\ b=5$.   30. $p=25,\ b=6$.

Exercise 88 (Page 347)

1. 34.   2. 14.   3. 34.   4. 22.
5. 20.   6. 36.   7. 64.   8. 8.
9. 194.   10. 34.   11. 194.   12. 198.
13. 52.   14. 76.   15. 234.   16. $10\sqrt{2}$.
17. $30\sqrt{3}$.   18. $2\sqrt{3}$.   19. $\sqrt{10}$.   20. 4.
21. $-2\sqrt{2}$.   22. (i) $\sqrt{6}$, (ii) $-\sqrt{6}$.   23. 34.
24. 22.   25. 36.   26. $\cdot 194$.   27. 52.
28. 76.   29. 4.   30. 4.   31. 194.
32. 98.   33. $-144\sqrt{5}$. 34. 4.   35. $\sqrt{10}$.
36. 52.   37. 47.

Exercise 89 (Page 352)

1. 6.   2. 5.   3. 8.   4. 3.
5. -7.   6. 33.   7. 10.   8. 9.
9. 9.   10. 1.   11. 5.   12. $2\tfrac{3}{16}$.
13. 12.   14. 5.   15. $\tfrac{3}{4}$.   16. $-\tfrac{1}{8}$.
17. 7.   18. 1.   19. 8.   20. 2.
21. $\tfrac{9}{2}$.   22. 3.   23. $6\tfrac{4}{5}$.   24. 49
25. $2\tfrac{8}{5}$.   26. 9.   27. 5.   28. 2.
29. 3.   30. 4.   31. ± 5.   32. ± 5.
33. ± 4.   34. 3.   35. $\tfrac{1}{2}$.   36. $\tfrac{5}{4}$.
37. 8.   38. 2.

Paper 1—Ex. 90 (Page 357)

1. $(x^2+3)(x^2-3)(x^2+1)(x+1)(x-1)$.   2. $5(a^2-a+1)$.

3. $\dfrac{a}{b}$.   4. $3\left(\dfrac{a}{b}+\dfrac{b}{a}\right)-5$.

5. 1   6. $\sqrt[3]{3}$ is greater; $\sqrt[18]{\dfrac{64}{81}}$

Paper 2—Ex. 91 (Page 358)

1. $(2x+a)(2x-a)(4x^2+2ax+a^2)(4x^2-2ax+a^2)$.
2. $252x^2y^2(x^2-y^2)(x^4+x^2y^2+y^4)$.
3. $\dfrac{x^3}{(x-y)(x^3+y^3)}$.
4. $(x+1)(x+7)(2x-3)$.
5. $\frac{3}{5}$.
6. $\sqrt{2}$.

Paper 3—Ex. 92 (Page 358)

1. $(x-2y-3)(x^2+4y^2+9+2xy+3x-6y)$.
2. $a-b$.
3. $\dfrac{x^3+y^3}{(x+y)^2}$.
4. $6x^2-13x+6$.
5. (ii) $\frac{1}{9}$.
6. $\dfrac{8+5\sqrt{2}}{2}$.

Paper 4—Ex. 93 (Page 359)

1. $3(x-2y)(1-x)(2y-1)$.
2. $525x^3y(x^3-y^3)(x+y)$.
3. $\dfrac{8x^7}{x^8-256y^8}$.
4. $\dfrac{x}{y}-1+\dfrac{y}{x}$.
5. (ii) 1.
6. $\sqrt{\frac{1}{6}}+\sqrt{\frac{5}{6}}$.

Paper 5—Ex. 94 (Page 359)

1. $(x+1)(x-1)^3$
2. $2x^2-2x$.
3. $x+y+z$.
4. $x^2-\dfrac{1}{x^2}+3$.
5. (ii) $3\frac{1}{2}$.
6. $\sqrt[4]{3}(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}})$.

Paper 6—Ex. 95 (Page 360)

1. $(a-2)^2(a+1)(a-5)$.
2. $(a-1)(a^4+a^3-2a-4)$.
3. $\dfrac{2(a^2+b^2)}{b^2}$.

$x^{-\frac{2}{3}}+y^{\frac{2}{3}}+z^{-\frac{4}{3}}-x^{-\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{1}{3}}z^{-\frac{2}{3}}+z^{-\frac{2}{3}}x^{-\frac{1}{3}}$.   6. 233.

Paper 7—Ex. 96 (Page 360)

1. $(x+1)(2x-1)^2$.
2. $3a^2-2ab+b^2$.
3. $a+b+c$.
6. $-\frac{1}{2}$.

Paper 8—Ex. 97 (Page 360)

1. $(a^2-5a-40)(a^2-5a-20)$.

2. $2x^3+5x^2-3x$.

3. $\dfrac{x^3-x-2}{x^3-x^2+3x-5}$.

4. $a^2-2+\dfrac{1}{a^2}$.

5. $\frac{2}{5}$.

6. 3.

Exercise 98 (Page 361)

1. $1:12$.

2. $4:3$.

3. $5:44$.

4. Ratio not possible.

5. $\frac{26}{19}$.

6. 8.

7. $-\frac{1}{4}$.

8. $\frac{5}{4}$.

9. $3:2$.

10. $5:2$.

11. $5:2$.

12. $4:3$.

13. $3:2$ or $5:3$.

14. $3:4$ or $5:2$.

15. $5:4$ or $2:3$.

16. $3:5$ or $5:3$.

17. $3:4$.

18. $1:2$.

19. $5:4$.

20. $2:3$ or $3:$

21. $3:2$.

22. $1:2$.

23. $15:11$.

24. $4:3$.

25. $11:19$.

26. $1:3$.

27. 7.

28. 17.

29. -170 and 806.

30. 12 and 9.

31. 25 and 30.

Exercise 99 (a) (Page 370)

1. 6.

2. 12.

3. $\dfrac{ab}{2}$.

4. $a^4+a^2b^2+b^4$.

5. 9.

6. 20.

7. ab.

8. $2(\sqrt{3}-1)$.

9. 8.

10. a^2b^2.

11. 3.

12. $a+b$.

13. $(a+b)\sqrt{a^2-ab+b^2}$.

14. $ab-\dfrac{1}{ab}$.

15. 3.

16. 4.

17. 2.

18. 4.

Exercise 99 (d) (Page 383)

8. 2.

9. 2.

10. 2.

11. 2.

12. $\dfrac{6(m^2-n^2)}{(2m-n)(m-2n)}$.

13. b.

14. m^2.

17. $\frac{9}{4}$.

18. 7.

19. $\frac{8}{9}$.

20. 5.

21. 10.

22. $\frac{1}{2}$.

23. $\frac{1}{4}$.

24. $\dfrac{a}{3}$.

Exercise 100 (a) (Page 391)

1. $ad = bc.$
2. $lq = mp.$
3. $l(a+b) = c(m+n).$
4. $k(p-q) = r(m-n).$
5. $b^2p + a^2q = 0.$
6. $a^2b + c = 0.$
7. $aq^2 + bp^3 = 0.$
8. $a^2d^2 = b^2c^2.$
9. $al^2 + blm + cm^2 = 0.$
10. $p + q^2 + q^2r = 0.$
11. $a^2 + 2y^2 - 2ay = b^2.$
12. $dl^3 - cl^2m + blm^2 - am^3 = 0.$
13. $y^2 = 4ax.$
14. $l^2x = a(y-m)^2 + bl(y-m) + cl^2.$
15. $l(a-c) = m(b-d).$
16. $(l-m)(cp-dq) = (p-q)(al-bm).$
17. $(a-b)(a-b-1) = 0.$
18. $ad - ab = 1.$
24. $a^3q^2 = b^3p^2.$
25. $a^4 = b^3.$
26. $l^3q^2 + m^3p^2 = 0.$
27. $a^m d^n = b^m c^n.$
28. $m^2 - n^2 = 4.$
29. $a^2 - b^2 = 4.$
30. $ab = 1.$
31. $4cd = 5.$
32. $p^2 - q^2 + \dfrac{1}{p^2} - \dfrac{1}{q^2} = 4.$
33. $a^2 - b^2 = 1.$
34. $ab + 1 = 0.$
35. $a^2d^2 - b^2c^2 = 4a^3c^2.$
36. $a^2 - b^2 = 2.$ 36-a. $x^2 - y^2 = 2.$ 37. $xy = 1.$
38. $z - y^2 = 2y^2z.$
39. $b^3 - a^3 + 2a^2b = 0.$
40. $k^4 - 4k^2 - l^4 + 2 = 0.$
41. $a^4 + 4a^2 - b + 2 = 0.$
42. $\dfrac{1}{l^3} + \dfrac{4}{l^2} - \dfrac{1}{m} + 2 = 0.$
43. $p^3 - q^3 - 3(p-q) = 0$
44. $a^3 - b^3 + 3(a-b) = 0.$
45. $a^3 + 3a^2b - b = 0.$
46. $p - 3pq^2 - q^3 = 0.$
47. $(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (c_1a_2 - c_2a_1)^2.$
48. $(fy - gx)(g-f) = (x-y)^2.$
49. $(xy' + yx')(yz' - zy) = (zx' + xz')^2.$
50. $(c - ad)^2 + b^2d = 0.$
51. $(a_1b_2 - b_1a_2)^2(b_1c_2 - c_1b_2) = (c_1a_2 - a_1c_2)^3.$
52. $(lq - mp)^2(qn + mr) + (pn + lr)^3 = 0.$
53. $a + b + c = 0.$ 54. $(bc' - b'c)^2(ab' - a'b) = (ca' - c'a)^3$
55. $(b_1c_2 - b_2c_1)^3(a_1b_2 - a_2b_1) = (c_1a_2 - c_2a_1)^4.$
56. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$ 57. $a^2x^2 - b^2y^2 = 1.$
58. $x^2 - y^2 = 1$ 59. $\dfrac{x^3}{a^2} + \dfrac{y^3}{b^3} = 1.$
60. $x^2 + y^2 = 1.$ 61. $a^2 - b^2 = 1.$ 62. $a^2 - b^2 = 1.$

Exercise 100 (b) (Page 404)

1. $a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0.$
2. $abc+2fgh-af^2-bg^2-ch^2=0.$
3. $br-cq+cp-ar+aq-bp=0.$
4. $a^3+b^3+c^3-3abc=0.$ 5. $ab+bc+ca \div 2abc=1.$
6. $2abc+ab+bc+ca-1=0.$ 7. $1+lm+mn+nl=0.$
8. $a^3+b^3+c^3-3abc=0.$ 9. $aq^2-bpq+cp^2=0.$
10. $la^2+mab+nb^2=0.$
11. $(b_1c_2-b_2c_1)(a_1b_2-a_2b_1)=(c_1a_2-c_2a_1)^2.$
12. $(c-ad)^2+b^3d=0.$
13. $(a_1b_2+b_1a_2)^2(b_1c_2-c_1b_2)=(c_1a_2-a_1c_2)^3.$
14. $(bc'-b'c)^2(ab'-a'b)=(ca'-c'a)^3.$
15. $2c^3=3ab^2-a^3.$ 16. $2l^3+m^3-3mn^2=0.$
17. $a^4-2a^2b^2-b^4+2c^4=0.$ 18. $a^4-4ac^3+3b^4=0.$
19. $m^6+3l^4m^2=4l^3n^3.$ 20. $a^6-2b^6+9a^2c^4-8a^3b^3=0.$
21. $abc=1.$ 22. $ace=bdf.$
23. $ab+bc+ca+1=0.$ 24. $a^2+b^3+c^3=abc+4.$
25. $a^2+b^2+c^2=abc+4.$ 26 $x^2+y^2=a^2.$
27. $x+a=0.$ 28. $x^2=y^2+2z^2.$
29. $ab^2=c^3.$ 30. $a^2+b^2+c^2+1=0.$

Exercise 101 (Page 412)

2. 10.   3. 17.   4. 13.
8. Mid pt. of AB is (3, −5), of BC (2, −8), of CA

$$(-3, -3).$$

Exercise 102 (Page 419)

Graphs : Q. 1 to 24 [Figures within brackets denote num!
of questions].
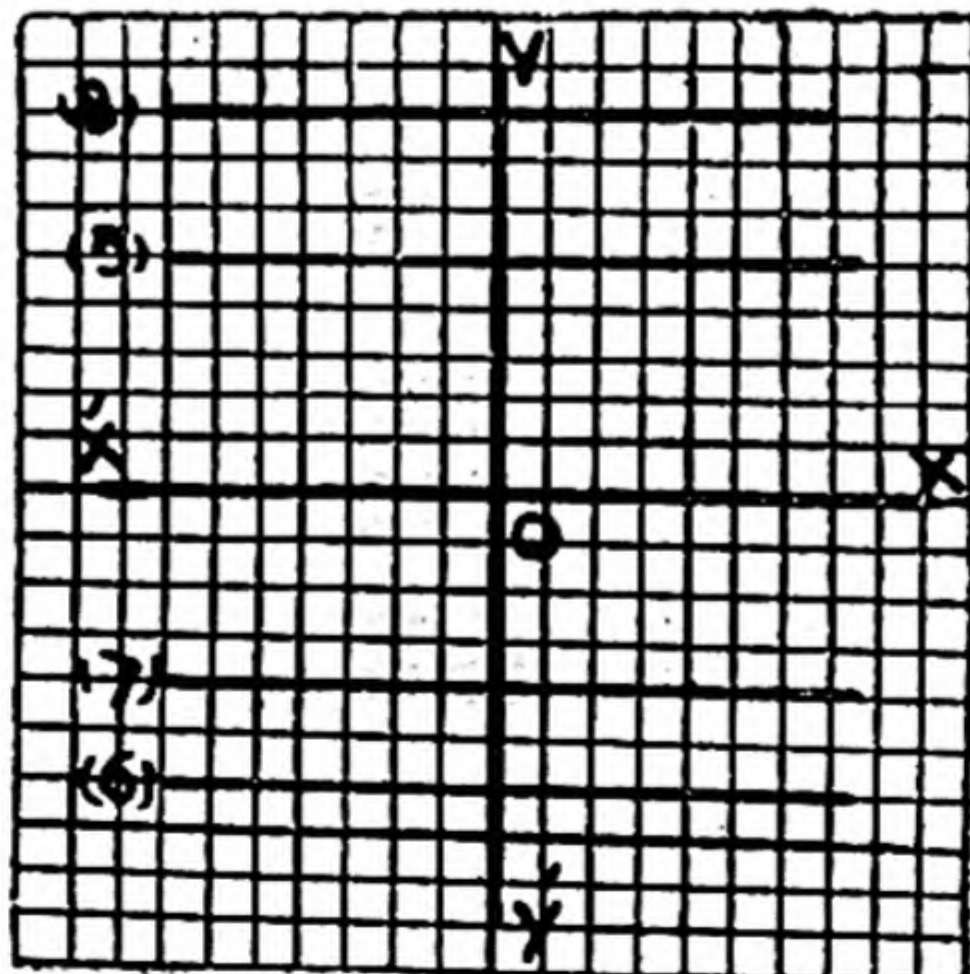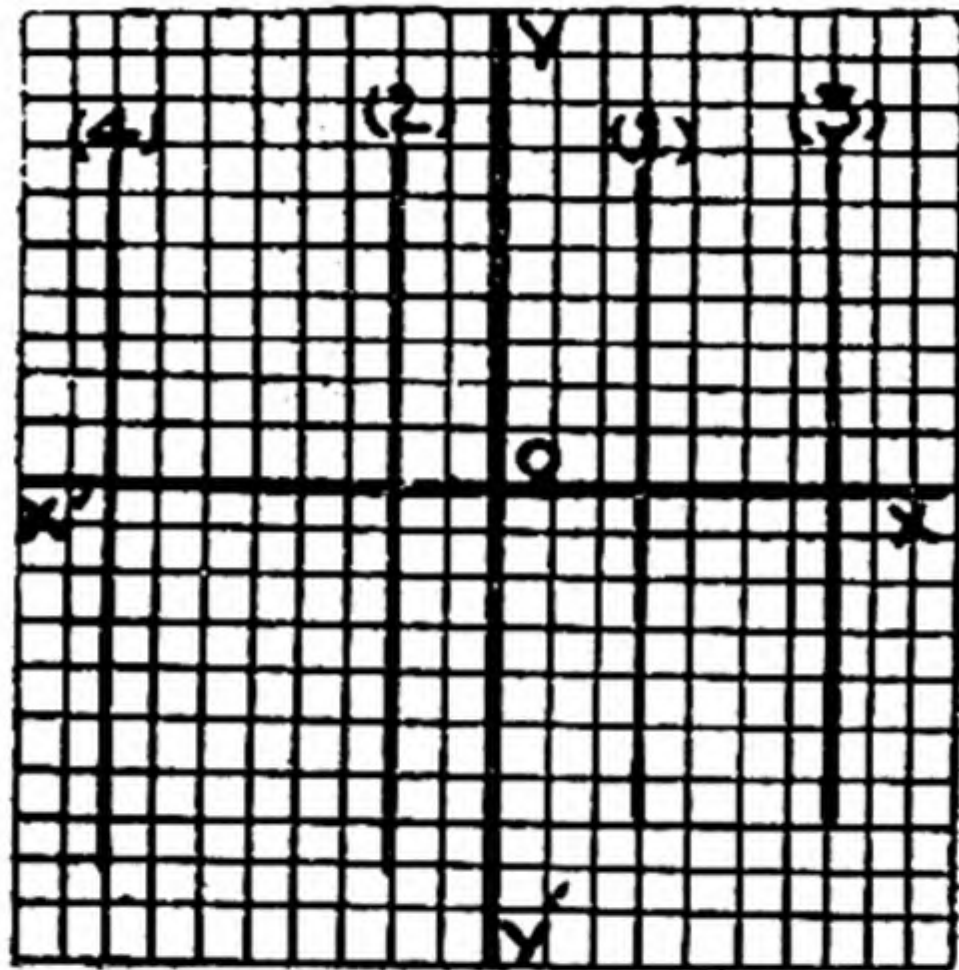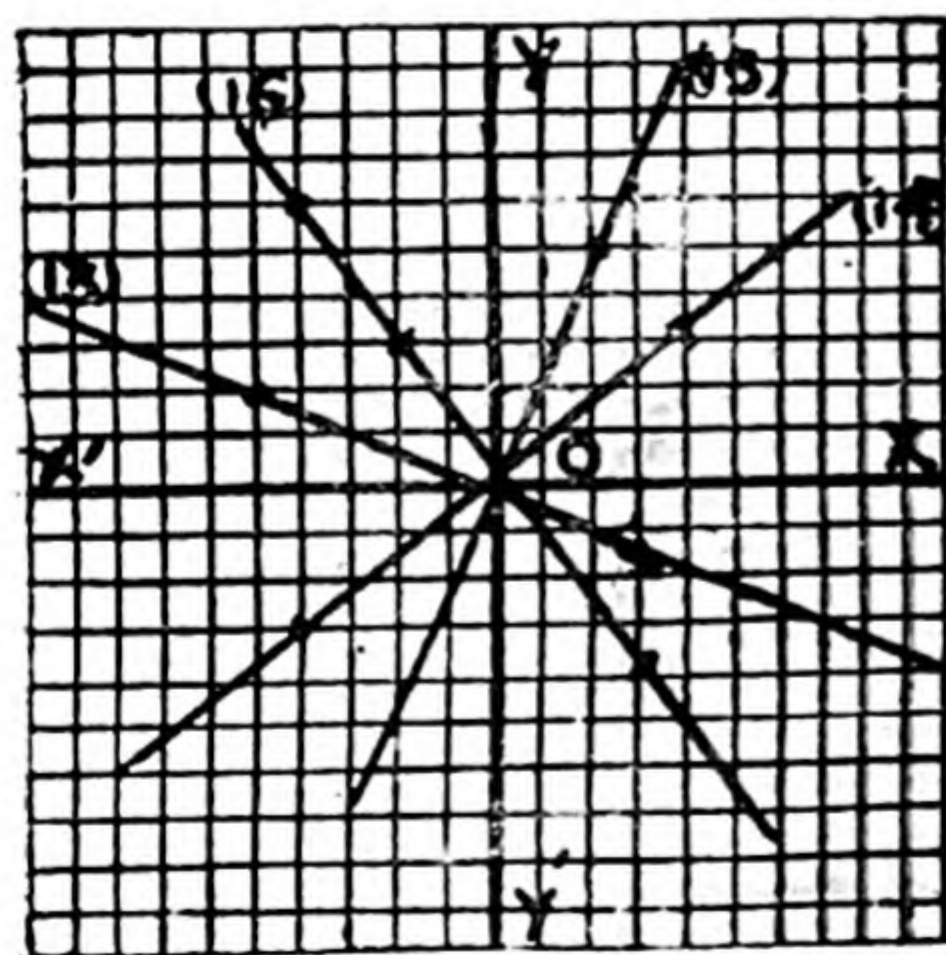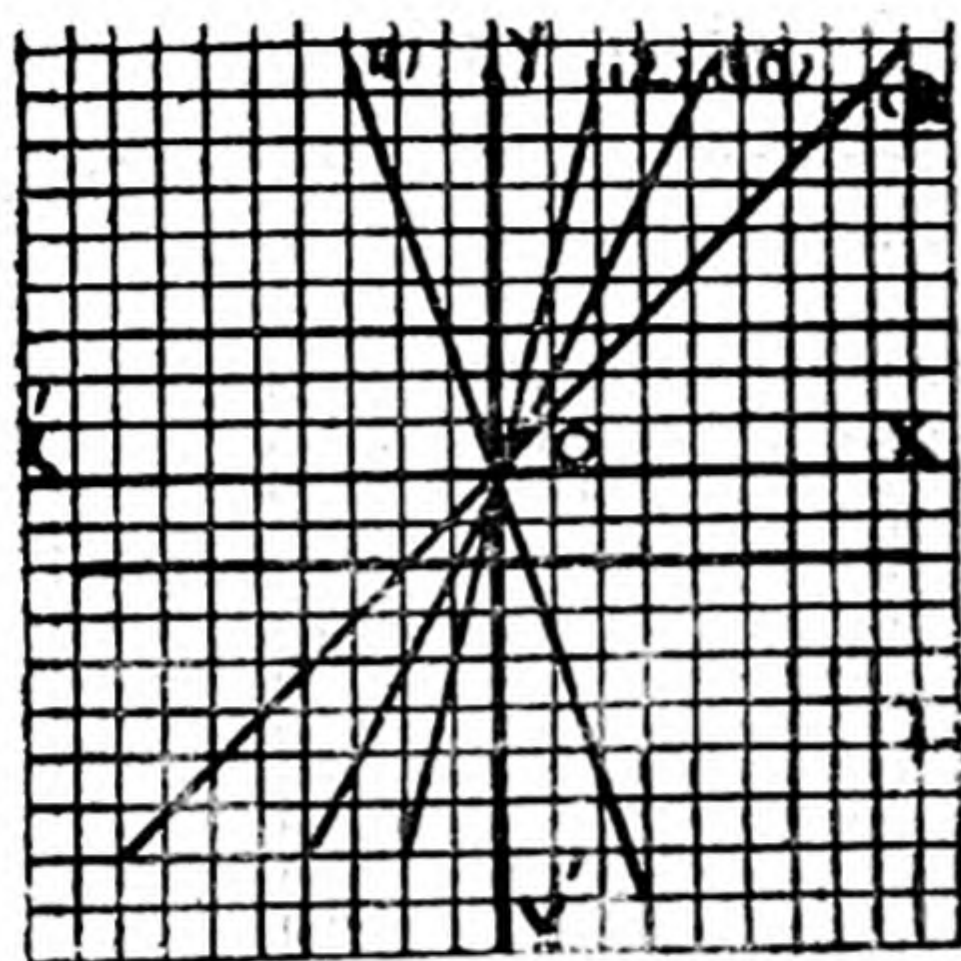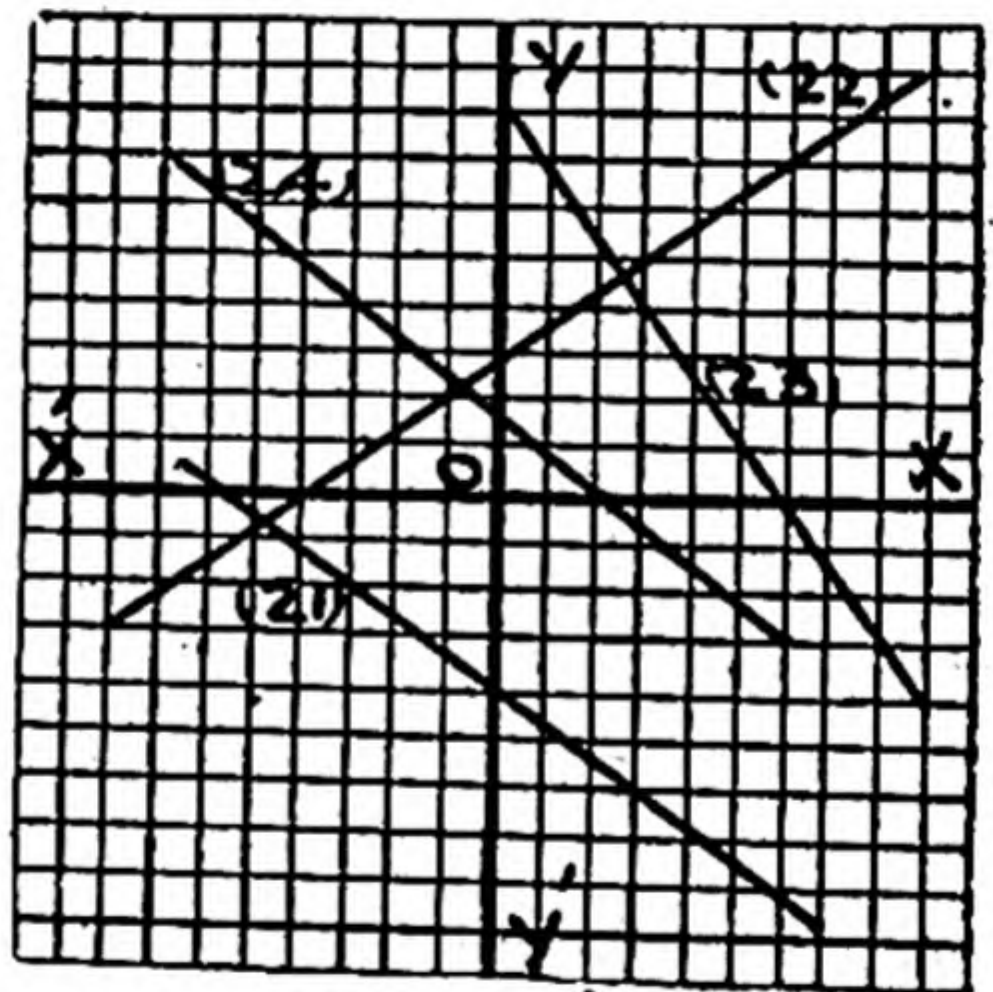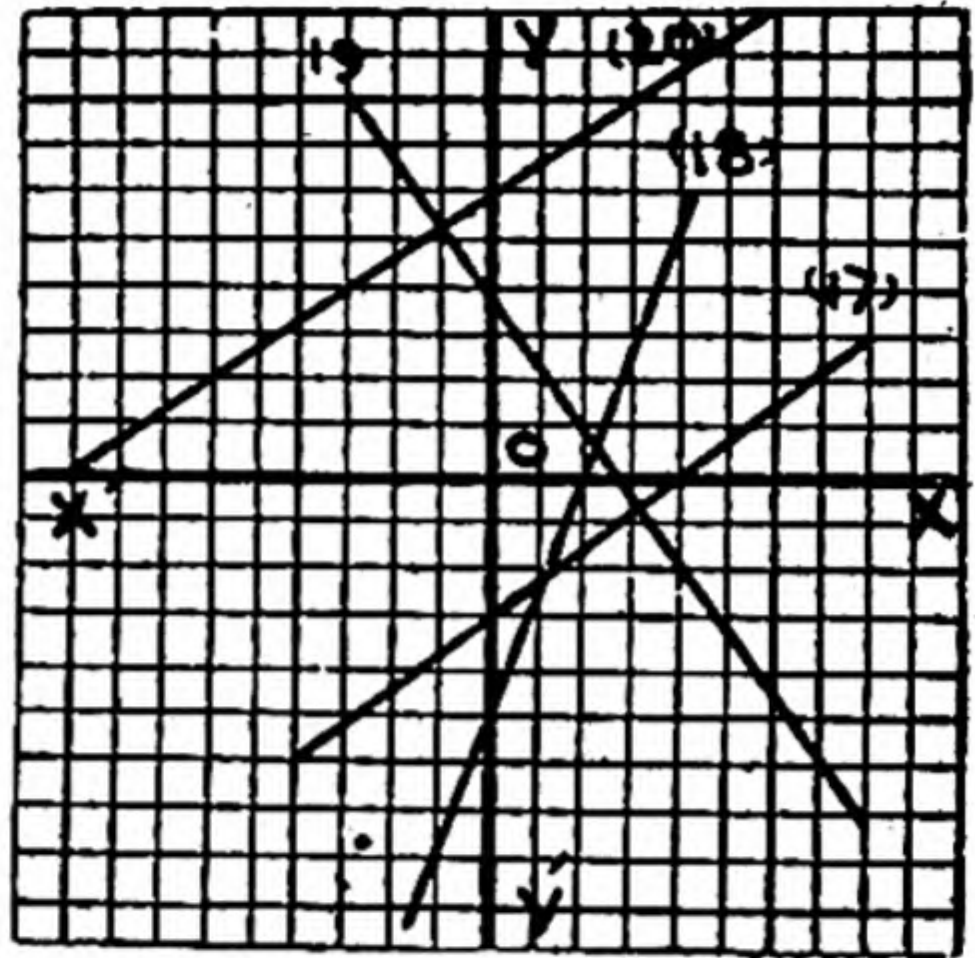
Intercepts (Q. 17 to 24).

17.   −3.   18.   −5.   19.   4.   **20.**   **6**

21.   −6.   22.   −4.   23.   6.   24.   2.

25.   0.   26.   4.   27.   7 ; −4.

28. $x=6, y=0$.   29.   10.

30.   11 ; Intercepts $-4\frac{1}{2}$ and $6\frac{1}{4}$.

Exercise 103 (Page 422)

1. $x=5, y=3.$
2. $x=6, y=-8.$
3. $x=4, y=2.$
4. $x=1, y=2.$
5. $x=1, y=1.$
6. $x=1.5, y=1.$
7. $x=8, y=3.$
8. $x=-3, y=8.$
9. $x=6, y=-4$; Intercepts 8 and $-16.$
10. (a) $x=-2, y=-3.$ (b) $-11.$
11. $x=-3, y=4$; Intercepts -9 and 6.
12. $x=\frac{7}{2}, y=\frac{5}{2}.$
13. $x=3, y=2$; one right angle.
14. $x=3, y=12.$
15. $(3, 0).$

Exercise 104 (Page 424)

1. $y=2x.$
2. $y=2x+6.$
3. $y=2x-6.$
4. $2x-3y=6.$
5. $2x+y=17.$
6. $3x-2y=8.$
7. $2x-3y+18=0.$
8. $y=13x+4.$
9. $x+2y-5=0, 2x-y=0, 4x+3y=0.$
10. $2x-y+3=0.$
11. $x+y=0.$
12. No   [The st. line is $3x-5y=16.$]
13. AB, $3x-2y=1$; CD, $2x-y=2$: $(3, 4).$
14. $x-3y+2=0$: 1 and $-1.$

Exercise 105 (Page 425)

1. $\dfrac{d-b}{a-c}.$
2. $\dfrac{a+b}{2}.$
3. $\dfrac{2c-3a}{a-5c}.$

4. $b+c.$
5. $\dfrac{(a-b)^2}{a+b}.$
6. $a+b.$

7. $a+b.$
8. $a-b.$
9. $-\dfrac{lm}{l+m+n}.$

10. $\dfrac{p^2-2pq+qr}{r-q}.$
11. $\dfrac{cd-ab}{a+b-c-d}.$

12. $\dfrac{2mn}{m+n}.$
13. $\dfrac{a^2+ab+b^2}{a+b}.$
14. $-\dfrac{a}{a+b}.$

15. $a-2b.$
16. $\frac{1}{2}(a+b)(c+d).$

17. $\dfrac{bc}{ad}$ 18. $\dfrac{bc}{a+b}$.   19. $\dfrac{2ab}{a+b}$

20. $-pq$.   21. $\dfrac{ac}{b}$.   22. p.

23. $m-n$.   24. $k+l$.   25. 8.   26. $2\frac{1}{3}$.

27. 4   28. $\frac{4}{3}$.   29. -7.   30. 4.

31. $1\frac{2}{3}$,   32. 9.   33. 16   34. 13.

35. $2\frac{3}{4}$.   36. $\frac{59}{63}$.   37. $-1\frac{9}{28}$.   38. $-5\frac{49}{14}$.

39. $1\frac{1}{6}$.   40. 15.   41. -1.   42. 3.

43. 4.   44. 6.   45 $1\frac{1}{2}$.   46. $1\frac{5}{3}$.

47. $\dfrac{a+b}{2}$.   48. $\dfrac{ab}{a+b-c}$.   49. $\dfrac{a+b}{2}$.

50. $\dfrac{a-c}{2}$.   51. 2.   52. 3.   53. $4\frac{2}{3}$.

54. $\frac{68}{27}$.   55. 24.   56. 1.   57. 7.

58. 2.   59. $\dfrac{psr+qsr-r^2-s^2}{s+r-ps-qr}$.   60. $\dfrac{p+q}{2}$.

61. -1.   62. 0.   63. $1\frac{1}{2}$.   64. $1\frac{4}{7}$.

65. 16.   66. 13.   67. $1\frac{1}{8}$.   68. 4.

69. $4\frac{1}{2}$.   70. 6.   71. 2.   72. $2\frac{1}{2}$.

Exercise 106 (Page 433)

1. $a+b+c$.   2. $p+q$.   3. $a^2+b^2+c^2$.

4. $ab+bc+ca$.   5. $-(a+b+c)$.   6. m^3+m^2+m.

7. 3 or -4.   8. $\frac{2}{5}$ or $\frac{3}{2}$.   9. 4 or $-\frac{9}{4}$.

10. 3 or $-\frac{1}{2}$.   11. $\dfrac{a+b+c}{3}$.   12. -2.

13. 2.   14. 0.

Exercise 107 (Page 435)

1. $x=\dfrac{b_1c_2-c_1b_2}{a_1b_2-b_1a_2}$, $y=\dfrac{c_1a_2-a_1c_2}{a_1b_2-b_1a_2}$.

2. $x=\dfrac{ap-bq}{a^2-b^2}$, $y=\dfrac{aq-bp}{a^2-b^2}$.

3. $x = \dfrac{q^2 - pr}{qr - p^2}$, $y = \dfrac{pq - r^2}{qr - p^2}$.

4. $x = \dfrac{l^2 + lm + m^2}{l + m}$, $y = -\dfrac{lm}{l + m}$.

5. $x = \dfrac{m + m'}{m'n + mn'}$, $y = \dfrac{n' - n}{m'n + mn'}$.

6. $x = 2p$, $y = 2q$.   7. $x = a + b$, $y = a - b$.

8. $x = a + b$, $y = a - b$.

9. $x = \dfrac{a^2 - b^2}{ac - bd}$, $y = \dfrac{a^2 - b^2}{ad - bc}$.

10. $x = -\dfrac{2b}{1 + b}$, $y = -\dfrac{2a}{1 + a}$.

11. $x = 3$, $y = 5$, or $x = 5$, $y = 3$.

12. $x = 6$, $y = 4$; or $x = 4$, $y = 6$.

13. $x = 8$, $y = 6$; or $x = -6$, $y = -8$.

14. $x = 5$, $y = -4$, or $x = 4$, $y = -5$.

15. $x = \frac{1}{2}$, $y = -\frac{1}{3}$ or $x = -\frac{1}{3}$, $y = \frac{1}{2}$.

16. $x = \frac{1}{2}$, $y = -\frac{1}{4}$ or $x = \frac{1}{4}$, $y = -\frac{1}{2}$.

17. $x = 5$, $y = 1$.   18. $x = 6$, $y = 2$.

19. $x = \frac{1}{7}$, $y = \frac{1}{4}$.   20. $x = \frac{1}{2}$, $y = \frac{1}{5}$.

21. $x = 2$, $y = 3$, $z = 4$ or $x = -2$, $y = -3$, $z = -4$.

22. $x = 4$, $y = 5$, $z = 6$ or $x = -4$, $y = -5$, $z = -6$.

23. $x = \frac{1}{2}$, $y = -\frac{1}{3}$, $z = -\frac{1}{5}$ or $x = -\frac{1}{2}$, $y = \frac{1}{3}$, $z = \frac{1}{4}$.

24. $x = \frac{2}{3}$, $y = -\frac{1}{2}$, $z = \frac{4}{5}$ or $x = -\frac{2}{3}$, $y = \frac{1}{2}$, $z = -\frac{4}{5}$.

25. $x = b + c - a$, $y = c + a - b$, $z = a + b - c$.

26. $x = \dfrac{b + c - a}{a}$, $y = \dfrac{c + a - b}{b}$, $z = \dfrac{a + b - c}{c}$.

27. $x = \dfrac{a}{2}$, $y = \dfrac{b}{2}$, $z = \dfrac{c}{2}$.

28. $x = \frac{1}{2}ca$, $y = \frac{1}{2}bc$, $z = \frac{1}{2}ab$.

29. $x = 1$, $y = 2$, $z = 3$.

30. $x = 2$, $y = 3$, $z = 4$, or $x = -2$, $y = -3$, $z = -4$.

31. $x = b - c$, $y = c - a$, $z = a - b$.

32. $x = a(b - c)$, $y = b(c - a)$, $z = c(a - b)$.

33. $x = bc(b - c)$, $y = ca(c - a)$, $z = ab(a - b)$.

34 $\quad x=-\dfrac{1}{(c-a)(a-b)}, \quad y=-\dfrac{1}{(b-c)(a-b)},$

$$z=-\dfrac{1}{(b-c)(c-a)}.$$

Exercise 109 (Page 445)

1. $a=3,\ b=-\frac{4}{3},\ c=0,\ d=-3.$
2. $a=0,\ b=-1,\ c=-\frac{3}{5},\ d=-4.$
3. $p=\frac{3}{2},\ q=\frac{9}{4}.$
4. $a=6,\ b=-30,\ c=36.$
5. $A=4,\ B=-1.$
6. $A=1,\ B=1.$
7. $A=7,\ B=-5.$
8. $P=2,\ Q=-3.$
9. $A=-2\ B=1,\ C=3.$
10. $A=0,\ B=2,\ C=3.$
11. $A=2,\ B=3,\ C=-14.$
12. $A=3,\ B=2,\ C=4.$
13. $A=2,\ B=7,\ C=1.$
14. $A=3,\ B=1,\ C=-3.$
15. $A=2,\ B=-4,\ C=3.$
16. $A=3,\ B=-2,\ C=1.$
17. $A=1,\ B=2.$
18. $A=2,\ B=6.$
19. $A=3,\ B=-4.$
20. $A=3,\ B=-1,\ C=2.$

Paper 1—Ex. 110 (Page 449)

1. $1.$
2. $x=16,\ y=8.$
4. $y^2+z^2+yz=0.$
5. $x=-5,\ y=-5.$

Paper 2—Ex. 111 (Page 449)

1. $x^2-\dfrac{1}{x^2}-2.$
2. $6.$
3. $2.$

4. $x^2+\dfrac{1}{x^2}=x+\dfrac{1}{x}+2.$

5. $x=3,\ y=-1$; Intercept $-2.$
6. $\frac{2}{9}.$

Paper 3 Ex. 112 (Page 450)

1. $\dfrac{x^2-y^2}{2}.$
2. $\dfrac{3-\sqrt5}{2}.$

4. $16a^2-b^2-16=0.$
5. $(1,\ 2).$

Paper 4— Ex 113 (Page 450)

1   $4a^2-16a+11$ 2   625   4 $x^2-y^2=2$ 5 $-\frac{4}{3}$

6   $\frac{3}{4}$

Paper 5   Ex. 114 (Page 451)

1   0.   2   62.   4. $c(x-a)=y\,(y-b)$.

5   Intercepts 8 and -6; abscissa -8.

Paper 6—Ex. 115 (Page 451)

1. $2x+1-\dfrac{3}{x}-\dfrac{1}{x^3}$ 2.   1.   4. $1+ab+bc+ca=0$

5.   7.   6. $A=-2,\ B=1,\ C=3$.

Paper 7—Ex. 116 (Page 452)

1. $-\dfrac{x^4+x^3y^3+y^4}{xy(x-y)}$.   2. $2\sqrt{3}$.

4. $a^4-4ac^3+3b^4=0$.   5.   0.

Paper 8 – Ex. 117 (Page 452)

1. $(a-b)(2a+b)(a+2b)$.   5. $\dfrac{ab}{a+b}$.

6. $A=8,\ B=-1,\ C=-5$.

Paper 9—Ex. 118 (Page 453)

1. $\dfrac{4a+3}{12a^2+8}$.   2.   121.   4. $\dfrac{x^2}{b^2}-\dfrac{y^2}{a^2}=1$

5.   At the point $(-15,\ -4)$.

Paper 10—Ex. 119 (Page 453)

1. $a=3$.

2. $4a^{-\frac{2}{3}}+b^{-2}+c^{-4}+2a^{-\frac{1}{3}}b^{-1}+2a^{-\frac{1}{3}}c^{-2}-b^{-1}c^{-2}$.

4. $12ab=1$ 5   1.   6. $A=3,\ B=-4$.

Answers to Matriculation Papers

1940 (Page 454)

1. (a) -13.   (b) $x=2, y=8$.   (b) 22 men, 14 boys.
2. (a) $x=3, y=-4$.   (b) $9x^2+4y^2+z^2+6xy+3xz-2yz$.

3. (a) $x^2+\dfrac{1}{x^2}-2$.   (b) x^2-2x+1.

4. (a) $(x+2)(x-2)(x^2+2x+4)(x^2-2x+4)$.
   (b) $(x^4-x^2+1)(x^2-x+1)(x^2+x+1)$.
   (c) $(b+c-a)(c+a-b)(a+b-c)(a+b+c)$.
5. (a) 8.   (b) -1.

6. (b) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

1941 (Page 455)

1. (a) -15.   (b) $x=\frac{1}{3}, y=-2$.   (c) 17.
2. (a) $(-1, 8)$.   (b) 198.
3. (a) $(x+3)(x-3)(x^2-3x+9)(x^2+3x+9)$.   (b) $x^2+xy-6y^2$.
4. $2x+3$; $(2x^2+x-8)(12x^3+28x^2+18x-8)$.
5. (a) 2.   (b) $(x-y)^3+(y-z)^3+(z-x)^3$.
6. (b). $a^2x^2-b^2y^2=1$.

1942 (Page 456)

1. (a) $x=1\frac{1}{3}$.   (b) $x=8, y=1$; $y=-1, y=9$.
2. 23 ; or 25 miles an hour.

3. (a) $x^2+\dfrac{x}{2}+4$. (b) $(x+2)(2x+7)$; $(1+2b)(1-2b)^2$.

4. $\dfrac{2x^3}{x^4-a^4}$.

5. $(c-a)^2=(b-d)(ad-bc)$.   7. 142.

1943 (Page 457)

1. (a) $x(x+1)(x-1)(x+2)(x-2)$.   (b) -11.   (c) $\frac{1}{9}$.
2. (a) $x=9$.   (b) $x=2, y=10$; $0, -1$.

3 100, 80 ; or 35, 71.   5. $-\dfrac{2}{x}$.   6. $x-8$.   7. 52.

1944 (Page 458)

1   (a) $x(3x+7y)(2x-5y)$ or $(1+a-b)(1-a+b)$.

    (b) $\dfrac{x^2-ax+a^2}{x-a}$ (c) 3^{3n}.

2.   (a) $-\frac{5}{3}$ (b) $x=3,\ y=4$; $(0,-2)$ and $(0, 2)$.

3.   A, 96 and B, 32 ; or 42 years and 18 years.

4. k.   5. $\dfrac{2y}{x+y}$.   6. (a) $b^3x=ay^3$.   7. (a) $\dfrac{5y+2}{4y-3}$.

8.   (a) x^2-3x+4.   (b) 12.

1945 (Page 460)

1   (i) $(x+y)(x-y)(x^2+xy+y^2)(x^3-xy+y^2)$.

2   $4x-5$ or $x+y$.   3. $x=144$;   or $x=y=z=12$.

4. $x=3,\ y=2$.   5. $\dfrac{ab}{a^2+v^2}$.   6. $a+\dfrac{1}{a}+6$.

7.   (a) x^3+3x.   8. $c^3=3c+d$ or $x-y$.

10   18, 27, 36 ; or 24.

1946 (Page 461)

1.   (a) 11.   (b) $\frac{1}{3}$, 3.   2.   (a) $(2x-5)(2x-1)$.   (b) 1.

3.   (a) $x^4-6x^3-5x^2+42x+40$.   (b) $\frac{1}{3}$.

4.   (b) $a^2(a^2+4)=b^2$.

5.   (a) 98.   (b) $x=-3,\ y=8$; intercept$=4$.

6.   (a) 45 years.

1947 (Page 462)

1   (a) $x=-10$.   (b) 671.   2.   (a) $2x^2+3x+2$.

    (b) (i) $2x(1+4x^2)(1+2x)(1-2x)$ (ii) $3x(5x+1)(x-5)$.

3.   (a) $5x-2-\dfrac{1}{x}$ 4.   (a) $\frac{1}{3}$.   (b) 2.

5   (a) $a^2-b^2=1$.   (b) $x=2,\ y=-5$.

6   (a) Length 15 ft. Breadth 12 ft..   (b) $\dfrac{1}{abc}$.

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